A Letters Journal Exploring the Frontiers of Physics

March 2008

EPL, **81** (2008) 64002 www.epljournal.org
doi: 10.1209/0295-5075/81/64002

Time scales separation for dynamo action

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received 13 September 2007; accepted in final form 21 January 2008
published online 21 February 2008

PACS 47.65.-d – Magnetohydrodynamics and electrohydrodynamics
PACS 47.65.Md – Plasma dynamos
PACS 47.20.-k – Flow instabilities

Abstract – The study of dynamo action in astrophysical objects classically involves two time scales: the slow diffusive one and the fast advective one. We investigate the possibility of field amplification on an intermediate time scale associated with time-dependent modulations of the flow. We consider a simple steady configuration for which dynamo action is not realised. We study the effect of time-dependent perturbations of the flow. We show that some vanishing low-frequency perturbations can yield exponential growth of the magnetic field on the typical time scale of oscillation. The dynamo mechanism relies here on a parametric instability associated with transient amplification by shear flows. Consequences on natural dynamos are discussed.

Introduction. – Most astrophysical objects exhibit magnetic fields. This is the case of planets, stars, as well as galaxies. The dynamo instability, characterised by an exponential growth of the magnetic field in a magneto-hydrodynamic flow, is expected to account for these fields. Among dynamo flows, a vast majority provides finite growth rate on the diffusion time only (known as "slow dynamos"). They are therefore not suitable for large astrophysical objects (stars and galaxies), for which the associated magnetic Reynolds number (measuring the relative strength of the induction to the diffusive term) is huge. This is the primary motivation for the investigation of fast dynamos, characterised by finite growth rate on the advective time scale rather than the diffusive time scale (the later being sometimes larger than the age of the universe).

While significant progress has been achieved in the case of moderate values of the magnetic Reynolds number [1,2], the fast dynamo limit has been tackled yet only with a few analytic flows [3,4], still remote from realistic configurations. Criteria to assess dynamo action (other than direct integration) are still missing. Most simple flows (such as uniform shear) fail to be dynamos and only yield transient amplification of the field, followed by ohmic decay.

We discuss the possibility of a third class of dynamo, for which the field grows on an intermediate time scale associated to flow modulations (for example due to a traveling wave). The efficiency of some time-dependent flows in the dynamo process has been recognised in various studies [4–8]. These studies focused on leading-order time dependence of the flow (either analytical, or in the form of a heteroclinic cycle), whereas we want to study the effects of vanishing perturbations on steady flows. We stress that the impact of small fluctuations on dynamo flows has already been addressed in other contexts. Gog *et al.* [8] considered the action of a (random) noise on the dynamo properties of time-dependent flows provided by heteroclinic cycles. Pétrélis and Fauve [9] and Peyrot *et al.* [10] investigated the effects of fluctuations on the dynamo threshold (respectively for the G.O. Roberts and the Ponomarenko flows).

The magnetic-field evolution in a conducting fluid is governed by the induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u}\times\mathbf{B}) + \eta\,\Delta\mathbf{B}. \qquad (1)$$

The rhs operator is not self-adjoint [11–14] and can therefore provide algebraic growth of an initial perturbation [15]. This transient magnetic amplification is typically observed in a uniform shear (as examplified by the Ω-effect). We will see how this effect can be used to yield dynamo instability under appropriate time-dependent disturbances.

Modelling. – We focus on a very simple setting, retaining only the essential physical characteristics of the above system (the transient amplification). Let us consider a two-dimensional flow $\mathbf{u} = (u_x(y,t),\, 0,\, 0)^t$ and further assume an initial magnetic field of the form $\mathbf{B}_0 = (B_x\, \cos(z),\, B_y\, \cos(z),\, 0)^t$ independent of the position $\mathbf{x}$. The field depends only on the space coordinate z, which provides a typical length scale L.

We assume that the flow consists of a uniform and constant shear $\partial_y u_x(y,t) \equiv \mathcal{S}$. The evolution of B_x and B_y in time is then governed by

$$\partial_t \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} -\eta & \mathcal{S} \\ 0 & -\eta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \mathcal{L} \begin{pmatrix} B_x \\ B_y \end{pmatrix}. \quad (2)$$

The time variation of the field strength is measured by

$$||\mathbf{B}||^2 = \frac{1}{2\pi} \int_0^{2\pi} B_x^2 + B_y^2 \,\mathrm{d}z = \frac{1}{2}\, \mathrm{e}^{-2\eta t} \left[(B_x(0) + \mathcal{S}\, t\, B_y(0))^2 + B_y(0)^2 \right], \quad (3)$$

which clearly highlights the transient algebraic amplification for an initial seed field along B_y.

We will now discuss the effect of a time-dependent perturbation of the flow. Two typical time scales can be identified in the original problem (2). The advective time scale $\tau_{\mathcal{U}} = L/\mathcal{U} = \mathcal{S}^{-1}$, and the diffusive time scale $\tau_\eta = L^2/\eta$. As we focus here on the limit of large magnetic Reynolds numbers (relevant to astrophysical dynamos), we shall preferably use the advective time scale. The system can be rescaled accordingly using $\varepsilon = \eta/\mathcal{U}L$ (the inverse of the magnetic Reynolds number), and

$$\mathcal{L}_{\mathcal{U}} = \begin{pmatrix} -\varepsilon & 1 \\ 0 & -\varepsilon \end{pmatrix}. \quad (4)$$

We will now consider small-amplitude, time-periodic, perturbations of the flow $\mathbf{u}$. These could for example represent the effect of a traveling wave. For simplicity, we assume that the perturbation modifies the direction of the flow but not its amplitude. The pulsation is described by a sinusoidal wave of amplitude α and frequency β. The new velocity field can therefore be expressed as $\mathbf{u}^{\mathrm{new}} = \mathcal{R}(\theta(t))\, \mathbf{u}$, where $\mathcal{R}(\theta)$ is the 2×2 rotation matrix of angle θ, and $\theta(t) = \alpha \sin(\beta t)$. The evolution of the magnetic field $\mathbf{B}$ is then governed by the linear operator

$$\mathcal{L}(t) = \mathcal{R}(\theta(t))\ \mathcal{L}_{\mathcal{U}}\ \mathcal{R}(\theta(t))^{-1}. \quad (5)$$

Note that we have just introduced a third time scale in the problem, $\tau_\beta \sim L/(\beta\mathcal{U})$.

We will investigate the field evolution and the possibility of dynamo instability, yet the system was made very simple: the field is independent of x and y, and the flow is planar ($u_z \equiv 0$). It is then natural to ponder the feasibility of dynamo action. In fact, the simplicity of the model will also prevent the application of the "two-dimensional field" and "planar velocity" anti-dynamo theorems [16,17]. As the field is independent of x and y, an expression in the form of a streamfunction would necessarily yield unbounded values: the field considered here does not decay to infinity. The above anti-dynamo theorems therefore do not apply to this setup.

Rewriting this system in the rotating frame relative to which the flow is steady yields

$$\partial_t \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} -\varepsilon & 1 + \dot{\theta}(t) \\ -\dot{\theta}(t) & -\varepsilon \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix}. \quad (6)$$

This problem can be tackled through the change of unknown $\mathbf{B}' = \mathrm{e}^{\varepsilon t}\ \mathbf{B}$, and a rescaling in time to the τ_β unit of time, $t' = \beta t$ which leads to

$$\partial_{t'} \begin{pmatrix} B'_x \\ B'_y \end{pmatrix} = \begin{pmatrix} 0 & 1/\beta + \alpha \cos t' \\ -\alpha \cos t' & 0 \end{pmatrix} \begin{pmatrix} B'_x \\ B'_y \end{pmatrix}. \quad (7)$$

Let us stress that, for simplicity, we have introduced this system in a very restricted setup, however such formulations can be extended to local behaviour in more complex flows, for instance near stagnation points of pulsed Beltrami waves [7].

Floquet analysis. – The stability of this system can be investigated depending on the values of α, β. We will in particular focus on the limit of small α, *i.e.* the limit of infinitesimal perturbations of the flow. This problem fits the framework of the Floquet analysis, and the stability is determined by the logarithm of the resolvent matrix eigenvalues. Numerical computation of these eigenvalues are reported fig. 1. For a perturbation of a definite temporal form β, finite growth rate can only be achieved for finite-amplitude modulations: there exists a threshold in the amplitude α below which σ' vanishes (and the unfiltered variable $\mathbf{B}$ decays). This reflects the impossibility of fast dynamo action (*i.e.* growth on the advective time scale) with the chosen flow. However, in the limit of time scales separation, that is

$$\tau_U \ll \tau_\beta \ll \tau_\eta, \quad (8)$$

which yields $\beta \to 0$, exponential growth can occur on the τ_β time scale, as revealed by the finite growth rate σ' at any value of α for small enough β. Tiny disturbances can yield exponentially growing solutions on a typical time scale provided by the perturbation (*i.e.* τ_β). This yields the notion of time scales separation. Remarkably, the growth rate does not always vanish in the limit of small α (*i.e.* infinitesimal perturbation).

Figure 1 reveals a scaling of the form $\alpha \propto \beta$, which points to the use of $\gamma = \alpha/\beta$. We investigate in fig. 2a the growth rate of (7) as a function of (α, γ). The system has an intricate behavior, with large regions of instability, even in the limit $\alpha \to 0$, at finite γ, separated by narrower

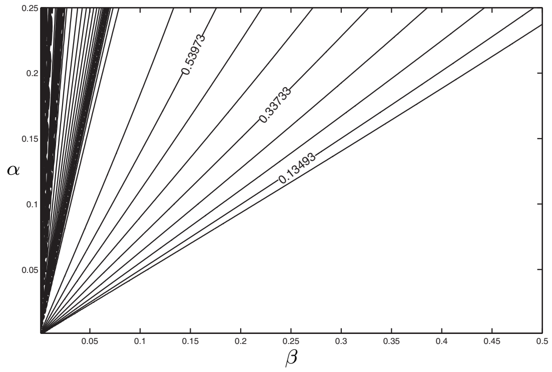

Fig. 1: Growth rate σ' (in the τ_β time scale) as a function of α and β.

neutrally stable bands. Finite growth rate on the τ_β time scale for large γ (small β at fixed α) is even clearer than in fig. 1. This behavior is reminiscent of the parametric instability that develops in forced mechanical systems. A typical example is provided by the Mathieu equation [18] (governing a vertically excited pendulum),

$$\ddot{\theta} + (\widetilde{\alpha} + \gamma \cos t)\ \theta = 0. \tag{9}$$

The classical stability diagram for this system is represented in fig. 2b. Note that system (7) can be rewritten in the scalar form

$$\ddot{B}'_x + \frac{\alpha^2 \sin t}{\gamma + \alpha^2 \cos t}\ \dot{B}'_x + \cos t\,(\gamma + \alpha^2 \cos t)\ B'_x = 0. \tag{10}$$

It is worth stressing that this system does not reduce in an appropriate limit to the unforced pendulum, *i.e.* $\gamma = 0$ in (9). An equivalence between both problems can, however, be established in the limit α, $\widetilde{\alpha} \to 0$ (which corresponds to vanishing gravity for the pendulum). This is highlighted in fig. 2c which presents cross-sections of figs. 2a and b for $\alpha = 0$ and $\widetilde{\alpha} = 0$, respectively.

The growth rate $\sigma'(\alpha, \gamma)$ seems unbounded as $\gamma \to \infty$, but this is only an effect of the chosen time scale τ_β, which becomes increasingly long compared to $\tau_{\mathcal{U}}$ in this limit. Translating to the advective time scale $\tau_{\mathcal{U}}$, the growth rate is given by $\sigma(\alpha, \gamma) = \beta \sigma'(\alpha, \gamma) = |\alpha| \sigma'(\alpha, \gamma)/|\gamma|$. The quantity $\sigma'' \equiv \sigma'(\alpha, \gamma)/|\gamma|$ is represented in fig. 3, which clearly shows that σ'' is uniformly bounded for all (α, γ) and reaches positive values. This validates the scaling $\alpha \sim \beta$, the optimum growth rate is achieved for $\gamma \approx 0.786$, and behaves like $\sigma \approx 0.454\,\alpha$ for small α.

One should keep in mind that ohmic decay has been filtered out from (6) to (7): exponential growth therefore requires $\sigma > \varepsilon$ and thus $\alpha > \varepsilon$. This corresponds to ensuring that a modified magnetic Reynolds number constructed as τ_η/τ_β remains larger than unity.

Interpretation. – It should be stressed that the instabilities reported here are only made possible by the non–self-adjoint nature of the $\mathcal{L}$ operator, *i.e.* of the induction equation. If the system was governed by a negative self-adjoint operator, it could be readily shown that a time dependence of the form (5) would imply negative values of

$$\tfrac{1}{2}\,\mathrm{d}_t |\mathbf{B}|^2 = \langle \mathcal{L}\mathcal{R}^{-1}(\theta(t))\,\mathbf{B} \,|\, \mathcal{R}^{-1}(\theta(t))\,\mathbf{B} \rangle \leqslant 0. \tag{11}$$

The role of transient growth in this parametric instability can be further analysed by taking advantage of the fact $\beta \to 0$. The system can then be investigated under the quasi-static approximation of constant $\dot{\theta} = \alpha_0$, and locally

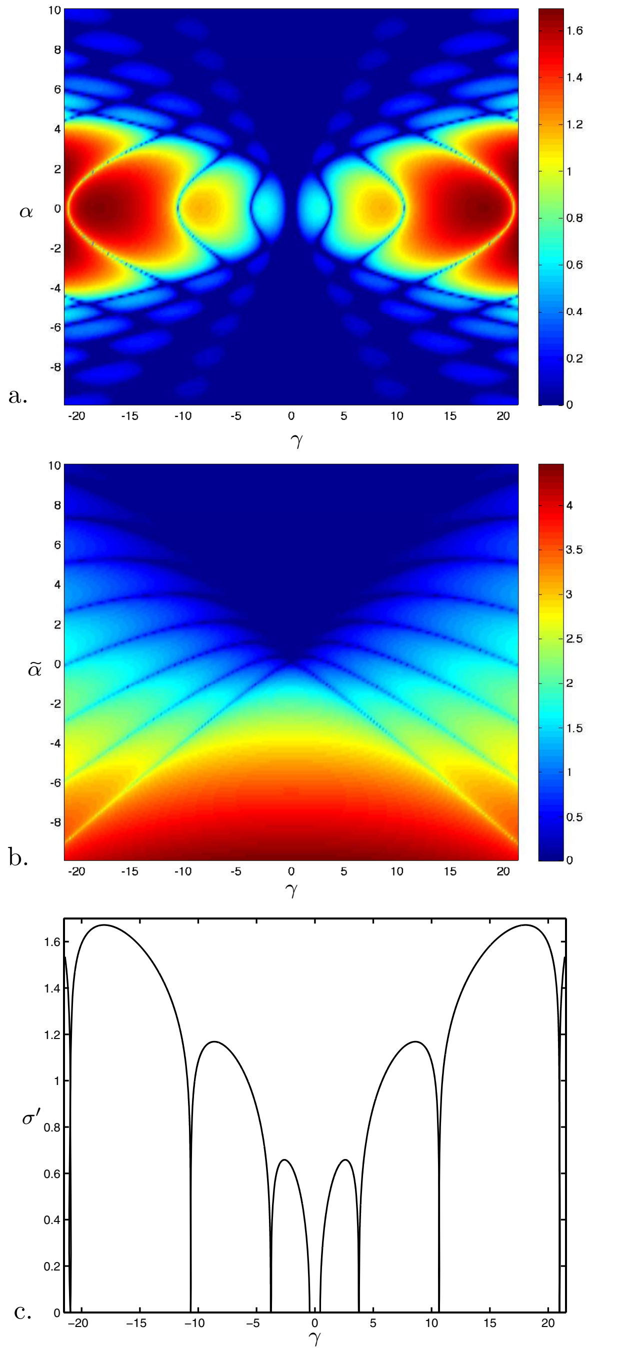

Fig. 2: (Color online) a. Growth rate σ' represented in the (γ, α)-plane. b. Growth rate for the Mathieu equation in the $(\gamma, \widetilde{\alpha})$-plane. c. Both solutions are identical when $\alpha = \widetilde{\alpha} = 0$.

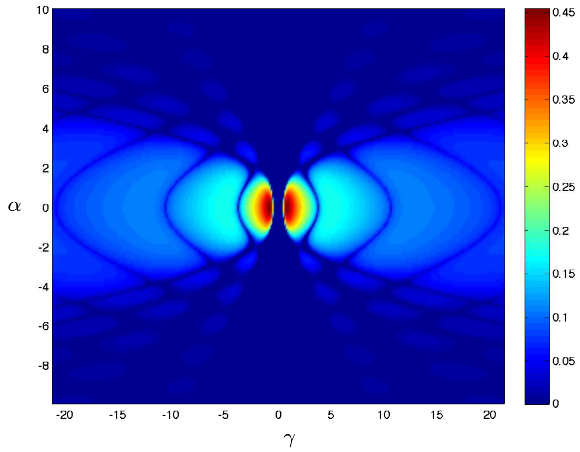

Fig. 3: (Color online) Growth rate $\sigma'' \equiv \sigma'(\alpha,\gamma)/|\gamma|$ in the (γ,α)-plane. Maxima are achieved at finite values of γ.

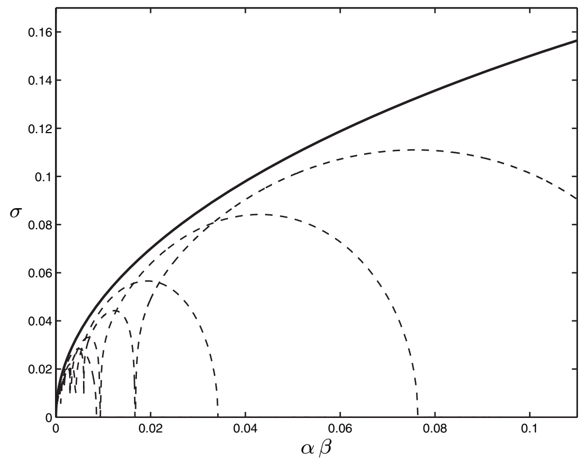

Fig. 4: Growth rate in the advective time scale *vs.* $\alpha\beta$. Dashed lines correspond to the result of a Floquet analysis on system (7) at fixed values of α, from right to left, respectively, $\alpha=0.25$, 0.1875, 0.125, 0.0625. The solid black curve corresponds to the estimate (16).

replaced near each time t_0 by

$$\mathcal{L}(t) = \mathcal{R}(\alpha_0\, t)\ \mathcal{L}_{\mathcal{U}}\ \mathcal{R}(\alpha_0\, t)^{-1}, \tag{12}$$

where $\alpha_0 = \alpha\beta\cos(\beta\, t_0)$. This system corresponds to a steady rotation of the shearing flow at angular velocity α_0 (such situation can occur in convectively driven dynamos, *e.g.* [19]). It is completely integrable, and admits exponentially growing solutions for $-1 < \alpha_0 < 0$,

$$\sigma = -\varepsilon + \sqrt{-\alpha_0\,(1+\alpha_0)}. \tag{13}$$

This formula is reminiscent of the classical Rayleigh-Pedley dispersion relation for rotating shear flows, in which α_0 is replaced by $2\,\alpha_0$ [20]. This similarity is due to the well-known formal analogy of the induction and vorticity equations. The governing pertubation equations do however differ.

This exponential growth can be directly related to the transient field amplification in the initial system (2). While an initial magnetic field is amplified by the shear as described by (3), its direction is altered. This results in a decreases of the phase with time. Let us consider this mechanism over a small increment of time δt. The initial field is amplified

$$||\mathbf{B}(t+\delta t)|| = \mathrm{e}^{\widehat{\sigma}\delta t}\,||\mathbf{B}(t)||, \tag{14}$$

and rotated by a small angle

$$\mathbf{B}(t+\delta t)/||\mathbf{B}(t+\delta t)|| = \mathcal{R}(\widehat{\alpha}\delta t)\,\mathbf{B}(t)/||\mathbf{B}(t)||. \tag{15}$$

If we now consider a system of the form (12), α_0 can be set to $\widehat{\alpha}$ so as to cancel the effect of rotation, which leaves us with $\mathbf{B}(t+\delta t) = \mathrm{e}^{\widehat{\sigma}\delta t}\,\mathbf{B}(t)$. Incrementing in time then yields an exponential increase of the amplitude with growth rate $\widehat{\sigma}$. Initially, the field which is the most amplified by (2) is $(1,1)^t/\sqrt{2}$, and corresponds to $\widehat{\sigma} = \frac{1}{2} - \varepsilon$ and $\widehat{\alpha} = -\frac{1}{2}$. This is precisely the maximum growth rate in (13), and establishes the direct connection between the change in the field orientation during transient amplification and the exponential growth obtained with (12).

The rotation rate $\widehat{\alpha}$ provides a typical time scale $\widehat{\alpha}^{-1}$ for systems exhibiting algebraic growth. This time scale will provide the physical interpretation of the instability process investigated here.

If we return to (6) and use the approximation (12), the rotation rate is provided by $\alpha\beta\cos(\beta t_0)$. For small enough $\alpha\beta$, formula (13) can be used over half a period, corresponding to $\cos(\beta t_0) < 0$. This expression being related to the fastest growing mode, the phase of the solution will act to reduce the realized growth rate. During the other half of the period $(\cos(\beta t_0) > 0)$, algebraic growth is achieved. A good approximation of the resulting growth rate can be obtained by simply considering the growth rate provided by (13) and $\alpha_0 = \alpha\,\beta$ on the first half-period and a zero growth rate for the reminder. As $\alpha\beta \ll 1$ this yields at leading order

$$\sigma = \tfrac{1}{2}\,\sqrt{\alpha\,\beta}. \tag{16}$$

Direct comparison of this expression (still on the advective time scale) with the Floquet integration is provided in fig. 4.

Expression (13) also sheds light on the requirement $\varepsilon \to 0$ for instability in the limit of infinitesimal perturbations. Indeed $\alpha\beta > \varepsilon^2$ is needed to allow positive values of σ.

The exponential instabilities investigated here can therefore be understood both in terms of parametric instabilities and in terms of perturbed algebraic amplification. It is interesting to note that the steady operator $\mathcal{L}$ is not an oscillator and has no characteristic frequency. Since it is associated with transient amplification, a typical time scale $\widehat{\alpha}^{-1}$ can, however, be defined as introduced above. The evolution of the unperturbed system provides, as for the pendulum (Mathieu equation), the natural time scale for parametric amplification.

Conclusions. – In the limit of large magnetic Reynolds number $Rm = \varepsilon^{-1}$, a distinction is usually made between dynamo action on the "slow" diffusive time scale τ_η and the "fast" advective time scale $\tau_{\mathcal{U}}$. We stress here the potential importance of a third time scale corresponding to flow modulations (here τ_β), on which exponential growth can be achieved. Natural dynamos often exhibit large scale shear (usually in the form of a differential rotation associated with the so-called "Ω-effect"). Such large scale shear are unable to produce dynamo action on their own. Although the model we present is extremely simplified (the shear is uniform and the field only depends on one spatial coordinate), we believe the conclusion that interaction of shear and rapid fluctuations can result in dynamo action is an important property. Although we restricted our attention to a simplified setup, interesting conclusions can be drawn concerning dynamo action. In the limit of large Rm, the flow needs to be measured with a precision Rm^{-1} for its dynamo property to be determined. Tiny fluctuations (as small as Rm^{-1}) can drastically change the dynamo properties of the flow. In astrophysical objects Rm^{-1} is often as small as 10^{-18}, so tiny fluctuations (but larger than Rm^{-1}) can modify a non-dynamo flow to provide exponential growth. Further studies are obviously needed to assess under which conditions this amplifications is realised in fully three-dimensional configurations.

The possibility to drive a dynamo through periodic modifications of the flow has already been pointed out, but such an approach was so far restricted to finite-amplitude modulations. We find that even small-amplitude oscillations can act to drive parametric instabilities, while the unperturbed operator is stable. More generally, many physical problems are governed by non–self-adjoint operators. The system we investigated being very simplified, it could be interpreted in a much more general framework than that of the induction equation (for instance in hydrodynamics, see [15,21]).

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