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Cross-references

Alfvén's Theorem and the Frozen Flux Approximation Anelastic and Boussinesq Approximations Core, Adiabatic Gradient Core Convection Core, Electrical Conductivity Core, Magnetic Instabilities Core-Mantle Boundary Topography, Implications for Dynamics Core-Mantle Boundary Topography, Seismology Core-Mantle Boundary, Heat Flow Across Core-Mantle Coupling, Electromagnetic Core-Mantle Coupling, Thermal Core-Mantle Coupling, Topographic Core Properties, Physical Core, Thermal Conduction Core Viscosity Inner Core Tangent Cylinder

CORE, BOUNDARY LAYERS

Basic ideas

It might be said that the term *boundary layer* means different things to different people. To someone observing fluid flowing past a flat plate, it might seem that there are two distinct regimes of motion. Far from the plate, the flow might seem too fast for the eye to follow, markers carried by the fluid appearing blurred as they speed past the plate; near the plate however the flow is so sluggish that it is easily followed by eye. The observer may call this the boundary layer, and the region beyond the free-stream or the mainstream, and he may feel that the interface between the two is reasonably sharp, so that he can call it the edge of the boundary layer. The theoretician will see no such sharp interface but will employ a mathematical technique, sometimes called matched asymptotics that similarly distinguishes an inner region near the plate from the outer region beyond. For him, the edge of the boundary layer is a region where the two solutions are required to agree with one another, i.e., to match.

To be successful, the asymptotic approach requires that the kinematic viscosity, v, is small, as measured by an appropriate nondimensional parameter, such as the inverse Reynolds number in the case of the flow past the plate, or the Ekman number, *E*, for the situations we encounter below. The thickness, δ , of the boundary layer is then small compared with the characteristic scale, *L*, of the system, so that $\varepsilon = \delta/L$ is small and vanishes with *v*; this does not imply that it is proportional to *v*.

The inner and outer solutions are developed as expansions in powers of ε . The relative size of successive terms in the expansions is determined by the matching process. Provided ε is sufficiently small, only a few terms in each expansion are needed to obtain useful solutions of acceptable accuracy. In what follows, we shall retain only the first, or "leading" term in the expansion of the inner solution, u, but will require the first and second terms in the expansion of the outer solution, U. The primary (leading order) part, U_0 , of the outer solution is independent of v and can therefore only satisfy one condition at a stationary impermeable boundary Γ , namely $U_{0\perp} \equiv \mathbf{n} \cdot \mathbf{U} = 0$, where **n** is the unit normal to Γ , directed into the fluid. The components $\mathbf{U}_{0\parallel}$ of \mathbf{U}_0 that are tangential to Γ will then in general be nonzero on z = 0, where z measures distance from Γ in the direction of **n**. The task of the boundary layer is to reconcile $\mathbf{U}_{0\parallel}$ at z = 0 with the no-slip condition: $\mathbf{u} = \mathbf{0}$ on Γ . This means that \mathbf{u}_{\parallel} must, through the action of viscosity, be reduced from $\mathbf{U}_{0\parallel}$ to zero in the distance δ . To achieve this deceleration, the viscous term in the momentum equation, $-v\nabla^2 \mathbf{u}_{\parallel}$, must be finite and nonzero. Since this force is of order $v\mathbf{u}_{\parallel}/\delta^2$, it follows that δ is proportional to $v^{1/2}$.

To derive the inner expansion, the stretched coordinate, $\zeta = z/\delta$, is introduced to replace the distance z from Γ . The boundary layer is then characterized by $\zeta = \mathcal{O}(1)$ and $\partial_{\zeta} \equiv \partial/\partial \zeta = \mathcal{O}(1)$, so that $\partial_z \equiv \partial/\partial z = \mathcal{O}(1/\delta) \gg 1/L$. In contrast, ∇_{\parallel} is much smaller; as in the mainstream, it is $\mathcal{O}(1/L)$. The matching principle asserts that the inner and outer expansions should agree with each other at the edge of the boundary layer, which is defined as a region where $z \ll L$ but $\zeta \gg 1$. [For example, if $z = \mathcal{O}(L\delta)^{1/2}$, then $z/L = \mathcal{O}((\delta/L)^{1/2}) \downarrow 0$ and $\zeta = \mathcal{O}((L/\delta)^{1/2}) \uparrow \infty$ as $\varepsilon \downarrow 0$.]

A significant consequence follows from mass conservation and the matching principle. Since variations in density across the thin boundary layer can be ignored at leading order, mass conservation requires that $\nabla \cdot \mathbf{u} = 0$. At leading order, this gives

$$\partial_{\zeta} \mathbf{u}_{\perp} + \nabla_{\mathbf{S}} \cdot \mathbf{u}_{\parallel} = 0.$$
 (Eq. 1a)

Here ∇_s is the surface divergence, which may be defined, in analogy with the better known definition of the three-dimensional divergence, as a limit. For any vector \mathbf{Q} depending on position \mathbf{x}_{\parallel} on Γ and directed tangentially to Γ ,

$$\nabla_{\mathbf{S}} \cdot \mathbf{Q} = \lim_{A \to 0} \frac{1}{A} \oint_{\gamma} \mathbf{Q} \cdot \mathbf{N} ds.$$
 (Eq. 1b)

Here γ is the perimeter of a small "penny-shaped disk" on Γ of area *A*; ds is arc length and **N** is the outward normal to γ lying in Γ . This is shown in Figure C35, which also gives the disk a small thickness



Figure C35 Efflux from the top of a "penny shape" volume encompassing the boundary layer. This compensates the volume flux deficit in the boundary layer through the sides of the volume.

 $z = \ell$. Since Nds = ds × n, where the vector element ds of arc length is in the right-handed sense with respect to n, an application of Stokes's theorem to Eq. (1b) gives

$$\boldsymbol{\nabla}_{\mathbf{S}} \cdot \mathbf{Q} = \lim_{A \to 0} \frac{1}{A} \oint_{\gamma} \, \mathrm{d} \mathbf{s} \cdot (\mathbf{n} \times \mathbf{Q}) = \lim_{A \to 0} \frac{1}{A} \int_{A} \, \mathrm{d} \mathbf{S} \cdot (\boldsymbol{\nabla} \times (\mathbf{n} \times \mathbf{Q})),$$

where $d\mathbf{S} = \mathbf{n}dS$ is the vector element of surface area on A. From which it follows that

$$\boldsymbol{\nabla}_{\mathrm{S}} \cdot \mathbf{Q} = \mathbf{n} \cdot \boldsymbol{\nabla} \times (\mathbf{n} \times \mathbf{Q}). \tag{Eq. 1c}$$

This is generally a more convenient form of the surface divergence than (1b).

Since Γ is impermeable, $\mathbf{u}_{\perp} = 0$ on $\zeta = 0$ and Eq. (1a) gives (on taking $\mathbf{Q} = \int_{0}^{\zeta} \mathbf{u}_{\parallel} d\zeta$)

$$\mathbf{u}_{\perp}(\zeta) = \mathbf{n} \cdot \boldsymbol{\nabla} \times \left(\int_{0}^{\zeta} \mathbf{u}_{\parallel} d\zeta \times \mathbf{n} \right).$$
 (Eq. 1d)

We now choose the "top" face of the penny in Figure C35 to be at the edge of the boundary layer, i.e., $\delta \ll \ell \ll L$. Then, by the asymptotic matching principle, the left-hand side of Eq. (1d) is both $u_{\perp}(\zeta \uparrow \infty)$ and $U_{\perp}(z \downarrow 0)$. Since the right-hand side of Eq. (1d) is $\mathcal{O}(\delta)$, this contribution to U is small compared with U_0 , i.e., it refers to the second, or "secondary," term U_1 in the expansion of U. This also establishes that the ratio of the secondary and primary terms in the expansion of U is $\mathcal{O}(\delta)$ and not, as might have been supposed a *priori*, $\mathcal{O}(v) = \mathcal{O}(\delta^2)$.

It follows from (1d) that, for z = 0,

$$\mathbf{U}_{1\perp} = \mathbf{n} \cdot \boldsymbol{\nabla} \times \left(\mathbf{Q}_{\parallel} \times \mathbf{n} \right), \tag{Eq. 2a}$$

where \mathbf{Q}_{\parallel} is now the volume flux deficit in the boundary layer:

$$\begin{split} \mathbf{Q}_{\parallel} &= \delta \int_{0}^{\infty} (\mathbf{u}_{\parallel} - \mathbf{U}_{\parallel}) \mathrm{d}\zeta, \quad \text{or} \quad \mathbf{Q}_{\parallel} = \int_{0}^{z/\delta \uparrow \infty} (\mathbf{u}_{\parallel} - \mathbf{U}_{\parallel}) \mathrm{d}z. \end{split} \tag{Eq. 2b, c}$$

The second of these equivalent forms, which will be used later, provides a convenient way of reminding us that the *z*-integration is only across the boundary layer and not across the entire fluid. In each expression, \mathbf{U}_{\parallel} is evaluated at z = 0 and therefore depends on \mathbf{x}_{\parallel} only.

Equation 2a provides a crucial boundary condition on U_1 . Further boundary conditions are not needed and in fact could not be imposed without overdetermining the solution. Not even the term $v\nabla^2 U_0$ appears in the equation governing U_1 , since this term is $\mathcal{O}(v) = \mathcal{O}(\delta^2)$, i.e., is asymptotically small compared with U_1 . For flow past a flat plate, the source (2a) at the boundary has the effect of displacing the effective boundary by an amount of order δ , which is then often called "the displacement thickness"; e.g., see Rosenhead (1963, Chapter V.5). For the Ekman and Ekman-Hartmann layers considered in "The Ekman layer" and "The Ekman-Hartmann layer" below, the effect of the secondary flow is more dramatic.

Ekman layers arise at the boundaries of highly rotating fluids; Ekman-Hartmann layers are their generalization in magnetohydrodynamics (MHD). The Ekman-Hartmann layer must reconcile both the mainstream flow **U** and the magnetic field **B** to conditions at a boundary Γ , where we assume that $\mathbf{u} = \mathbf{0}$ and $\mathbf{b} = \mathbf{B}_{\Gamma}$. It follows from $\nabla \cdot \mathbf{b} = 0$ that b_{\perp} is the same everywhere in the boundary layer. It therefore coincides with $B_{\Gamma\perp}$ at the wall and with $B_{0\perp}$ at the edge of the boundary layer. The remaining components of **b** change rapidly through the boundary layer: $\mathbf{b} = \mathbf{B}_0$ at its "upper" edge and $\mathbf{b} = \mathbf{B}_{\Gamma}$, at its "lower" edge. This occurs because the current density, \mathbf{j}_{\parallel} , in the boundary layer is large. In analogy with Eq. (2a), there is a secondary current flow $J_{1\perp}$ between the boundary layer and the mainstream given by

$$J_{1\perp} - J_{\Gamma\perp} = \mathbf{n} \cdot \boldsymbol{\nabla} \times (\mathbf{I}_{\parallel} \times \mathbf{n}), \qquad (\text{Eq. 3a})$$

where I_{\parallel} is the electric current deficit:

$$\begin{split} \mathbf{I}_{\parallel} &= \delta \int_{0}^{\infty} (\mathbf{j}_{\parallel} - \mathbf{J}_{\parallel}) d\zeta, \quad \text{or} \quad \mathbf{I}_{\parallel} = \int_{0}^{z/\delta \to \infty} (\mathbf{j}_{\parallel} - \mathbf{J}_{\parallel}) dz. \end{split} \tag{Eq. 3a, b}$$

If the exterior region, z < 0, is electrically insulating, $J_{\Gamma} = 0$, so bringing Eq. (3a) even closer to Eq. (2a). When attention is focused on the mainstream magnetic field alone, I is the symptom of the boundary layer that has most significance, and it is thought of as a "current sheet." In a similar way, the volumetric deficit Q may be regarded as a "vortex sheet."

The theory necessary to evaluate **Q** and **I** will be described in "The Ekman layer." It will be found that, as far as the effects of viscosity are concerned, the relevant boundary layer scale is the Ekman layer thickness, $\delta_{\rm E} = (\nu/\Omega)^{1/2} = E^{1/2}L$, where $E = \nu/\Omega L^2$ is the Ekman number and Ω is the angular speed of the system.

Geophysical overtones

The mathematical concepts introduced above need to be handled with care when applied to a complex object like the Earth's core. Various intricate physical effects can occur in the core and a mathematical model by essence requires some simplifying assumptions. Let us first note that the dynamics of the Earth's core is often modeled by a set of coupled physical quantities: the magnetic field, the fluid velocity, and a driving mechanism (usually thermal and/or chemical). This introduces (see Anelastic and Boussinesq approximations) at least three diffusivities. The magnetic diffusivity largely dominates the two others. It follows that boundary layers (associated with low diffusivities) can develop both based on the smallness of viscosity (as discussed above) and on the smallness of say the thermal diffusivity (compositional diffusivity being even smaller). The thickness of such thermal (compositional) boundary layers, δ_{κ} , depends on the thermal (compositional) diffusivity (or the relevant turbulent diffusivity), κ , of core material. These layers are conceptually significant in understanding how heat enters and leaves the FOC. We shall not consider them in this article and refer the reader to Anelastic and Boussinesq approximations.

Assume that $L = 2 \times 10^6$ m is a typical large length scale of core motions and magnetic fields (see *Core motions*). If the molecular viscosity v is 10^{-6} m²/s, then E is about 10^{-15} . This suggests that an asymptotic solution to core MHD is fully justified as far as the effects of viscosity are concerned. A mainstream solution would be expected away from the boundaries and a boundary layer would develop near the mantle and near the solid inner core.

This line of thought is also the basis for the determination of core surface motions from observations of the main geomagnetic field and its secular variation. On the assumption that electric currents generated in the mantle by the geodynamo are negligible, the observed fields can be extrapolated downwards to the CMB and used to provide information about the fluid motions "at the top of the core," using Alfvén's theorem. A difficulty however remains: the fluid in contact with the mantle co-rotates with it by the no-slip condition. There *is* no relative motion! Realistically, one can hope only to determine the mainstream flow at the edge of the boundary layer. The connection between the fields at the CMB to the fields at the edge of the boundary layer was considered by Backus (1968), who presented an analysis of Ekman-Hartmann type. Hide and Stewartson (1972) made further developments of the theory.

The boundary layer thickness, δ_E , obtained from the above estimate of the Ekman number is however only about 10 cm and it is natural to wonder whether such a thin layer can have any effect whatever on the dynamics of a fluid body that is about ten million times thicker! Turbulence provides amelioration. It is widely accepted that the molecular viscosity is inadequate to transport large-scale momentum in the core, its role being subsumed by small-scale turbulent eddies. These, and the associated momentum flux, are highly anisotropic through the action of Coriolis and Lorentz forces (Braginsky and Meytlis, 1990). Nevertheless, a crude ansatz is commonly employed: the molecular v is replaced by an isotropic turbulent viscosity having the same order of magnitude as the molecular magnetic diffusivity, that is to say about 1 m²/s. Then $E \approx 10^{-9}$ and the asymptotic approach still seems secure. Even ignoring the possible intrinsic instability of the boundary layer (see "Stability of the Ekman-Hartmann layer"), it is reasonable to suppose that the small-scale eddies would penetrate the boundary layer so that a better estimate of $\delta_{\rm E}$ would be 100 m. That such a layer would have a dynamical effect on the dynamics of the core then seems less implausible.

There are, however, skeptics who believe that, even if the boundary layers are 100 m thick, they are still too thin to be of geophysical interest. Kuang and Bloxham (1997) removed them by replacing the noslip conditions on the core-mantle boundary (CMB) and inner core boundary (ICB) by the conditions of zero tangential viscous stress. This step obviously eliminates viscous coupling between the fluid outer core (FOC) and mantle and between the FOC and the solid inner core (SIC). Then, in the absence of other coupling mechanisms, the rotation of the mantle and the SIC does not affect the dynamics of the FOC and spin-up does not occur (see "The Ekman layer"). Geophysical justification for the step rests on estimates indicating that, owing to the large electrical conductivity of the SIC and despite the relatively small electrical conductivity in the mantle, the magnetic couples between the FOC and both the mantle and the SIC still greatly exceed the viscous couples, which can therefore be disregarded by adopting the zero viscous stress conditions. In this article we shall ignore such complications and adopt the "traditional approach" where no-slip conditions apply at both boundaries. See also Core-mantle coupling.

Although the smallness of *E* makes the asymptotic approach to MHD very attractive, purely analytic methods are not powerful enough to provide the solutions needed; numerical methods must be employed. One might nevertheless visualize a semianalytic, seminumerical approach in which the computer finds an inviscid mainstream flow **U** and magnetic field **B** that satisfy the boundary conditions (2a) and (3a). This too is not straightforward, since solutions for v = 0 raise important numerical difficulties and to restore *v* to the mainstream would be tantamount to repudiating the boundary layer concept.

At this stage it is natural to wonder what use asymptotic methods have in core MHD. In the present state of algorithmic development and computer capability, the answer may seem to be, "Not much!" The boundary layer concept is, however, valuable in locating regions requiring special attention in numerical work, such as free shear layers and boundary layer singularities ("Free shear layers"). Moreover, the only way of verifying that the numerical simulations have reduced the effect of viscosity on core dynamics to a realistic level is by confronting them with the expectations of asymptotic theory.

The Ekman layer

Consider an incompressible fluid of uniform density, ρ , moving steadily with velocity \mathbf{u} , relative to a reference frame rotating with constant angular velocity $\mathbf{\Omega}$. In this reference frame, the Coriolis force per unit volume, $2\rho\mathbf{\Omega} \times \mathbf{u}$, is balanced by the pressure gradient $-\nabla p$ and the viscous stresses $\rho v \nabla^2 \mathbf{u}$, where v is the kinematic viscosity. The normal component of this balance gives, to leading order in the boundary layer expansion, $\nabla_{\perp} p = \mathbf{0}$, which implies that the pressure, p, throughout the boundary layer coincides with the pressure, P, in the mainstream at the edge of the boundary layer. The tangential components give dominantly

$$2\Omega_{\perp}\mathbf{n} \times \mathbf{u}_{\parallel} = -\nabla_{\parallel}(P/\rho) + v\partial_{zz}\mathbf{u}_{\parallel}, \qquad (\text{Eq. 4})$$

which shows that only the component, $\Omega_{\perp} \equiv \Omega \cdot \mathbf{n}$, of angular velocity normal to the boundary is significant at leading order. We assume here that $\Omega_{\perp} \neq 0$.

Inside the *Ekman layer* governed by Eq. (4), the interplay between the additional viscous forces needed to meet the no-slip boundary condition causes the flow to be deflected from the direction of U, leading to the well known Ekman spiral of the velocity $\mathbf{u}(z)$ as z/δ increases from zero to infinity; see Figure C36.

As a consequence of the Ekman spiral there is a transverse mass transport, which is quantified by the volume flux deficit

$$\mathbf{Q}_{\parallel} = \frac{1}{2} \delta_{\mathrm{E}} [-\mathbf{U}_{0} + (\mathrm{sgn} \Omega_{\perp}) \mathbf{n} \times \mathbf{U}_{0}], \quad \text{where} \quad \delta_{\mathrm{E}} = \sqrt{(\nu/|\Omega_{\perp}|)}$$
(Eq. 5a, b)

is the appropriately redefined Ekman layer thickness which, over curved boundaries such as the CMB and ICB, depends on x_{\parallel} through Ω_{\perp} . The velocity $U_{1\perp}$ follows directly from Eq. (2a); see also Greenspan (1968, p. 46):

$$\mathbf{U}_{1\perp} = \mathbf{n} \cdot \mathbf{u} = \frac{1}{2} \mathbf{n} \cdot \nabla \times \{ \delta_{\mathrm{E}} [\mathbf{n} \times \mathbf{U}_0 + (\mathrm{sgn}\Omega_{\perp})\mathbf{U}_0] \}, \qquad (\mathrm{Eq.}\ 6a)$$

which, when the mainstream velocity \mathbf{U}_0 is geostrophic and boundary is planar normal to the rotation vector $(\mathbf{n} \times \mathbf{\Omega} = \mathbf{0})$, takes the simpler form

$$\mathbf{U}_{1\perp} = \mathbf{n} \cdot \mathbf{u} = \frac{1}{2} \delta_{\mathrm{E}}(\mathrm{sgn}\Omega_{\perp}) \mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{U}_{0}. \tag{Eq. 6b}$$

This relates the normal flow to the vorticity $\nabla \times U_0$ of the primary mainstream flow (see also Pedlosky, 1979).

The phenomenon described by Eq. (6) is often called Ekman pumping or *Ekman suction* depending on whether $u_{\perp} >$ or < 0. It should again be stressed that the secondary flow induced by the mainstream boundary condition (6) is scaled by the boundary layer thickness $\delta_{\rm E}$ and is therefore small. It is worth emphasizing however that this modification of the effective boundary conditions introduces dissipation into the mainstream force balance, and this generally provides the dominant dissipation mechanism for a sufficiently large-scale flow. More precisely, for all flows in the mainstream characterized by a length scale larger than $LE^{1/4}$, dissipation within the boundary layers will dominate over the bulk effects of viscosity. Indeed, Ekman pumping/suction is a particularly significant process in determining the evolution of the angular momentum of a rotating fluid, which occurs on the spin-up timescale, $\tau_{s-u} = E^{-1/2}/\Omega = E^{1/2}L^2/\nu$, i.e., on a timescale intermediate between the rotation period $2\pi/\Omega$ (the day) and the viscous diffusion time L^2/v (which, for the molecular v, would exceed the age of the Earth).



Figure C36 Side (*left*) and top (*right*) views of the Ekman layer profile. The velocity executes a spiral from zero velocity on the boundary to the mainstream velocity **U** at the edge of the boundary layer. Copyright 2007 from 'Mathematical Aspects of Natural Dynamos' by E. Dormy and A.M. Soward (eds.). Reproduced by permission of Routledge/Taylor & Francis Group, LLC.

The theory of the Ekman layer adumbrated above assumes that conditions are steady. It can be adapted to time-dependent situations provided they do not change too rapidly, i.e., provided that they occur on timescales longer than the rotation period $2\pi/\Omega$. When considering the effects on core flow of the lunisolar precession, in which Ω changes on the diurnal timescale, the present theory is inapplicable.

The Ekman-Hartmann layer

The situation described in "The Ekman layer" is now generalized to MHD; the fluid is electrically conducting and a magnetic field is present. In the mainstream, the electric current density, **J**, is given by Ohm's law:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}), \tag{Eq. 7a}$$

where **E** is the electric field and σ is the electrical conductivity. The Lorentz force per unit volume, $\mathbf{J} \times \mathbf{B}$, is therefore of order $\sigma B^2 U$. In the Earth's core, this is comparable with the Coriolis force $2\rho \mathbf{\Omega} \times \mathbf{U}$. The Elsasser number, $\Lambda = \sigma B^2 / \rho \Omega$, is therefore of order unity. The asymptotic limit of interest is therefore $E \downarrow 0$ with $\Lambda = \mathcal{O}(1)$. Since $\Lambda = M^2 E$, where $M = BL(\sigma / \rho v)^{1/2}$ is the Hartmann number, it follows that $M \sim E^{-1/2} \uparrow \infty$ as $E \downarrow 0$.

Both **B** and the magnetic field, **b**, in the boundary layer obey Gauss's law, and to dominant order $\nabla \cdot \mathbf{b} = 0$ gives, as before ("Basic ideas"), $\partial_{\zeta} \mathbf{b}_{\perp} = \mathbf{0}$, from which $\mathbf{b}_{\perp} = B_{\Gamma \perp} = \mathbf{B}_{0 \perp}$, at the edge of the mainstream. We shall suppose that $\mathbf{B}_{0 \perp} \neq \mathbf{0}$. Ohm's law in the boundary layer is essentially the same as (7a) but, in our notation, it is written as

$$\mathbf{j} = \sigma(\mathbf{e} + \mathbf{u} \times \mathbf{b}). \tag{Eq. 7b}$$

We again suppose steady conditions so that $\mathbf{E} = -\nabla \Phi$ and $\mathbf{e} = -\nabla \phi$, where Φ and ϕ are the electric potentials in the mainstream and boundary layer. The above expressions in terms of electric potentials correspond to a low magnetic Reynolds number description. It is important to remember we are here concerned with the magnetic Reynolds based on the boundary layer scale. This is a very small quantity in the case of the Earth's core. To leading order, the normal component of Eq. (7b) is $e_{\perp} = -\partial_{\zeta}\phi = 0$ and this shows that, throughout the boundary layer, ϕ coincides with Φ both at the edge of the mainstream and on the boundary itself. This generally depends on \mathbf{x}_{\parallel} , so that a current $\mathbf{J}_{\Gamma} = \sigma_{\Gamma} \mathbf{E}_{\Gamma}$ flows in the stationary wall, if its conductivity σ_{Γ} is nonzero. To leading order, the components of Eq. (7b) parallel to Γ together with Ampère's law, $\mu \mathbf{j} = \nabla \times \mathbf{b}$, where μ is the magnetic permeability, imply

$$\mu^{-1}\mathbf{n} \times \partial_z \mathbf{b}_{\parallel} = \sigma(-\nabla_{\parallel}\phi + \mathbf{u}_{0\parallel} \times \mathbf{B}_{0\perp}). \tag{Eq. 8}$$

[The last term in Eq. (7b) also contributes $U_{1\perp} \times b_{0\parallel}$ but, as this is asymptotically smaller than the term $u_{0\parallel} \times B_{0\perp}$ retained, it has therefore been discarded.]

The determination of the boundary layer structure requires the equation of motion to be satisfied too. This differs from Eq. (4) by the addition of the Lorentz force. This may be decomposed into a magnetic pressure gradient and the divergence of the Maxwell stress **bb**/ μ . The magnetic pressure can be absorbed into *p* to form a total pressure which, by the same argument as before, is constant across the boundary layer to leading order. The components of the equation of motion parallel to Γ are governed by

$$2\Omega_{\perp}\mathbf{n} \times (\mathbf{u}_{\parallel} - \mathbf{U}_{0\parallel}) = -(\mathbf{B}_{\perp}^2/\rho\mu\eta)(\mathbf{u}_{\parallel} - \mathbf{U}_{0\parallel}) + \nu\partial_{zz}\mathbf{u}_{\parallel} \quad \text{(Eq. 9)}$$

see Gilman and Benton, (1968) and Loper, (1970).

Equations 8 and 9 determine the structure of the Ekman-Hartmann layer which, in the limit $\Lambda \downarrow 0$, becomes the Ekman layer considered in "The Ekman layer" (see Figure C37). For $\Lambda \uparrow \infty$, the Coriolis force



Figure C37 Streamwise component of the velocity in the boundary layer for different values of the Elsasser number, illustrating the smooth transition from an Ekman-type boundary layer to the Hartmann-type boundary layer.

is unimportant and the theory reduces to that governing the Hartmann layer, which is a well known boundary layer that arises in the study of MHD duct flow at large Hartmann number; see, e.g., Roberts (1967), Müller and Bühler (2001). It is then found that

$$\mathbf{u}_{\parallel} = \mathbf{U}_{0\parallel} [1 - \exp(z/\delta_{\mathrm{H}})]$$
(Eq. 10a, b)

where $\delta_{\rm H} = \sqrt{(\rho v / \sigma)/|B_{\perp}|}$ is the Hartmann layer thickness. For a flat boundary, there is no flux deficit and $U_{1\perp}$ therefore vanishes. Equations (8) and (9) also show that

$$\Lambda = B_{\perp}^2 / \rho \mu \eta \Omega_{\perp} = \left(\delta_{\rm E} / \delta_{\rm H}\right)^2 \tag{Eq. 11}$$

is a convenient redefinition of the Elsasser number measuring the relative importance of Lorentz and Coriolis forces in determining the boundary layer structure. For small A, the Ekman spiral persists together with the associated boundary layer pumping. As *E* increases through O(1)values, these effects decrease until at large A the flow becomes unidirectional without any associated boundary layer pumping.

Free shear layers

Supposing that the Earth's fluid core occupies the spherical $r_i \le r \le r_0$, we redefine the Ekman and Hartmann numbers by

$$E = v/r_0^2 \Omega = \delta_{\rm E}^2/r_0^2$$
 and $M^2 = r_0^2 |\mathbf{B}|^2 / \rho v \mu \eta = r_0^2 / \delta_{\rm H}^2$
(Eq. 12a, b)

respectively, and redefine the Elsasser number (11) as $\Lambda = M^2 E$. In what follows, (r, θ, φ) will be spherical coordinates in which the colatitude θ is zero at the north pole. We shall also employ cylindrical polar coordinates (s, φ, z) .

As explained earlier, Ekman-Hartmann layers are generally present on both the CMB, $r = r_0$ and the ICB $(r = r_i)$. Since horizontal variations are negligible in comparison with the very rapid variation across the layers, the curvature of the spherical boundaries does not affect our earlier results as these depend only on the components of Ω and **B** normal to the boundaries. Where these vanish, the theory adumbrated in "The Ekman layer" and "The Ekman-Hartmann layer" breaks down, and boundary layer singularities arise. To focus on these, we at first consider only the nonmagnetic case, M = 0. At the equator of the inner sphere, where the Ekman layer is singular, Eq. (2a) ceases to apply and a new force balance is struck, and a new type boundary layer arises called the equatorial Ekman layer. This is intimately linked to a free shear layer, which surrounds the tangent cylinder, i.e., the imaginary cylinder touching the inner sphere at the equator and having generators parallel to the rotation axis (see *Inner core: tangent cylinder*). This shear layer exhibits a complicated asymptotic structure that is best illustrated by the Proudman-Stewartson problem (Proudman, 1956; Stewartson, 1957, 1966), which concerns the slow steady axisymmetric flow induced by rotating the solid inner core at a slightly faster rate than the outer solid mantle. The flow pattern that occurs in this fundamental configuration illustrates the interplay between the various boundary and shear layers. It is important to stress that these layers do not take into account magnetic effects. Such effects are important in the Earth core and lead to a variety of shear layers whose details are not discussed here (see Kleeorin *et al.* 1997).

The mainstream flow is dominantly geostrophic and azimuthal: $U_G = U_G(s)\hat{\varphi}$, where $\hat{\varphi}$ denotes the unit vector in the φ -direction. Outside the tangent cylinder, $s = r_i$, the fluid co-rotates with the outer sphere, and there is no Ekman layer on the CMB. Within the tangent cylinder, $U_G(s)$ adjusts its value so that the Ekman suction, $-U_r(r_i) = \mathcal{O}(E^{1/2}U_G)$, into the Ekman layer on the ICB equals the Ekman pumping, $U_r(r_o)$, out of the Ekman layer on the CMB. This generates a secondary flow $U_z(s)\hat{z}$ in the mainstream that has only a z-component. It depends on s alone; such flows are termed geostrophic. In the northern (southern) hemisphere it is negative (positive).

As the tangent cylinder is approached from within, U_G tends to the inner sphere velocity. The jump in the geostrophic velocity across the tangent cylinder is smoothed out in an exterior quasigeostrophic layer $(s > r_i)$ in which the effects of Ekman pumping on the outer core boundary $\sim \Omega E^{1/2} \partial_s U_G$ is now balanced in the axial vorticity equation by lateral friction $v \partial_{sss} U_G$ in a layer of width $\mathcal{O}(r_0 E^{1/4})$, the Stewartson $E^{1/4}$ -layer. There is a comparable but thinner $E^{2/7}$ -layer inside $(s < r_i)$, whose main function is to smooth out $\partial_s U_G$ and so achieve continuity of stress.

These quasigeostrophic layers do not resolve all the flow discontinuities. The secondary mainstream flow, $U_z(s)\hat{z}$, feeds the Ekman layers on the ICB with fluid. In each hemisphere, the associated mass flux deficit, \mathbf{Q}_{\parallel} , is directed towards the equator and builds up as the equator is approached. Mass conservation demands that this fluid be accounted for, and this is one of the main functions of the shear layer at the tangent cylinder. The fluid is ejected towards the CMB as a jet in an inner ageostrophic layer of width $\mathcal{O}(r_0 E^{1/3})$, the Stewartson $E^{1/3}$ -layer.

As the equator is approached, the Ekman layer becomes singular: from (5b), $\delta_{\rm E} \sim \mathcal{O}(|\theta - \pi/2|^{-1/2}E_i^{1/2}r_i) \to \infty$, as $\theta \to \pi/2$, where $E_i = v/r_i^2 \Omega = E(r_0/r_i)^2$ is the Ekman number based on the inner core radius. When $|\theta - \pi/2| = \mathcal{O}(E_i^{1/5})$ however, this expression for $\delta_{\rm E}$ becomes of the same order as the distance, $\mathcal{O}((\theta - \pi/2)^2 r_i)$, of the point concerned in the Ekman layer from the tangent cylinder. This defines the lateral extent, $\mathcal{O}(E_i^{1/5}r_i)$, of the equatorial Ekman layer, i.e., the distance over which the solution (5a) fails. The radial extent of the equatorial Ekman layer is $\mathcal{O}(E_i^{2/5}r_i)$.

The MHD variant of Proudman-Stewartson problem reveals various other shear layers (Hollerbach, 1994a; Kleeorin *et al.*, 1997; Dormy *et al.*, 2002), while a nonaxisymmetric version exhibits even more structure (Hollerbach, 1994b; Soward and Hollerbach, 2000). Other investigations have been undertaken in plane layer (Hollerbach, 1996) and cylindrical (Vempaty and Loper, 1975, 1978) geometries.

Stability of the Ekman-Hartmann layer

The laminar Ekman-Hartmann layer profiles are determined by the linearized equations 8 and 9. This corresponds to a small Re approximation, where Re = UL/v is the Reynolds number and L is some characteristic length, possibly the core radius r_0 , but perhaps smaller. In view of the low viscosity in the Earth's liquid core, and of its large size, Re is naturally huge. In the boundary layers, however, the relevant length scale is the boundary layer width δ . Thus the corresponding Reynolds number $\text{Re}_{\text{BL}} = U\delta/\nu$ (often referred to as the "boundary layer Reynolds number") is much less than $\text{Re}_0 = Ur_0/\nu$ for the full core. Since the width of the Ekman-Hartmann layer is based on the normal components Ω_{\perp} and B_{\perp} of both the rotation and the magnetic field, Re_{BL} depends on position, \mathbf{x}_{\parallel} . Both these components decrease with latitude (at any rate for a magnetic field having dipole symmetry) and so the boundary layer width δ increases as the equator is approached, just as we explained in connection with the equatorial Ekman layer. Consequently the boundary layer Reynolds number Re_{BL} increases in concert.

Evidently the boundary layer Reynolds number Re_{BL} is much smaller than Re_0 and for geophysical parameter values it may be sufficiently small to justify the linear approach described in the previous sections. In that circumstance a linear stability analysis may be undertaken to determine a critical value of Re_{BL} (say Re_c , which the boundary layer becomes unstable (usually as a traveling wave). It is well known that boundary layer profiles with inflection points are generally prone to instability (e.g., Schlichting and Gersten, 2000). So, on the one hand, the Hartmann layer (10), which lacks an inflection point, is extremely stable to disturbances up to high values of Re_{BL} while, on the other hand, the Ekman and Ekman Hartmann profiles, which spiral, can develop instabilities at moderate Re_c .

Much effort has been devoted identifying the critical Reynolds number Re_c and the associated traveling wave mode of instability, whose orientation is determined by its horizontal wave vector \mathbf{k}_{\parallel} . Comprehensive Ekman layer stability studies have been undertaken in both the case of vertical rotation $\Omega_{\parallel} = \mathbf{0}$ (Lilly, 1966) and oblique rotation $\Omega_{\parallel} \neq \mathbf{0}$ (Leibovich and Lele, 1985).

The Ekman-Hartmann layer stability characteristics have been investigated in the context of the Earth's liquid core. For a model with normally directed magnetic field and rotation ($\Omega_{\parallel} = B_{\parallel} = 0$), it has been shown for Earth core values of Ω and *B* (Gilman, 1971) that Re_{BL} is less than the critical value Re₀ necessary for an instability to grow. Since Re_{BL} is so small, the linearization leading to Eqs. (8) and (9) is justified. The more general orientation with $\Omega_{\parallel} \neq 0$ and $B_{\parallel} \neq 0$ appropriate to the local analysis of a shell with an axisymmetric dipole magnetic field has also been studied. Desjardins *et al.* (2001) show that, while the Ekman-Hartmann layer is stable in the polar caps, an equatorial band extending some 45° both north and south of the equator could develop instabilities.

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Cross-references

Alfvén's Theorem and the Frozen Flux Approximations Anelastic and Boussinesq Approximations Core Motions Core-Mantle Boundary Topography, Implications for Dynamics Core-Mantle Boundary Topography, Seismology Core-Mantle Boundary, Heat Flow Across Core-Mantle Coupling, Electromagnetic Core-Mantle Coupling, Thermal Core-Mantle Coupling, Topographic Geodynamo Inner Core Tangent Cylinder Magnetohydrodynamics

CORE, ELECTRICAL CONDUCTIVITY

Core processes responsible for the geomagnetic field dissipate energy by two competing mechanisms that both depend on the electrical conductivity, σ_e , or equivalently the reciprocal quantity, resistivity, $\rho_e = 1/\sigma_e$. The obvious dissipation is ohmic heating. A current of density *i* (amperes/m²) flowing in a medium of resistivity ρ_e (ohm m) converts electrical energy to heat at a rate $i^2 \rho_e$ (watts/m³). Thus one requirement for a planetary dynamo is a sufficiently low value of ρ_e (high σ_e) to allow currents to flow freely enough for this dissipation to be maintained. In a body the size of the Earth, this means that the core must be a metallic conductor. However, a metal also has a high thermal conductivity, introducing a competing dissipative process (see *Core, thermal conduction*). The stirring of the core that is essential to dynamo action maintains a temperature gradient that is at or very close to the adiabatic value (see also *Core, adiabatic gradient*) and conduction of heat down this gradient is a drain on the energy that would otherwise be available for dynamo action.

Heat transport by electrons dominates thermal conduction in a metal and the thermal and electrical conductivities are related by a simple expression (the Wiedemann-Franz law; see *Core, thermal conduction*). Thus, while the viability of a dynamo depends on a conductivity that is high enough for dynamo action, it must not be too high. In reviewing planetary dynamos, Stevenson (2003) concluded that high conductivity is a more serious limitation. It is evident that if the Earth's core were copper, instead of iron alloy, there would be no geomagnetic field.

The conductivity of iron

By the standards of metals, iron is a rather poor electrical conductor. At ordinary temperatures and pressures its behavior is complicated by the magnetic properties, but these have no relevance to conduction under conditions in the Earth's deep interior. We are interested in the properties of nonmagnetic iron, which means iron above its Curie point, the temperature of transition from a ferromagnetic to a paramagnetic (very weakly magnetic) state (1043 K), or in one of its nonmagnetic crystalline forms, especially the high pressure form, epsilon-iron $(\in$ -Fe). Extrapolations from high temperatures and high pressures both indicate that the room temperature, zero pressure resistivity of nonmagnetic iron would be about 0.21 $\mu\Omega$ m. This is slightly more than twice the value for the familiar, magnetic form of iron and more than ten times that of a good conductor, such as copper. This is one starting point for a calculation of the conductivity of the core. A more secure starting point is the resistivity of liquid iron, just above its zero pressure melting point (1805 K), 1.35 $\mu\Omega$ m, although this is only marginally different from a linear extrapolation from 0.21 $\mu\Omega$ m at 290 K, because melting does not have a major effect on the resistivity of iron.

Effects of temperature and pressure

For a pure metal, resistivity increases almost in proportion to absolute temperature, but increasing pressure has an opposite effect. Phonons, quantized thermal vibrations of a crystal structure, scatter electrons, randomizing the drift velocities that they acquire in an electric field. The number of phonons increases with temperature, shortening the average interval between scattering events and increasing resistivity. Pressure stiffens a crystal lattice, restricting the amplitude of thermal vibration, or, in quantum terms, reducing the number of phonons at any particular temperature. It is convenient to think of the vibrations as transient departures from a regular crystal structure and that electrons are scattered by the irregularities. Although this is a highly simplified view it conveys the sense of what happens. Temperature increases crystal irregularity and pressure decreases it.

The temperature and pressure effects are given a quantitative basis by referring to a theory of melting, due in its original form to F.A. Lindemann. As modified by later discussions, Lindemann's idea was that melting occurs when the amplitude of atomic vibrations