Instability of Ekman-Hartmann layers near the Core Mantle Boundary

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Abstract

We investigate the instability of mixed Ekman-Hartmann boundary layers arising in rotating incompressible magnetohydrodynamics flows in a parameter regime relevant to the Earth liquid core. Relying on the small depth of the layer, we perform a local study in a half space at a given co-latitude $\theta \neq \pi/2$, and assume a mean dipolar axial magnetic field with internal sources. Instabilities are driven, for high enough Reynolds number, by the quadratic term in the momentum equation.

Nonlinear stability can be proven using energy methods in the neighborhood of the poles [2]. Next, following the work of D. Lilly [6], we restrict our analysis to the linear growth phase. We describe the dependence of the critical Reynolds number in terms of θ and Elsasser number (measuring the relative strength of Lorentz and Coriolis forces). It turns out that no matter how large the Elsasser number is, there exists a critical band centered on the equator in which instabilities can occur. For geophysically relevant values of parameters, this band could extend until some 45 degrees away from the equator. This establishes the possibility of boundary layer instabilities near the core-mantle boundary (CMB).

We finally present a first attempt of interaction with field maps at the CMB and core flows derived from the secular variation of the field [4]. We investigate a possible relation between boundary layer instabilities and rapid geomagnetic impulses (also called "jerks") observed some eight times over the last century.

1. Introduction

We investigate the instability of mixed Ekman-Hartmann boundary layers arising in rotating incompressible magnetohydrodynamic flows in a parameter regime relevant to the Earth liquid core. The magnetohydrodynamic flow in the Earth's core is believed to support a self–excited dynamo process generating the Earth's magnetic field.

One can try to model the core by a spherical shell Ω filled with a a conducting fluid of density ρ, kinematic viscosity ν, conductivity σ, which rotates rapidly with angular velocity Ω_o. We will only consider here phenomena occuring close to the outer bounding sphere. Important parameters are the Ekman number E, the Rossby number ε, the Elsasser number Λ and the magnetic Reynolds number θ defined introducing the magnetic diffusivity $\eta = (\sigma\mu_o)^{-1}$, a typical velocity U, length scale L and magnetic field $\mathcal{B}$ as

$$E = \frac{\nu}{2\,\Omega_o L^2}, \qquad \varepsilon = \frac{U}{2\,\Omega_o L},$$

$$\Lambda = \frac{\mathcal{B}^2}{2\,\rho\Omega_o\mu_o\eta}, \qquad \theta = \frac{UL}{\eta}.$$

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We perform a local study in a half space at a given co-latitude $\theta \neq \pi/2$. The incompressible conducting fluid is assumed to be governed by the incompressible Navier–Stokes equations coupled with Maxwell's equations, in which displacement currents are neglected. Outside the Earth core, the mantle Ω^c is considered as an electrical insulator and the magnetic field is therefore harmonic.

At the core mantle boundary $\partial\Omega$, we require the velocity of the fluid to vanish and the tangential component of the electric field and magnetic field to be continuous. Since we consider perturbations of a mean dipolar magnetic field $\mathbf{B}_o$, we split $\mathbf{B}$ into two parts $\mathbf{B} = \mathbf{B}_o + \theta\mathbf{b}$, where $\mathbf{b}$ denotes the scaled perturbation. Thus, equations write as follows

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\nabla p}{\varepsilon} - \frac{E}{\varepsilon}\Delta\mathbf{u} + \frac{\mathbf{e}_\Omega \times \mathbf{u}}{\varepsilon} = \frac{\Lambda}{\varepsilon}\operatorname{curl}\mathbf{b}\times\mathbf{B} + \frac{\Lambda\theta}{\varepsilon}\operatorname{curl}\mathbf{b}\times\mathbf{b}, \tag{1}$$

$$\partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{u} + \frac{\operatorname{curl}\mathbf{u}\times\mathbf{B}_o}{\theta} + \frac{\Delta\mathbf{b}}{\theta}, \tag{2}$$

$$\operatorname{div}\mathbf{u} = 0, \qquad \operatorname{div}\mathbf{u} = 0, \tag{3}$$

and in Ω^c,

$$\operatorname{curl}\mathbf{b} = 0, \quad \operatorname{curl}\mathbf{E} = -\theta\partial_t\mathbf{b}, \tag{4}$$

$$\operatorname{div}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{b} = 0. \tag{5}$$

We consider in the sequel the following orderings for $E, \Lambda, \theta, \varepsilon$

$$\varepsilon \to 0, \quad \Lambda = \mathcal{O}(1), \quad \varepsilon\theta \to 0, \quad E \sim \varepsilon^2. \tag{6}$$

These limits are relevant for the Earth's core, for which we use the following estimates:

$$\begin{aligned}
\mathcal{B} &\sim 5.10^5\ nT,\\
\rho &\sim 10^4\ kg.m^{-3},\\
\mu_o &\sim 4\pi.10^{-7}\ T.m.A^{-1},\\
\eta &\sim 1.1\ m^2.s^{-1},\\
\nu &\sim 10^{-6}\ m^2.s^{-1},\\
\Omega &\sim 7.3.10^{-5}\ rad.s^{-1},\\
U &\sim 10^{-4}\ m.s^{-1}.
\end{aligned}$$

This yields adimensionnal numbers of

$$\begin{aligned}
\Lambda &\sim 0.25,\\
\varepsilon &\sim 4.10^{-7},\\
E &\sim 1.1.10^{-15},\\
\theta &\sim 3.1.10^{2},\\
Re_o &\sim 16.8.
\end{aligned}$$

2. Nonlinear stability

First, we rigorously prove the nonlinear stability, provided the Reynolds number defined on boundary–layer characteristics is smaller than a critical value. Is is shown that the normal component of the magnetic field increases the critical Reynolds number for instability and that the nonlinear stability cannot be established everywhere at the Earth's core surface.

Let us first introduce the method on the pure Ekman case:

$$\varepsilon(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p - E\Delta\mathbf{u} + \mathbf{e}_\Omega \times \mathbf{u} = 0,$$

$$\operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u}_{|z=0} = 0,$$

where

$$E \sim \varepsilon^2 \to 0, \qquad \beta = \frac{E^{1/2}}{\varepsilon}.$$

In the interior, away from the boundary

$$\partial_t \mathbf{u}_o^{int} + \mathbf{u}_o^{int} \cdot \nabla \mathbf{u}_o^{int} + \beta \mathbf{u}_o^{int} + \nabla p = 0,$$

$$\operatorname{div} \mathbf{u}_o^{int} = 0, \qquad \mathbf{u}_o^{int} \cdot \mathbf{n}_{|z=0} = 0.$$

Nonlinear stability will be proven in the following sense:

$$\sup_{t \geq 0} \int |\mathbf{u}(t) - \mathbf{u}^{int}|^2 \leq C \int |\mathbf{u}(0) - \mathbf{u}^{int}(0)|^2.$$

The main idea of the proof relies on a formal asymptotics expansion of the solution

$$\mathbf{u} \sim \mathbf{u}_N = \sum_{k=0}^{N} \varepsilon^k \left(\mathbf{u}_k^{int}(t,x,y,z) + \mathbf{u}_k^{BL}\left(t,x,y,\frac{z}{E^{1/2}}\right)\right).$$

An approximate solution satisfies:

$$\varepsilon(\partial_t \mathbf{u}_N + \mathbf{u}_N \cdot \nabla \mathbf{u}_N) + \nabla p_N - E\Delta \mathbf{u}_N + \mathbf{e} \times \mathbf{u}_N = O(\varepsilon^{N+1}),$$

$$\operatorname{div} \mathbf{u}_N = 0, \qquad \mathbf{u}_{N|z=0} = 0.$$

The next step is to estimate the energy of the difference $\mathbf{v} = \mathbf{u} - \mathbf{u}_N$:

$$\begin{aligned}
\frac{d}{dt}\int \frac{|v|^2}{2} + \frac{E}{\varepsilon}\int |\nabla \mathbf{v}|^2 &\leq \left|\int \mathbf{v}\cdot(\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{u}_N \cdot \nabla \mathbf{u}_N)\right| + \ldots \\
&\leq C \sup_{z\geq 0}\left| z\mathbf{u}^{BL}\left(\frac{z}{E^{1/2}}\right)\right| \int |\nabla \mathbf{v}|^2 + \ldots \\
&\leq CE^{1/2} \sup |\mathbf{u}^{int}| \int |\nabla \mathbf{v}|^2 + \ldots
\end{aligned}$$

Then, a stability criterium can be deduced

$$Re_{Ekman} = \frac{\varepsilon \sup |\mathbf{u}^{int}|}{E^{1/2}} = \frac{\varepsilon}{E}\sup|\mathbf{u}^{int}|E^{1/2} = \frac{UL_{Ekman}}{\nu} \leq Re_c, \tag{7}$$

where $L_{Ekman} = E^{1/2}$ denotes the size of the Ekman boundary layer.

In the general case, one gets

$$Re_{BL} = \frac{|\mathbf{U}^{BL}|_\infty L^{BL}}{\nu} \leq Re_c^{BL}. \tag{8}$$

For the Ekman–Hartmann case, the equation far from the boundary writes again as

$$\partial_t \mathbf{u}_o^{int} + \mathbf{u}_o^{int} \cdot \nabla \mathbf{u}_o^{int} + \beta \mathbf{u}_o^{int} + \nabla p = 0,$$

$$\operatorname{div} \mathbf{u}_o^{int} = 0, \qquad \mathbf{u}_o^{int} \cdot \mathbf{n}_{|z=0} = 0,$$

with

$$\beta = \sqrt{\frac{2E}{\varepsilon^2 \tan(\tau/2)}}$$

$$\tan\frac{\tau}{2} = \frac{1}{\Lambda + \sqrt{1+\Lambda^2}}.$$

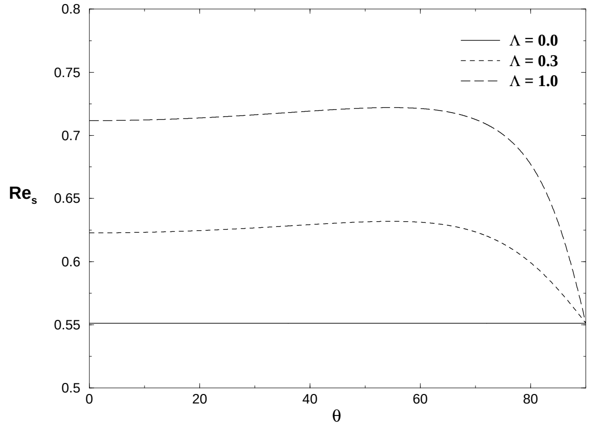

Figure 1: Representation of an analytical result from our previous study (Re_s), here using present conventions. If the Reynolds number Re attached to the boundary layer is lower than Re_s, non-linear stability is demonstrated. We concentrate in the sequel on boundary layer Reynolds number above Re_s.

As a result, the Ekman–Hartmann layer is stable if

$$\|U_\infty\| \frac{\varepsilon}{\sqrt{E}} \leq Re_s(\Lambda, \theta_o), \tag{9}$$

since then:

$$\sup_{t \geq 0} \int \left(|\mathbf{u}(t) - \mathbf{u}_s|^2 + \frac{\Lambda\theta}{\varepsilon} |b(t) - b_s|^2 \right) \leq \int \left(|\mathbf{u}(0) - \mathbf{u}_s|^2 + \frac{\Lambda\theta}{\varepsilon} |b(0) - b_s|^2 \right).$$

This expression is illustrated on figure 2. (see [2] for detailed analysis).

3. Linear Instability

We now numerically investigate the linear instability of the layer. We study the dependence of the critical Reynolds number in terms of latitude and the Elsasser number (which measures the relative strength of Lorentz forces and Coriolis forces). We focus on the parameter range relevant for the Earth's core (6). We now consider the linearized system (where λ denotes the boundary layer's scale)

$$\frac{\varepsilon}{\lambda}(\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U}) - \frac{E}{\lambda^2} \Delta \mathbf{u} + \frac{\nabla p}{\lambda} = \frac{\Lambda}{\lambda} (\operatorname{curl} \mathbf{b}) \times \mathbf{e}' - \mathbf{e} \times \mathbf{u} + \frac{\Lambda\theta}{\lambda} \left((\operatorname{curl} \mathbf{B}) \times \mathbf{b} + (\operatorname{curl} \mathbf{b}) \times \mathbf{B} \right),$$

$$\frac{\theta}{\lambda}(\partial_t \mathbf{b} + \mathbf{U} \cdot \nabla \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{b} \cdot \nabla \mathbf{U} - \mathbf{B} \cdot \nabla \mathbf{u}) = \frac{1}{\lambda} \operatorname{curl} (\mathbf{u} \times \mathbf{e}') + \frac{1}{\lambda^2} \Delta \mathbf{b},$$

$$\operatorname{div} \mathbf{u} = 0, \qquad \operatorname{div} \mathbf{b} = 0,$$

and seek travelling wave type solutions ($f(z) \exp(i\alpha(y' - ct))$). Angles are specified on figure 3.. We want to minimize Re_i depending on the parameters α, γ, θ.

In the case $\theta_o = 0$ and in the absence of electromagnetic coupling, Lilly [6] showed that the Ekman flow is linearly stable to two dimensional disturbances when the Reynolds number $Re_o = \varepsilon\sqrt{2/E}$ exceeds the critical value 54.16. The purpose of this work is to extend Lilly's results to incompressible MHD flows at a given colatitude $\theta_o \in [0, \pi/2)$ for dipolar static magnetic field.

After validation of against Lilly's study and Leibovich and Lele [7], we obtain the curve for critical Reynolds number Re_i for instability versus colatitude (figure 3.). The angle for the

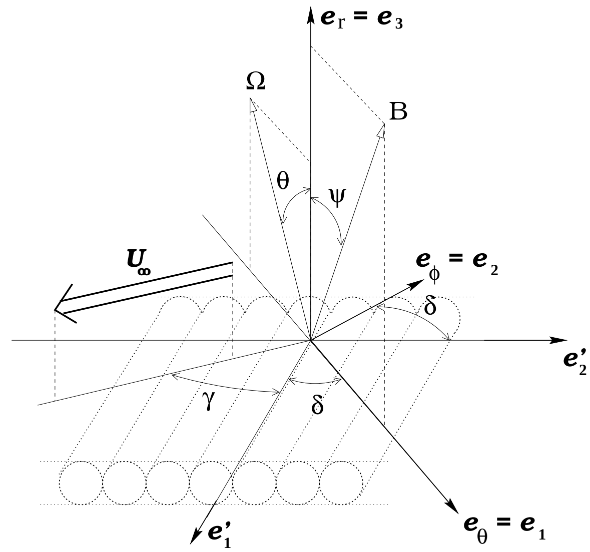

Figure 2: Geometry of the local study. Rotation vector and magnetic field respectively make an angle θ and ψ with the normal to the boundary. Travelling wave solutions are sought for an external velocity $\mathbf{U}_\infty$. These quantities respectively make an angle δ and $\gamma_O = \delta + \gamma$ with the plante $(\Omega, \mathbf{B})$.

travelling wave solution with $\mathbf{e}_\theta$ (namely δ) is represented on figure 3.. Note the bifurcation in two branches past a critical colatitude (as observed in the non–magnetic case by Leibovich and Lele). Figure 5.a shows the product $\alpha \times c_r$, this is the pulsation of the travelling wave, near the equator waves propagate in longitude, above and below they propagate towards the poles. Figure 5.b displays $\delta + \gamma$, this gives the angle of $\mathbf{U}_\infty$ with $\mathbf{e}_\theta$, note that this angle is close to -10 degrees near the equator.

4. Linear instability from modeled large scale field over the years

We finally present a first attempt of interaction with field maps at the CMB and core flows derived from the secular variation of the field [4]. Because the mantle is insulating, the principal field as mesured in observatories can be expanded in spherical harmonics and downward continued to the core surface, i.e. the boundary where we seek instabilities (see figure 4.). Consecutive models in time suggest a surface flow to account for the field variations (additional hypothesis are however needed to ensure uniqueness of this reconstructed flow: the flow is assumed to be of large scale and tangentialy geostrophic). We want to investigate possible relations between boundary layer instabilities and magnetic field variations. Of particular interests are rapid geomagnetic impulses (also called "jerks") observed some eight times over the last century (1969, 1979, and 1992 are among the most recent related events [5]). These events being very rapid in time, a small scale instability represents a very good candidate to account for these observations.

The model needs to include an additional angle ψ_2 now that the magnetic field does not necessarily lie in the meridional plan (see figure 4.). The computed growth rate are represented on figure 8, whereas the large scale velocity is drawn on figure 9, both in 1980. The complete results will be published in [4]. It is clear that center of vortices correspond to stable regions. As expected from the results of the previous section, the most unstable zone is located in a neighborhood of the equator in the Pacific ocean.

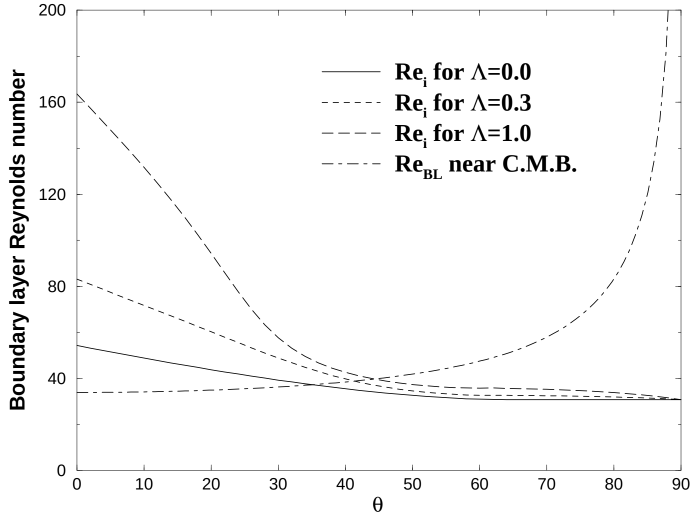

Figure 3: Boundary layer Reynolds number for instability for three different values of the Elsasser number versus co-latitude θ. An estimation of the boundary layer Reynolds number near the Core-Mantle Boundary is also represented for comparison.

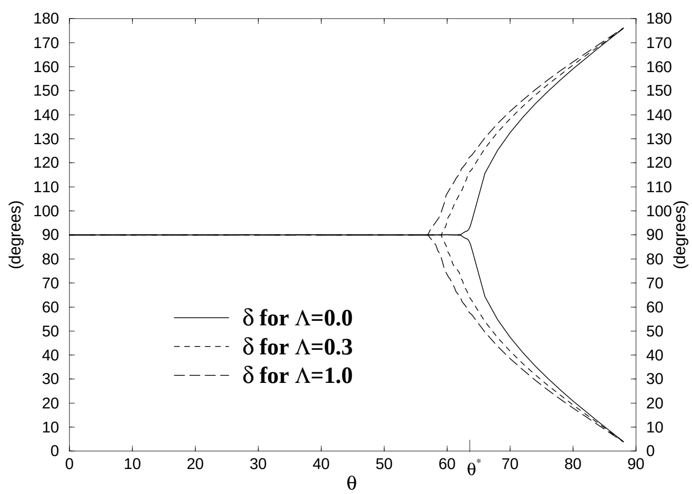

Figure 4: Angles δ for instability with respect to e_θ represented versus the co-latitude θ. The instability develops in the e_ϕ direction near the pole. Past a critical co-latitude (decreasing with Λ) two branches of solutions exist. The instability is aligned with e_θ near the equator.

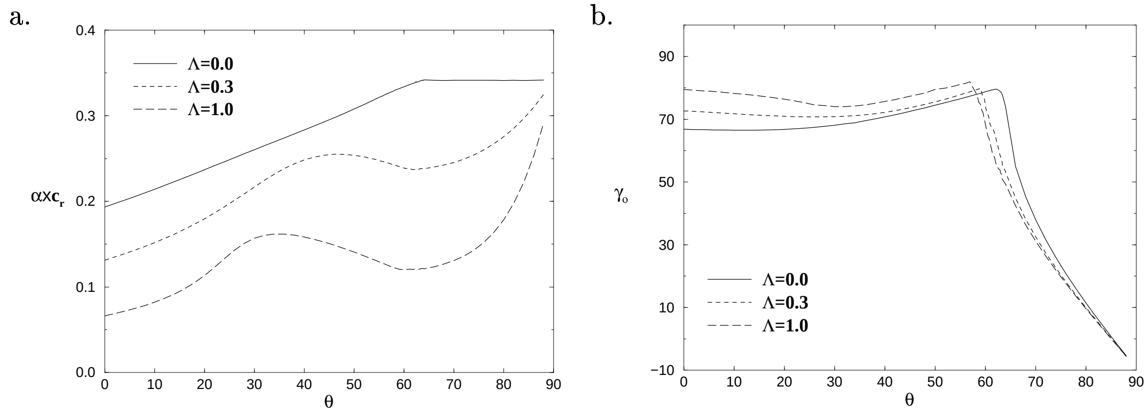

Figure 5: a. phase velocity $\alpha \times c_r$ b. $\gamma_o = \delta + \gamma$ (between the e_θ direction and $\mathbf{U}_\infty$). Both quantities are represented versus co-latitude θ.

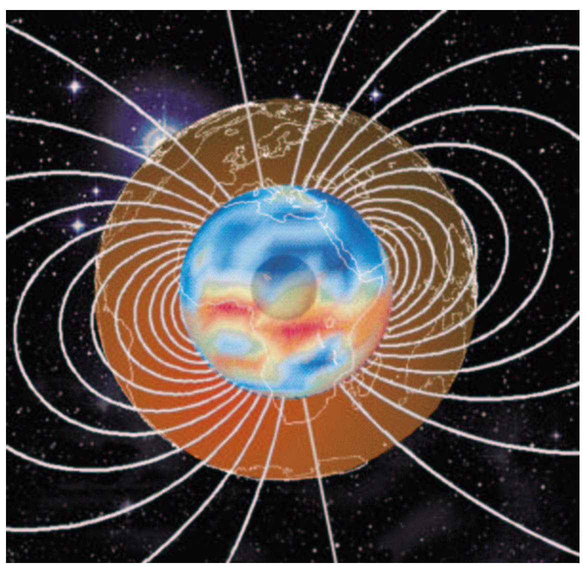

Figure 6: The principal magnetic field known at the surface of the Earth can be downward prolongated through the insulating mantle (the color code presents the radial component of the prolongated field). This klnowledge of the field at the outer boundary of the core with the fluid flow derived from its variations, provides the starting point for a refined boundary layer instability study.

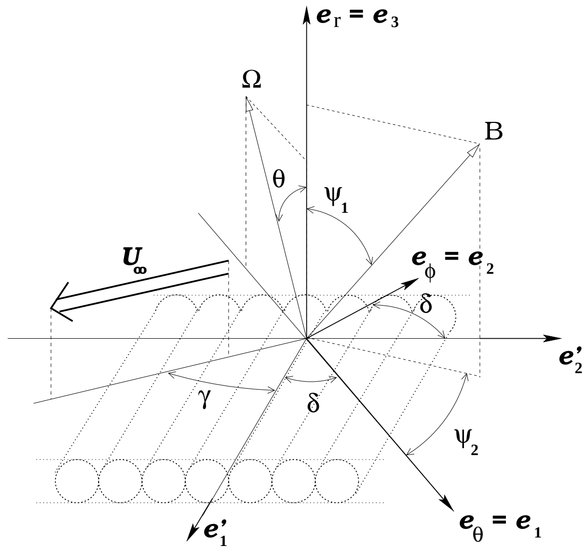

Figure 7: Geometry for the local instability study based on actual magnetic field and core flow models. An additional angle ψ_2 is added now that the field is not purely dipolar axial.

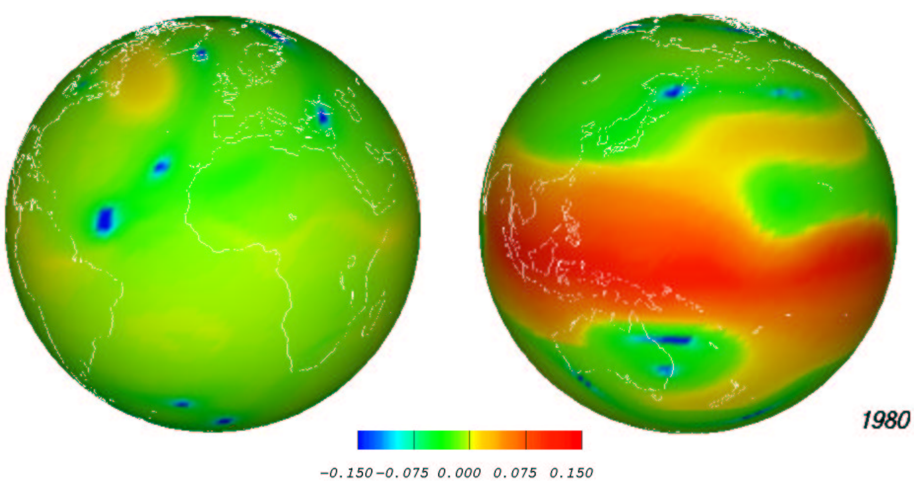

Figure 8: Growth rate of linear instabilities in 1980.

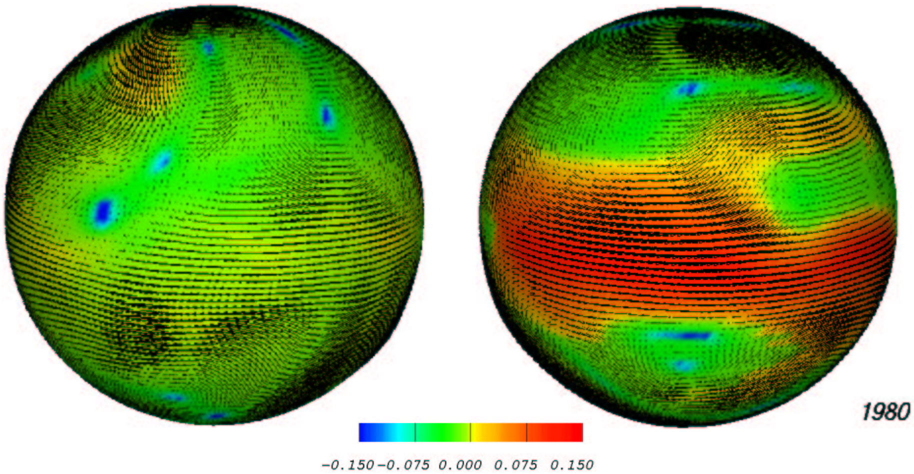

Figure 9: Growth rate of linear instabilities with represented with the large scale velocity field in 1980.

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