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On magnetic boundary conditions for non-spectral dynamo simulations

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We address issues associated with non-local magnetic boundary conditions for non-spectral dynamo simulations. We introduce an integro-differential formulation for a domain bounded by an insulating outer domain. We show how to combine the flexibility of a local discretisation with a rigorous formulation of magnetic boundary conditions in arbitrary geometries. This formulation substantiates from mathematical point of view a new method for numerical solution of magnetohydrodynamic problems with non-local boundary conditions based on coupling finite volumes and boundary elements. Finally, we discuss practical efficiency of this new method.

Keywords: Boundary conditions; Dynamo modeling; Numerical simulation

1. Introduction

Over the last few years numerical dynamo models for the planets and stars have made impressive progress. The results from spectral models (based on poloidal-toroidal decomposition and spherical harmonic expansion) have been most impressive (e.g. Glatzmaier and Roberts 1995, Christensen *et al*. 1999, Christensen *et al*. 2001, Brun *et al*. 2004). Yet there seems to be a general impression that a new generation of model is needed to allow further progress (decrease the Ekman number in the case of the Earth, increase the Reynolds number in the case of the Sun and decrease the magnetic Prandtl number in both cases). Several groups are now working on the development of methods based on a local dicretisation of the governing equations (finite differences, finite-volumes, finite-elements, spectral-elements, ...). These developments are in large part motivated by the possibility of local mesh refinement near sharp structures and by an efficient implementation on massively parallel computers. These approaches have been widely used in the past for fluid mechanics problems, yet their application to dynamo problems raises new issues, such as magnetic boundary conditions. While a proper treatment in the spectral domain is well

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established, implementation of magnetic boundary conditions on such grid methods raises new issues.

Alternative approaches to resolve such boundary conditions without expansion into spherical functions have been proposed in Jepps (1975) and Ivanova (1976). These are discussed and compared by Pavel Hejda (1982). Both approaches are based on poloidal-toroidal decomposition for the magnetic and velocity fields. We have recently proposed a numerical algorithm to handle this problem using primitive variables (Iskakov *et al.* 2004), and therefore avoiding the associated increase in differential order. Discussions with other teams incited us to provide a solid theoretical justification for such an approach. This is the aim of the present article. We will address here the issue of non-local magnetic boundary conditions with primitive variables (rather than the standard poloidal-toroidal decomposition). We substantiate our method from a mathematical point of view.

2. Governing equations

We shall consider here a simple problem of induction in a conducting domain Ω_c of finite resistivity surounded by an insulating domain Ω_i. The magnetic permeability is assumed to be uniform in all space (say $\mu = \mu_0$).

The magnetic fields in a flow of conducting fluid is governed by the magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \eta(\nabla \times \mathbf{B}), \tag{1a}$$

where η is the magnetic diffusivity, in the insulator it is described by

$$\nabla \times \mathbf{B} = \mathbf{0}. \tag{1b}$$

These equations are complemented in both regions by Gauss law

$$\nabla \cdot \mathbf{B} = 0. \tag{1c}$$

Equations (1a, b) reveal a change from time-dependent to time independent on the boundary between conducting and non-conducting domains. This raises particular issues from the numerical point of view. Inside the conducting domain the behavior of the magnetic field is characterized by a typical diffusion or advection time. On the other hand the characteristic time in the insulator is vanishing. In this situation in order to calculate magnetic field at a given point on the boundary one has to use information about the global behavior of the magnetic field over the whole boundary. Such boundary conditions are known as global or non-local boundary conditions.

3. Jump conditions for the induction equation

To solve a partial differential equation (PDE) in a bounded domain, one often needs *boundary conditions*, which select the solution of physical relevance, and ensure the well-posedness of the PDE from the mathematical viewpoint. To assess how many boundary conditions are needed is not always straightforward, even for basic equations. Let us simply recall that the following transport equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad x > 0, \quad \text{and} \quad u|_{t=0} = u_0 \quad t = 0, \tag{2}$$

requires no boundary condition when $a \leq 0$, and one boundary condition (e.g. the value of u at $x=0$) when $a > 0$.

In particular, one has to pay attention to the difference between *internal* and *external* boundaries. On an outer boundary, one generally specifies some variable values, leading for instance to Dirichlet-type or Neumann-type boundary conditions. In the case of an internal boundary between two regions 1 and 2, one specifies *jump conditions*, that characterize the difference between variable values in regions 1 and 2. "Generically", the number of these jump conditions is $n_1 + n_2$, where n_1 and n_2 are the numbers of outer conditions required for the PDE solution in regions 1 and 2.

As a simple illustration, the parabolic problem

$$\frac{\partial u}{\partial t} - \kappa_1\frac{\partial^2 u}{\partial x^2} = f_1, \quad x \in -(1,0), \quad \text{and} \quad \frac{\partial u}{\partial t} - \kappa_2\frac{\partial^2 u}{\partial x^2} = f_2, \quad x \in (0,1), \tag{3}$$

modeling the heat diffusion between two regions of different conductivities κ_1 and κ_2, requires a boundary condition on each outer boundary $x = \{\pm 1\}$: either the temperature, the flux, or their combination. Therefore, at the interface $\{x = 0\}$, two jump conditions are needed; natural ones are continuity of temperature, and continuity of flux.

Let us now focus on the induction equation (1a). Assuming that the magnetic diffusivity η takes two different values $\eta_\pm$, in two regions $\Omega_\pm$ separated by an interface Γ, jump conditions should be imposed on Γ.

By comparison with (3), one could expect 6 jump conditions for the vector-valued equations (1a). However, such a comparison does not hold. Contrary to the heat equation, the induction equation alone is not parabolic. Indeed, in $\Omega_\pm$, the resistive term

$$\nabla \times \left[\eta_\pm(\nabla \times \cdot)\right] = -\eta_\pm \nabla^2 + \eta_\pm \nabla(\nabla \cdot). \tag{4}$$

The second term in the right hand side of (4) prevents this operator from being elliptic. To cancel this term, one must take into account the divergence free condition (1c). This provides an additional constraint, which in fact reduces the number of jump conditions.

This is usually seen through the standard decomposition of $\mathbf{B}$ in poloidal and toroidal components, which reduces equation (1a) to two scalar equations. Therefore only 2 independent value conditions at the outer boundaries, and 4 independent jump conditions on the surface of discontinuity are required.

These jump conditions can be directly obtained from (1a). Let us rewrite (1a) using the electric field $\mathbf{E}$,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \eta\nabla \times \mathbf{B} = \mathbf{E} + \mathbf{u} \times \mathbf{B}. \tag{5a, b}$$

Assuming that all terms in equation (5a) are integrable over $\Omega = \Omega_- \cup \Omega_+$, which is physically expected, (5a) can be integrated over a contour crossing the boundary. Taking the limit of infinitely small contour length in the normal direction, one concludes that the tangential components of the electric field are continuous at the contact boundary

$$[\![\mathbf{E} \times \mathbf{n}]\!] = \mathbf{0}. \tag{6}$$

where **n** is the normal to the surface of discontinuity.

Let us now consider the equation of magnetostatics (5b),

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \qquad \mu_0 \mathbf{j} := \eta^{-1}(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{7a, b}$$

In the absence of surface density currents, one can again assume all terms in the above equation to be integrable over Ω. Proceeding as previously yields another two jump boundary conditions: the tangential components of the magnetic induction are continuous

$$[\![\mathbf{B} \times \mathbf{n}]\!] = \mathbf{0}. \tag{8}$$

From a physical point of view, these conditions imply that the energy flux and the normal current through the boundary are continuous

$$[\![(\mathbf{B} \times \mathbf{E}) \cdot \mathbf{n}]\!] = 0, \qquad [\![\mathbf{j} \cdot \mathbf{n}]\!] = 0. \tag{9a, b}$$

Conditions (6, 8) are sufficient to fully determine **B** in $\Omega_\pm$ (Roberts 1967, Jackson 1972). One can rigorously demonstrate this sufficiency by using a variational formulation of (1a). Let $\boldsymbol{\psi} = \boldsymbol{\psi}(\mathbf{x}, t)$ be a smooth vector-valued test function defined on $\overline{\Omega} \times \mathbb{R}^+$, where $\overline{\Omega} = \Omega \cup \partial\Omega$. Multiplying (5a) by $\boldsymbol{\psi}$, and integrating over $[0, T] \times (\Omega_- \cup \Omega_+)$, one gets

$$\int_0^T \int_{\Omega_- \cup \Omega_+} \frac{\partial \mathbf{B}}{\partial t}(\mathbf{x}, t) \cdot \boldsymbol{\psi}(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t + \int_0^T \int_{\Omega_- \cup \Omega_+} \nabla \times \mathbf{E}(\mathbf{x}, t) \cdot \boldsymbol{\psi}(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0, \tag{10}$$

Integrating by parts, in both space and time yields

$$\begin{aligned} &- \int_0^T \int_{\Omega_- \cup \Omega_+} \mathbf{B}(\mathbf{x}, t) \cdot \frac{\partial \boldsymbol{\psi}}{\partial t}(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t + \int_0^T \int_{\Omega_- \cup \Omega_+} \mathbf{E}(\mathbf{x}, t) \cdot \nabla \times \boldsymbol{\psi}(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x}\, dt \\ &+ \int_{\Omega_- \cup \Omega_+} \mathbf{B}(\mathbf{x}, T) \cdot \boldsymbol{\psi}(\mathbf{x}, T)\, \mathrm{d}^3\mathbf{x} - \int_{\Omega_- \cup \Omega_+} \mathbf{B}(\mathbf{x}, 0) \cdot \boldsymbol{\psi}(\mathbf{x}, 0)\, \mathrm{d}^3\mathbf{x} \\ &+ \int_0^T \int_\Gamma [\![\mathbf{E}(\mathbf{x}, t) \times \mathbf{n}]\!] \cdot \boldsymbol{\psi}(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0. \end{aligned} \tag{11}$$

To conclude uniqueness, one needs an energy estimate on **B**. Jump condition (6) provides such an estimate by cancelling the last integral of (11). Then, due to (8), there exists a sequence $(\boldsymbol{\psi}_n)_{n\in\mathbb{N}}$ of smooth functions defined on $\overline{\Omega} \times \mathbb{R}^+$ such that

$$\boldsymbol{\psi}_n \xrightarrow[n\to+\infty]{} \mathbf{B}, \qquad \nabla \times \boldsymbol{\psi}_n \xrightarrow[n\to+\infty]{} \nabla \times \mathbf{B}, \tag{12a, b}$$

in such a way that, taking the limit in (11), yields

$$0 = \tfrac{1}{2}\int_{\Omega_-\cup\Omega_+} |\mathbf{B}(\mathbf{x},T)|^2\,\mathrm{d}^3\mathbf{x} - \tfrac{1}{2}\int_{\Omega_-\cup\Omega_+} |\mathbf{B}(\mathbf{x},0)|^2\,\mathrm{d}^3\mathbf{x} + \int_0^T\int_{\Omega_-\cup\Omega_+} \mathbf{E}(\mathbf{x},t)\cdot(\nabla\times\mathbf{B}(\mathbf{x},t))\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t. \tag{13}$$

Given this estimate, one can conclude that $\mathbf{B}$ is unique. Indeed, assume two solutions $\mathbf{B}_1$ and $\mathbf{B}_2$ of (1a), (6), (8), with the same initial conditions. Then $\tilde{\mathbf{B}} = \mathbf{B}_1 - \mathbf{B}_2$ is still a solution of (1a), (6), (8), with homogeneous initial conditions ($\tilde{\mathbf{B}}(\mathbf{x},0) = 0$).

From a mathematical point of view, the energy estimate (13) is

$$\tfrac{1}{2}\int_{\Omega_-\cup\Omega_+} |\tilde{\mathbf{B}}(\mathbf{x},T)|^2\,\mathrm{d}^3\mathbf{x} + \int_0^T\int_{\Omega_-\cup\Omega_+} \eta|\nabla\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t = \int_0^T\int_{\Omega_-\cup\Omega_+} \left(\mathbf{u}\times\tilde{\mathbf{B}}\right)\cdot\left(\nabla\times\tilde{\mathbf{B}}\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}t. \tag{14}$$

Using Cauchy-Schwarz inequality

$$\tfrac{1}{2}\int_{\Omega_-\cup\Omega_+} |\tilde{\mathbf{B}}(\mathbf{x},T)|^2\,\mathrm{d}^3\mathbf{x} + \int_0^T\int_{\Omega_-\cup\Omega_+} \eta|\nabla\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t \le \left(\int_0^T\int_{\Omega_-\cup\Omega_+} |\mathbf{u}\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_{\Omega_-\cup\Omega_+} |\nabla\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t\right)^{1/2}, \tag{15}$$

and its immediate consequences

$$\min_{\mathbf{x}\in\Omega}(\eta)\int_0^T\int_{\Omega_-\cup\Omega_+} |\nabla\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t \le \left(\int_0^T\int_{\Omega_-\cup\Omega_+} |\mathbf{u}\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_{\Omega_-\cup\Omega_+} |\nabla\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t\right)^{1/2}, \tag{16a}$$

$$\left(\int_0^T\int_{\Omega_-\cup\Omega_+} |\nabla\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t\right)^{1/2} \le \frac{1}{\min\limits_{\mathbf{x}\in\Omega}(\eta)}\left(\int_0^T\int_{\Omega_-\cup\Omega_+} |\mathbf{u}\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t\right)^{1/2}, \tag{16b}$$

one deduces from (15) and (16b)

$$\tfrac{1}{2}\int_{\Omega_-\cup\Omega_+} |\tilde{\mathbf{B}}(\mathbf{x},T)|^2\,\mathrm{d}^3\mathbf{x} + \int_0^T\int_{\Omega_-\cup\Omega_+} \eta|\nabla\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t \le \frac{1}{\min_{\mathbf{x}\in\Omega}(\eta)}\int_0^T\int_{\Omega_-\cup\Omega_+} |\mathbf{u}\times\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t \le \frac{C}{\min_{\mathbf{x}\in\Omega}(\eta)}\int_0^T\int_{\Omega_-\cup\Omega_+} |\tilde{\mathbf{B}}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t, \tag{17}$$

where the constant C depends on $\mathbf{u}$. It then follows from Gronwall's lemma (e.g. Reinhard 1989) that $\tilde{\mathbf{B}} = \mathbf{0}$.

4. Jump conditions in the limiting case

Let us now consider jump conditions in the limiting case when one of the domains tends to be an insulator. Noting $\eta_i := \eta_-$ (resp. $\eta_c := \eta_+$) the diffusivity in the insulator (resp. in the conductor), we investigate the limit $\eta_i \to +\infty$.

In the insulator $\mathbf{B}$ satisfies (1bc). Since $\mathbf{B}$ is both divergence and curl free,

$$\mathbf{B} = -\nabla\phi, \qquad \nabla^2\phi = 0. \tag{18a, b}$$

Such problems require only one outer condition, one therefore expects 3 jump conditions on the boundary between the conducting and insulating domains.

To describe the transition from 4 to 3 jump conditions, one can expand magnetic field on both sides of the boundary, in powers of $\varepsilon = (\eta_i)^{-1}$. Denoting $\mathbf{B}_c$ the field in the conductor and $\mathbf{B}_i$ the field in the insulator, one can write

$$\mathbf{B}_c = \mathbf{B}_c^{(0)} + \varepsilon\mathbf{B}_c^{(1)} + \varepsilon^2\mathbf{B}_c^{(2)} + \cdots, \qquad \mathbf{B}_i = \mathbf{B}_i^{(0)} + \varepsilon\mathbf{B}_i^{(1)} + \varepsilon^2\mathbf{B}_i^{(2)} + \cdots. \tag{19a, b}$$

Substituting these series into equations (1) and into the jump conditions (6, 8), one obtains at leading order of ε:

$$\nabla\times\mathbf{B}_i^{(0)} = \mathbf{0}, \qquad \nabla\cdot\mathbf{B}_i^{(0)} = 0, \tag{20a, b}$$

$$\frac{\partial\mathbf{B}_c^{(0)}}{\partial t} - \eta_c\nabla^2\mathbf{B}_c^{(0)} - \nabla\times(\mathbf{u}\times\mathbf{B}_c^{(0)}) = \mathbf{0}, \qquad \nabla\cdot\mathbf{B}_c^{(0)} = 0. \tag{20c, d}$$

Since $\mathbf{E} = \varepsilon^{-1}\nabla\times\mathbf{B} - \mathbf{u}\times\mathbf{B}$ in the insulator, jump condition (6) becomes degenerate

$$(\nabla\times\mathbf{B}_i^{(0)})\times\mathbf{n}\big|_\Gamma = \mathbf{0}\times\mathbf{n}, \tag{21}$$

whereas (8) provides two independent jump conditions

$$\mathbf{B}_i^{(0)}\times\mathbf{n}\big|_\Gamma = \mathbf{B}_c^{(0)}\times\mathbf{n}\big|_\Gamma. \tag{22}$$

Higher order terms reveal the sharp structures necessary to accomodate any departure from the leading balance.

Note that (21) is redundant with the equation on $\mathbf{B}_i^{(0)}$, so that we are left with two scalar jump conditions (22). The third jump condition stems from the divergence-free condition on $\mathbf{B}_c^{(0)}$ and $\mathbf{B}_i^{(0)}$: integrating over an infinitely thin penny shaped disk containing a small surface element yields

$$\mathbf{B}_i^{(0)}\cdot\mathbf{n}\big|_\Gamma = \mathbf{B}_c^{(0)}\cdot\mathbf{n}\big|_\Gamma. \tag{23}$$

This provides the three needed conditions

$$[\![\mathbf{B}]\!] = \mathbf{0}. \tag{24}$$

5. Integro-differential formulation

5.1. *Boundary problem in the conductor*

We will treat here the conductor in a differential form. While this is the case in most of the recent developments, alternative formulations can be proposed (e.g. Stefani *et al.* 2000, Xu *et al.* 2004).

Continuity of **B** accross the boundary allows us to reconnect the solution of the induction equation in the conductor with the solution of an elliptic problem in the insulator. On the boundary of the conducting domain $\partial\Omega_c$ equation (5a) can be used only for the normal component of **B** since tangental components of $\nabla \times \mathbf{E}$ are discontinuous. To supplement this equation, two boundary conditions need to be provided for the tangential components of **B**. Since **B** is continuous on the boundary, these boundary conditions for the induction equation can be obtained from the elliptic problem in the insulator. In case of non-local boundary conditions one thus expects these to be of the form

$$\mathbf{B} \times \mathbf{n} = \mathcal{F}(\mathbf{B} \cdot \mathbf{n}), \quad \mathbf{x} \in \partial\Omega_c, \tag{25}$$

where $\mathcal{F}$ is some integral operator on the boundary surface.

5.2. *Integral formulation in the insulator*

In order to solve the exterior problem, we will rely on the integral formulation. The differential approaches can also be used in this context, with issues associated to bounding the exterior domain (e.g. Chan *et al.* 2001, Matsui and Okuda 2005, Guermond *et al.* 2003, Alouges 2001).

The form of the integral operator (25) needs to be specified from the solution of the elliptic problem (18) in the insulating domain. The insulator being unbounded, an additional statement is needed for the magnetic field at infinity

$$\phi \to \mathrm{O}(r^{-2}), \qquad r \to \infty. \tag{26}$$

Neumann boundary condition for the elliptic problem (18) are provided by the normal component of the magnetic field on the boundary $\partial\Omega$

$$\left.\frac{\partial\phi}{\partial n}\right|_{\partial\Omega} = -B_n, \qquad (B_n : \partial\Omega \to \mathbb{R}). \tag{27}$$

It follows from potential theory that the harmonic function ϕ at any point inside the domain can be represented through the surface integral of its boundary values (e.g. Bonnet 1995)

$$\phi(\mathbf{x}) = -\int_{\partial\Omega} \left(\phi(\mathbf{y}) \frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{y}) + B_n(\mathbf{y})\, G(\mathbf{x}, \mathbf{y}) \right) \mathrm{ds}(\mathbf{y}), \quad \mathbf{x} \in \Omega_i, \tag{28}$$

where $G(\mathbf{x},\mathbf{y}) = -1/(4\pi|\mathbf{x}-\mathbf{y}|)$ is the fundamental solution of the Laplace equation with unit normal $\mathbf{n}$ directed into the insulator.

If $\partial\Omega$ is smooth enough (28) is still valid in the limiting case when $\mathbf{x} \in \partial\Omega$. However a singularity occurs at $\mathbf{y} = \mathbf{x}$. To eliminate this singularity, one rewrites (28) as

$$\left(1 + \int_{\partial\Omega} \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y})\,\mathrm{ds}(\mathbf{y})\right)\phi(\mathbf{x}) \\ = -\int_{\partial\Omega}\left((\phi(\mathbf{y}) - \phi(\mathbf{x}))\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) + B_n(\mathbf{y})\,G(\mathbf{x},\mathbf{y})\right)\mathrm{ds}(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega, \tag{29}$$

or

$$\tfrac{1}{2}\phi(\mathbf{x}) = -\,\mathrm{PV}\int_{\partial\Omega}\left(\phi(\mathbf{y})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) + B_n(\mathbf{y})\,G(\mathbf{x},\mathbf{y})\right)\mathrm{ds}(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega, \tag{30}$$

where the integral is taken as the Cauchy principal value (noted with PV).

Consequently the tangential components of the magnetic field are

$$\mathbf{B}(\mathbf{x}) \times \mathbf{n} = -\nabla\phi(\mathbf{x}) \times \mathbf{n}, \quad \mathbf{x} \in \partial\Omega. \tag{31}$$

The solutions of integral equations (30, 31) implicitly provide two boundary conditions (25) for the induction equation. One can thus reformulate the governing equations in terms of an integro-differential problem

$$\begin{cases} \mathbf{E} = \eta\nabla \times \mathbf{B} - \mathbf{u} \times \mathbf{B}, \quad \nabla\cdot\mathbf{B} = 0, \quad \mathbf{x} \in \Omega_c \cup \partial\Omega_c, \\ \dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad \mathbf{x} \in \Omega_c, \\ \dfrac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{n}) = -\,(\nabla \times \mathbf{E})\cdot\mathbf{n}, \quad \mathbf{x} \in \partial\Omega_c, \\ \mathbf{B} \times \mathbf{n} = -\,\nabla\phi \times \mathbf{n}, \quad \mathbf{x} \in \partial\Omega_c, \\ \phi(\mathbf{x}) = -2\,\mathrm{PV}\displaystyle\int_{\partial\Omega}\left(\phi(\mathbf{y})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) + B_n(\mathbf{y})\,G(\mathbf{x},\mathbf{y})\right)\mathrm{ds}(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega_c, \end{cases} \tag{32}$$

where Ω_c is the conducting domain and $\partial\Omega_c$ is its boundary with an insulator.

5.3. *Formulation in terms of E*

For numerical reasons, it can be convenient to rewrite the above system in terms of the electric fields since it then does not involve the divergence-free constraint (1c) and simplifies implicit numerical integration of the resulting discrete system.

Let us introduce, for convenience, the vector $\dot{\mathbf{B}} = \partial\mathbf{B}/\partial t$; then the induction equation (5a, b) can be written as

$$\frac{\partial \mathbf{E}}{\partial t} = \eta\nabla \times \dot{\mathbf{B}} - \frac{\partial}{\partial t}(\mathbf{u} \times \mathbf{B}), \qquad \dot{\mathbf{B}} = -\nabla \times \mathbf{E}. \tag{33a, b}$$

On the boundary $\partial\Omega_c$, equation (33b) is satisfied only for the normal component of $\dot{\mathbf{B}}$ (since the tangential components of $\nabla \times \mathbf{E}$ are discontinuous)

$$\dot{\mathbf{B}} \cdot \mathbf{n} = -(\nabla \times \mathbf{E}) \cdot \mathbf{n}, \qquad \mathbf{x} \in \partial\Omega_c. \tag{34}$$

To supplement the system (33a, 34) on the boundary, two additional conditions are needed for the tangential components of $\dot{\mathbf{B}}$. As previously, these are obtained from the solution of an elliptic problem in the insulator in the form

$$\dot{\mathbf{B}} \times \mathbf{n} = \mathcal{F}(\dot{\mathbf{B}} \cdot \mathbf{n}), \quad \mathbf{x} \in \partial\Omega_c, \tag{35}$$

where $\mathcal{F}$ is the same integral operator on the boundary surface as in (25).

An alternative integro-differential formulation can then be introduced in terms of the electric field

$$\begin{cases} \dfrac{\partial \mathbf{E}}{\partial t} = \eta \nabla \times \dot{\mathbf{B}} - \dfrac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} - \mathbf{u} \times \dot{\mathbf{B}}, \qquad \dfrac{\partial \mathbf{B}}{\partial t} = \dot{\mathbf{B}}, \quad \mathbf{x} \in \Omega_c \cup \partial\Omega_c, \\ \dot{\mathbf{B}} = -\nabla \times \mathbf{E}, \quad \mathbf{x} \in \Omega_c, \\ \dot{\mathbf{B}} \cdot \mathbf{n} = -(\nabla \times \mathbf{E}) \cdot \mathbf{n}, \quad \mathbf{x} \in \partial\Omega_c, \\ \dot{\mathbf{B}} \times \mathbf{n} = -\nabla\phi \times \mathbf{n}, \quad \mathbf{x} \in \partial\Omega_c, \\ \phi(\mathbf{x}) = -2\,\mathrm{PV} \displaystyle\int_{\partial\Omega} \left(\phi(\mathbf{y}) \frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{y}) + \dot{B}_n(\mathbf{y})\, G(\mathbf{x}, \mathbf{y}) \right) \mathrm{ds}(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega_c. \end{cases} \tag{36}$$

Such formulation allows implicit numerical time-stepping scheme preserving the divergence of $\mathbf{B}$.

6. Discussion

Equations (36) combine the flexibility of a local discretisation with a rigorous formulation of magnetic boundary conditions next to the insulator in arbitrary geometries. In Iskakov *et al.* (2004), we have presented a numerical method which can be interpreted as a discrete version of this formulation. The method is based on combination of finite volumes and boundary elements. The last two equations in (36) are handled using the Boundary Element Method (BEM), i.e. the boundary surface is subdivided into small elements and each element contains several nodes where the boundary variables and coordinates are defined. Then the tangental components of the magnetic field on the boundary are calculated through its normal component by multiplying with the BEM matrix which represents the integral operator $\mathcal{F}$ in (35).

To establish the accuracy of this approach as a test we reproduce the analytical solution for the decaying dipole field in a sphere. The relative error between the analytical and numerical values for the dipole decay rate σ and for the decaying field solution $\mathbf{B}$ are plotted in figure 1 for decreasing mesh spacing. The graph illustrates second order accuracy.

To calculate the boundary conditions at N^2 points on the bounding surface using the BEM matrix multiplication, one obviously needs $O(N^4)$ operations. This is a lot in

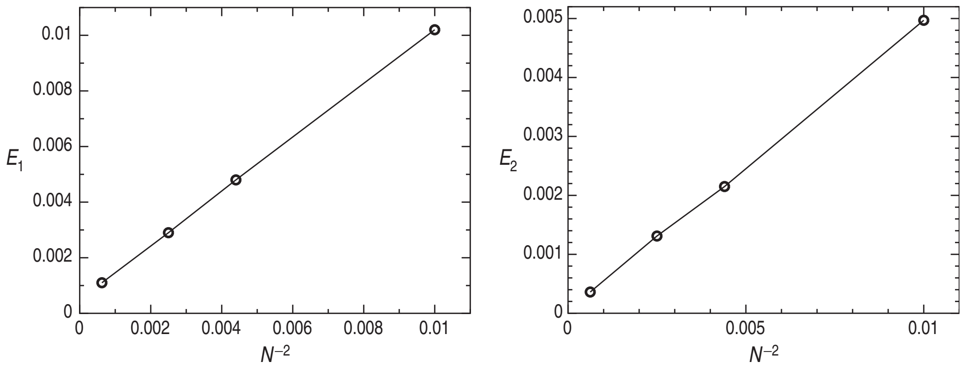

Figure 1. Relative error between the analytical and the numerical dipole decay rate $E_1(N) = |\pi^2 - \sigma(N)|/|\sigma(N)|$ (left) and relative error between the numerical solution $\mathbf{B}(N)$ and the eigenfunction $\mathcal{E}$ in the L_2 norm: $E_2(N) = \|\mathcal{E} - \mathbf{B}(N)\|_2/\|\mathbf{B}(N)\|_2$ (right) for various mesh sizes.

comparison with extending the computational mesh into the insulating domain and numerically solving the Laplace problem using a fast Poisson solver (which requires only $O(N^3)$ operations). A consistent and accurate procedure for extending the mesh into the insulator can be found for example in Alouges (2001). Here, we will discuss how the integral approach can be modified to become more practical and efficient.

The BEM discretization of the last equation in (36) leads to the system of linear equations

$$\phi_{\tilde{k}} = \sum_k C_{\tilde{k}}^k \phi_k + \sum_k D_{\tilde{k}}^k \dot{B}_{n_k}, \tag{37}$$

or equivalently in matrix form

$$\bar{\Phi} = C\,\bar{\Phi} + D\,\bar{\dot{B}}_n, \tag{38}$$

where the coefficients

$$C_{\tilde{k}}^k = c_k \frac{x_{\tilde{k}} - x_k}{\|x_{\tilde{k}} - x_k\|^3}, \qquad D_{\tilde{k}}^k = \frac{d_k}{\|x_{\tilde{k}} - x_k\|}, \tag{39}$$

describe pairwise interaction between points on the surface in terms of the gravitational or elecrostatic potential and c_k and d_k are geometrical factors describing the boundary elements.

The system (37) is nonsparse and non-symmetric, but has asymptotically limited condition numbers. For such systems, the generalized conjugate residual algorithm (GCRA) is known to converge rapidly (Rokhlin 1985). The cost of GCRA is asymptotically determined by the number of operations required for applying the matrices C and D to the vector. For N^2 points on the surface these operations can be achieved very efficiently through the fast multipole method (FMM) by $O(N^2)$ operations (Greengard and Rokhlin 1997, Cheng *et al.* 1999). Then the cost of solving non-local magnetic boundary conditions (last equation in (32) or in (36)) also becomes

$O(N^2)$ operations provided that the numerical mesh on the boundary is fixed and recalculation of the coefficients (39) is not required.

Finally, most natural objects (planets, stars and galaxies) investigated in the field of dynamo modeling correspond to axisymmetric computationnal domains. Under such circumstances an alternative hybrid approach can be proposed since the elliptic problem (18) can be transformed in spectral space in the longitudinal direction. Modes then decouple and only the latitudinal interactions require an integral treatment. The resulting cost is then reduced from standard $O(N^4)$ operations down to $O(N^3)$ operations.

In conclusion, unless some knowledge of the field in the insulator is needed during the computation, the integral formulation provides a rigorous way to compute induction in a bounded domain without using spherical harmonics. Its efficiency can be strongly increased by using a fast multipole approach ($O(N^2)$ operations for N^2 points on the surface), or simply by taking advantage of the azimuthal symmetry of the domain ($O(N^3)$ operations). Gridding methods for the insulator are however to be preferred if the whole potential field needs to be known during the simulation.

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