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Stability of mixed Ekman–Hartmann boundary layers

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Abstract. In this paper we study the nonlinear stability of Ekman–Hartmann-type boundary layers in a rotating magnetohydrodynamics flow in a half-space and between two planes. We prove rigorously that if the Reynolds number defined on boundary-layer characteristics is smaller than a critical value, the boundary layer is nonlinearly stable. It is shown that the normal component of the magnetic field increases the critical Reynolds number for instability.

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1. Introduction and physical motivation

The stability of boundary-layer shear flows in magnetohydrodynamic rotating systems is of some general interest. We will concentrate our efforts in this study to the parameter range relevant for the Earth's core. The magnetohydrodynamic flow (MHD) in the Earth's core is believed to support a self-excited dynamo process generating the Earth's magnetic field. Although one has very few means of access to the deep interior of our planet, most of the parameters characterizing the dynamics in the core are quite well known [20, 21]. One can try to model the core by a spherical shell Ω filled with a conducting fluid of density ρ, kinematic viscosity ν, conductivity σ, which rotates rapidly with angular velocity Ω_0. We will only consider here phenomena occuring close to the outer bounding sphere. Important parameters are the Ekman number E, the Rossby number ε, the Elsasser number Λ and the magnetic Reynolds number θ defined introducing a typical velocity U and magnetic field B as

$$E = \nu\Omega_0^{-1}L^{-2}, \quad \varepsilon = U\Omega_0^{-1}L^{-1}, \quad \Lambda = \mathcal{B}^2\rho^{-1}\Omega_0^{-1}\mu_0^{-1}\eta^{-1}, \quad \theta = UL\eta^{-1}. \tag{1}$$

The Earth's core is believed to be in the asymptotic regime of small Ekman number ($E \simeq 10^{-15}$) and Rossby number ($\varepsilon \simeq 10^{-7}$).

Here we present the analytical study of a simplified problem. The stability of an Ekman–Hartmann layer is investigated at the boundary with a half-space $\mathbb{R}^2 \times [0, +\infty[$. We will consider the limit of small Rossby number ε at fixed Elsasser number Λ. It is natural to let $\varepsilon\theta$ go to zero as it appears to be the rescaled size of the self-induced magnetic field. Finally, we will let the Ekman number go to zero, and enforce it to be of size ε^2, in order to have a bounded and nonvanishing Ekman pumping term.

The stability of the Ekman–Hartmann layer at the core–mantle boundary is a critical issue in understanding how it may affect the main flow and thus the dynamo process. The stability of this layer is often assumed *a priori* in numerical works.

As the resolution of the Ekman–Hartmann boundary layers is difficult to achieve numerically [11], some models use free-slip boundary conditions to suppress those layers. Recently, Kuang and Bloxham [18] have highlighted the important effects of the boundary layers on the main flow (and field) in a computation of a hyper-viscous dynamo flow at moderate Ekman numbers.

Some other models suppress inertial and viscous effects in the momentum equation, this leads to the 'magnetostrophic' equilibrium, with the consequence of the Taylor constraint [23]. This simplification can be used to study the induction in the Earth's core [15], but leads to an underdetermination of the geostrophic flow. In practice one needs to restore viscous effects in boundary layers only, through pumping, giving a prescription for this flow.

Let us now describe the stability result. We define a boundary-layer Reynolds number by

$$Re_{BL} = u\frac{\varepsilon}{\sqrt{E}} \tag{2}$$

where u is a typical value of the rescaled velocity (and therefore of order 1). This number is the product of the typical value of the viscosity by the size $\sqrt{E}$ of the Ekman layer, divided by the viscosity E/ε. Notice that we build this Reynolds number on the size of the Ekman layer at $\Lambda = 0$ and not on the size of the Ekman–Hartman layer. This point of view clearly emphasizes the stabilizing role of the magnetic effects and the fact that the stability is controlled by only two dimensionless parameters, namely Λ and the particular combination $\varepsilon/\sqrt{E}$ (and of course on the colatitude).

As E is of order ε^2, Re_{BL} remains constant in the limiting process under consideration. We prove that the Ekman–Hartmann boundary layer is *linearly* and *nonlinearly stable* provided

$$Re_{BL} < Re_s(\Lambda, \theta_0) \tag{3}$$

where θ_0 is the colatitude, and give an explicit formula for Re_s. Of course this does not prove that the layer is unstable for $Re_{BL} \geqslant Re_s$ since Re_s is a poor bound. However, this bound seems physically nonempty, and Re_s is plotted in figure 1. We recall that as u being by definition of order 1 and as $\varepsilon \sim 10^{-7}$ and $E \sim 10^{-15}$, Re_{BL} is of order 1 and therefore completely falls within the values of Re_s given by figure 1.

This estimate on the critical Reynolds number is, however, far from being optimal. In the Ekman case ($\Lambda = 0$) for $\theta_0 = 0$, instabilities appear near $Re_{BL} = 55$ [19], and moreover the magnetic field has a stabilizing effect. In [8], using the methods introduced by Lilly [19] in the pure Ekman case ($\Lambda = 0$) we have computed the critical Reynolds number as a function of θ_0 and Λ for which linear instability occurs for the complete MHD problem.

The stabilizing effect of the magnetic field as well as the destabilizing effect of low latitudes can also be deduced from such an analysis.

2. The governing equations

Let Ω be a three-dimensional domain, with smooth boundaries (typically a ball, a half-space or region between two parallel planes), which will be called the core, Ω^c being the mantle to fit geophysical terminology.

In Ω, we consider the following MHD model, where we assume the fluid to be incompressible. We do not consider buoyancy effects here (see [16] for a discussion of the

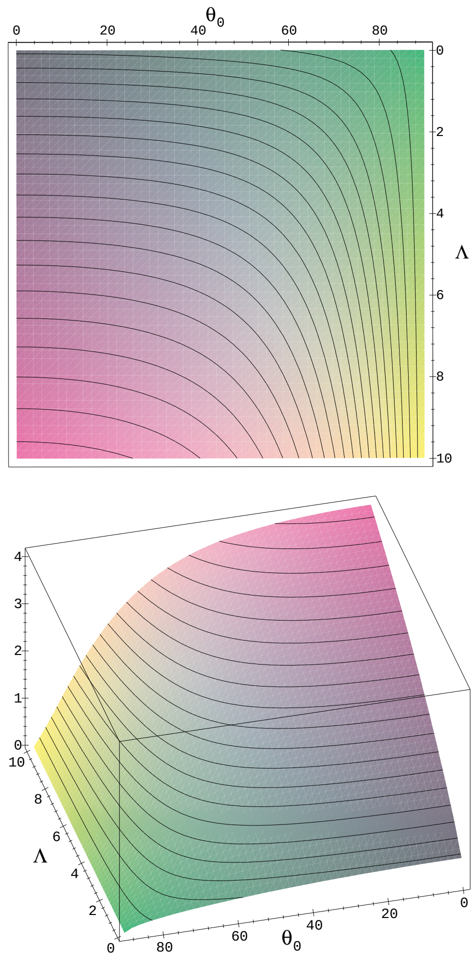

Figure 1. Re_s as a function of Elsasser Λ and colatitude θ_0. The stabilizing role of the normal component of the magnetic induction is clearly illustrated.

linear thermal Ekman layer). We neglect displacement currents in Maxwell's equations, and take into account the Coriolis effect

$$
\begin{aligned}
&\rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) - \mu\Delta\boldsymbol{u} + \nabla p + \rho\Omega_0 \boldsymbol{e}\times\boldsymbol{u} = \boldsymbol{j}\times\boldsymbol{B},\\
&\boldsymbol{j} = \mu_0^{-1}\mathrm{curl}\,\boldsymbol{B}, \quad \mathrm{curl}\,\boldsymbol{E} = -\partial_t\boldsymbol{B}, \quad \boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}),\\
&\mathrm{div}\,\boldsymbol{B} = 0, \quad \mathrm{div}\,\boldsymbol{u} = 0.
\end{aligned} \tag{4}
$$

$\boldsymbol{e}$ denotes a constant unit vector, direction of rotation, $\boldsymbol{j}$ denotes current density which is related through Ohm's law to the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$. The electrical conductivity σ, the fluid dynamic viscosity μ and density ρ are positive constants. As a result, we can eliminate $\boldsymbol{j}$ and $\boldsymbol{E}$ in the above system and obtain

$$
\begin{aligned}
&(\partial_t \boldsymbol{u} + \boldsymbol{u}.\nabla\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \frac{\nabla p}{\rho} + \Omega_0\boldsymbol{e}\times\boldsymbol{u} = \frac{1}{\rho\mu_0}\mathrm{curl}\,\boldsymbol{B}\times\boldsymbol{B},\\
&\partial_t\boldsymbol{B} = \mathrm{curl}\,(\boldsymbol{u}\times\boldsymbol{B}) + \eta\Delta\boldsymbol{B},\\
&\mathrm{div}\,\boldsymbol{B} = 0, \quad \mathrm{div}\,\boldsymbol{u} = 0.
\end{aligned} \tag{5}
$$

Where the magnetic diffusivity η is defined as $(\sigma\mu_0)^{-1}$ and the kinematic viscosity ν is defined as $\nu = \mu/\rho$.

Outside the shell, the mantle Ω^c is considered as an electrical insulator and the magnetic field is assumed to be harmonic:

$$\text{curl } \boldsymbol{B} = 0, \quad \text{curl } \boldsymbol{E} = -\partial_t \boldsymbol{B}, \quad \text{div } \boldsymbol{E} = 0, \quad \text{div } \boldsymbol{B} = 0. \tag{6}$$

At the core–mantle boundary $\partial\Omega$, we impose the velocity of the fluid to vanish and the normal component of the Poynting vector $\boldsymbol{E} \times \boldsymbol{B}$ to be continuous.

Introducing typical quantities

$$\boldsymbol{u} = U\boldsymbol{u}', \quad \boldsymbol{B} = \mathcal{B}\boldsymbol{B}', \quad \boldsymbol{E} = \mathcal{E}\boldsymbol{E}', \quad p = \pi p', \quad \boldsymbol{x} = L\boldsymbol{x}', \quad t = Tt',$$

and dropping the primes, we adimensionalize (5) as follows:

$$\frac{L}{UT}\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{\pi}{\rho U^2}\nabla p + \frac{\Omega_0 L}{U}\boldsymbol{e}\times\boldsymbol{u} - \frac{\nu}{UL}\Delta\boldsymbol{u} = \frac{\mathcal{B}^2}{\rho\mu_0 U^2}\text{curl } \boldsymbol{B}\times\boldsymbol{B}, \; \frac{L}{UT}\partial_t\boldsymbol{B}$$
$$= \text{curl } (\boldsymbol{u}\times\boldsymbol{B}) + \frac{\eta}{UL}\Delta\boldsymbol{B}, \qquad \text{div } \boldsymbol{B} = 0, \quad \text{div } \boldsymbol{u} = 0. \tag{7}$$

Taking

$$T = \frac{L}{U}, \quad \mathcal{E} = \mathcal{B}U, \quad \pi = \rho U\Omega_0 L, \quad \varepsilon = \frac{U}{\Omega_0 L}, \quad E = \frac{\nu}{\Omega_0 L^2}, \quad P_m = \frac{\nu}{\eta},$$

and

$$\Lambda = \frac{\mathcal{B}^2}{\rho\Omega_0\mu_0\eta}, \quad \theta = \frac{P_m\varepsilon}{E} = \frac{UL}{\eta},$$

we rewrite (7)

$$\begin{aligned} &\partial_t\boldsymbol{u} + \boldsymbol{u}.\nabla\boldsymbol{u} + \frac{\nabla p}{\varepsilon} + \frac{\boldsymbol{e}\times\boldsymbol{u}}{\varepsilon} - \frac{E}{\varepsilon}\Delta\boldsymbol{u} = \frac{\Lambda}{\varepsilon\theta}\text{curl } \boldsymbol{B}\times\boldsymbol{B} \\ &\partial_t\boldsymbol{B} = \text{curl } (\boldsymbol{u}\times\boldsymbol{B}) + \frac{1}{\theta}\Delta\boldsymbol{B}, \qquad \text{div } \boldsymbol{B} = 0, \quad \text{div } \boldsymbol{u} = 0, \end{aligned} \tag{8}$$

and in Ω^c, we have

$$\text{curl } \boldsymbol{B} = 0, \quad \text{curl } \boldsymbol{E} = -\partial_t\boldsymbol{B}, \quad \text{div } \boldsymbol{E} = 0, \quad \text{div } \boldsymbol{B} = 0. \tag{9}$$

The numbers $\varepsilon, E, P_m, \Lambda, \theta$ are respectively called Rossby, Ekman, magnetic Prandtl, Elsasser and magnetic Reynolds numbers.

Next, we split the magnetic field $\boldsymbol{B}$ into two parts a large-scale, time-independent field $\boldsymbol{B}_0 = \boldsymbol{e}'$ and a scaled perturbation $\boldsymbol{b}$ such that

$$\boldsymbol{B} = \boldsymbol{e}' + \theta\boldsymbol{b},$$

so that (8) becomes in Ω

$$\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{\nabla p}{\varepsilon} - \frac{E}{\varepsilon}\Delta\boldsymbol{u} + \frac{\boldsymbol{e}\times\boldsymbol{u}}{\varepsilon} = \frac{\Lambda}{\varepsilon}\text{curl } \boldsymbol{b}\times\boldsymbol{e}' + \frac{\Lambda\theta}{\varepsilon}\text{curl } \boldsymbol{b}\times\boldsymbol{b} \tag{10}$$

$$\partial_t\boldsymbol{b} + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \boldsymbol{b}\cdot\nabla\boldsymbol{u} + \frac{\text{curl } (\boldsymbol{u}\times\boldsymbol{e}')}{\theta} + \frac{\Delta\boldsymbol{b}}{\theta}, \qquad \text{div } \boldsymbol{b} = 0, \quad \text{div } \boldsymbol{u} = 0, \tag{11}$$

and in Ω^c

$$\text{curl } \boldsymbol{b} = 0, \quad \text{curl } \boldsymbol{E} = -\theta\partial_t\boldsymbol{b}, \quad \text{div } \boldsymbol{E} = 0, \quad \text{div } \boldsymbol{b} = 0. \tag{12}$$

The boundary conditions with an insulator can be written as

$$\boldsymbol{u} = 0 \quad \text{and} \quad (\boldsymbol{E}\times\boldsymbol{b})\cdot\boldsymbol{n} \quad \text{is continuous.} \tag{13}$$

Notice in particular that on the fluid's side, we have curl $\boldsymbol{b} = \boldsymbol{E}$.

We consider in the following the orderings for $E, \Lambda, \theta, \varepsilon$:

$$\varepsilon \to 0, \quad \Lambda = \mathcal{O}(1), \qquad \varepsilon\theta \to 0 E \sim \varepsilon^2. \tag{14}$$

Notice that this includes $\theta \to 0$, $\theta = \mathcal{O}(1)$ or $\theta \to +\infty$ with $\varepsilon\theta \to 0$.

These limits are relevant to the Earth's core [10, 11]. Typical values for the Earth's core are $\varepsilon \sim 10^{-7}$, $\Lambda = \mathcal{O}(1)$, $\varepsilon\theta \sim 10^{-4}$, $E \sim 10^{-15}$, which fit (14).

3. Statement of the results

Let us first consider the case $\Omega = \mathbb{R}^2 \times [0, +\infty[$ (half-plane $z \geqslant 0$) and let $\boldsymbol{e}$ and $\boldsymbol{e}'$ be the vectors given in (e_1, e_2, e_3) basis by

$$\boldsymbol{e} = (-\sin\theta_0, 0, \cos\theta_0)$$

and

$$\boldsymbol{e}' = \frac{e'}{\|e'\|} \qquad \text{with} \quad e' = (\sin\theta_0, 0, 2\cos\theta_0),$$

where we identify (e_1, e_2, e_3) with (e_θ, e_ϕ, e_r) at a given colatitude θ_0. The normalized imposed magnetic field $\boldsymbol{e}'$ is assumed to be dipolar, even though any other case could have been considered. Let $U_\infty = (u_{1,\infty}, u_{2,\infty})$ be a given velocity at infinity $z = +\infty$. Let $(\boldsymbol{u}_s(x, y, z), \boldsymbol{b}_s(x, y, z))$ be the Ekman–Hartmann layer (see section 4.2 for analytic expressions) which matches the boundary conditions at $z = 0$ and satisfies $(u_{s,1}, u_{s,2}) = U_\infty$ at $z = +\infty$.

Theorem 3.1. *The Ekman–Hartmann layers are stable provided*

$$\|U_\infty\| \frac{\varepsilon}{\sqrt{E}} \leqslant Re_s(\Lambda, \theta_0),$$

where Re_s is given analytically in section 4.4. More precisely, under this condition, if $(\boldsymbol{u}, \boldsymbol{b})$ is another solution of (10), (11)

$$\sup_{t\geqslant 0} \int \left(|\boldsymbol{u}(t) - \boldsymbol{u}_s|^2 + \frac{\Lambda\theta}{\varepsilon} |\boldsymbol{b}(t) - \boldsymbol{b}_s|^2 \right) \leqslant \int \left(|\boldsymbol{u}(0) - \boldsymbol{u}_s|^2 + \frac{\Lambda\theta}{\varepsilon} |\boldsymbol{b}(0) - \boldsymbol{b}_s|^2 \right).$$

Let us now turn to the mathematical approach of the problem. By mathematical approach, we mean partial differential equation (PDE) type mathematics. The aim is to describe the convergence of solutions of (10), (11) in the limit (14), dealing with all the nonlinearities and boundary conditions. We would like to emphasize here the differences between the approaches of PDE people and physicists: mathematicians try to prove convergence of *time dependent, fully nonlinear* solutions of (10), (11) to solutions of some reduced systems (without small parameters) on arbitrarly large time intervals (rarely globally in time), the limit system being also *fully nonlinear* (as complex as two-dimensional Euler's equations). On the other side, physicists are more interested in *global in time* stability of *time independent boundary-layer profiles* (the stability in Lyapunov or dynamical sense). Each theorem is followed by a small comment to make the link with physical concerns.

In what follows, $\Omega = \mathbb{R}^2 \times [0, 1]$, and to simplify the analysis, $\boldsymbol{e} = \boldsymbol{e}' = e_3$, perpendicular to the boundary of Ω (a similar analysis could probably be done for different vectors $\boldsymbol{e}$ and $\boldsymbol{e}'$ provided they are not tangent to $\partial\Omega$). In section 4 we make the formal analysis of the limit $\varepsilon \to 0$. As usual in antisymmetric perturbations of parabolic systems, we have to distinguish between well-prepared an ill-prepared initial data [6, 14]. We prove, for well-prepared initial data, that $\boldsymbol{j}^\varepsilon = \text{curl}\, \boldsymbol{b}^\varepsilon$ goes to 0 and that $\boldsymbol{u}^\varepsilon$ converges to a two-dimensional vector field $\boldsymbol{u}_0^{int}$ independent on z, satisfying a damped Euler equation

$$\partial_t \boldsymbol{u}_0^{int} + (\boldsymbol{u}_0^{int} \cdot \nabla)\boldsymbol{u}_0^{int} + \beta \boldsymbol{u}_0^{int} + \nabla p_0^{int} = 0, \tag{15}$$

$$\text{div}\, \boldsymbol{u}_0^{int} = 0, \tag{16}$$

where $\beta \boldsymbol{u}_0^{int}$ is a damping term, the sum of a viscous Ekman type pumping and a magnetic effect.

$$\beta = \sqrt{\frac{2E}{\varepsilon^2 \tan \frac{\tau}{2}}}$$

with

$$\tan \frac{\tau}{2} = \frac{1}{\Lambda + \sqrt{1 + \Lambda^2}}.$$

In section 4.2 we construct the Ekman–Hartmann layer, following for instance [1], and in section 4.3 we construct an approximate solution $(\boldsymbol{u}_\varepsilon^{app}, \boldsymbol{j}_\varepsilon^{app})$ starting from $\boldsymbol{u}_0^{int}$, an approximate solution which satisfies (10), (11) up to very small (in ε) error terms. It is classical that for $\beta = 0$, Sobolev norms of solutions of (15), (16) have a double exponential behaviour in time. But for $\beta > 0$ we prove the following.

Theorem 3.2. *Let $\beta > 0$ and let $s \geqslant 2$. There exists a positive function Γ_s such that every solution $\boldsymbol{u}_0^{int}$ of (15), (16) satisfies*

$$|\boldsymbol{u}_0^{int}(t, .)|_{L^\infty(\mathbb{R}^2)} \leqslant \Gamma_s(|\boldsymbol{u}_0^{int}(0, .)|_{H^{s+1}(\mathbb{R}^2)})\mathrm{e}^{-\beta t}. \tag{17}$$

This theorem is just the mathematical expression of the damping effect of the Ekman–Hartmann pumping: if the limit flow is initially smooth, it remains smooth and decreases exponentially in time. This is an improvement with respect to [14], since we can now get convergence results which are *global* in time, which was not the case in [14].

Let us now introduce the function

$$\Xi(k) = \sqrt{1 + k^2}\sqrt{k} \int_0^{+\infty} z(|\cos(zk)| + |\sin(zk)|)\mathrm{e}^{-z}\, \mathrm{d}z, \tag{18}$$

the critical Reynolds number for stability

$$Re_s(\Lambda) = \frac{1}{\Xi\left(\frac{1}{\Lambda + \sqrt{1+\Lambda^2}}\right)} \tag{19}$$

and the boundary-layer Reynolds number of $\boldsymbol{u}_0^{int}$ at time t,

$$Re_{BL}(t) = |\boldsymbol{u}_0^{int}(t, \cdot)|_{L^\infty(\mathbb{R}^2)} \frac{\varepsilon}{\sqrt{E}}. \tag{20}$$

We prove in section 5 the following convergence results.

Theorem 3.3. *Let $\boldsymbol{u}_0^{int}(0, x, y)$ be a given $H^s(\mathbb{R}^2)$ function, with $s > 5$. Let $\boldsymbol{u}_0^{int}(t, x, y)$ be the global solution of (15), (16) with initial data $\boldsymbol{u}_0^{int}(0, x, y)$. Let $\boldsymbol{u}_0^\varepsilon$ and $\boldsymbol{b}_0^\varepsilon$ be given sequences of $L^2(\Omega)$ and $L^2(\mathbb{R}^3)$ functions, respectively, such that*

$$\|\boldsymbol{u}_0^\varepsilon(x, y, z) - \boldsymbol{u}_0^{int}(0, x, y)\|_{L^2(\Omega)}^2 + \frac{\Lambda\theta}{\varepsilon}\|\boldsymbol{b}_0^\varepsilon(x, y, z)\|_{L^2(\mathbb{R}^3)}^2 \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

and let $\boldsymbol{u}^\varepsilon$, $\boldsymbol{b}^\varepsilon$ be global weak solutions of (10), (11) with initial data $\boldsymbol{u}_0^\varepsilon$ and $\boldsymbol{b}_0^\varepsilon$. Then

$$\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}_0^{int}\|_{L^\infty([0,T],L^2(\mathbb{R}^2))} + \frac{\Lambda\theta}{\varepsilon}\|\boldsymbol{b}^\varepsilon\|_{L^\infty([0,T],L^2(\mathbb{R}^2))} \to 0$$

for every T such that

$$\sup_{0\leqslant t\leqslant T} Re_{BL}(t) < Re_s(\Lambda). \tag{21}$$

This is just the mathematical formulation of the stability of Ekman–Hartmann layers, expressed in a time dependent framework of theorem 3.1.

We emphasize the fact that we are only considering so-called 'well-prepared initial data' (that is sequences of initial data u_0^ε which converge to a z independent function $\boldsymbol{u}_0^{int}$ as $\varepsilon \to 0$).

Notice that the boundary layers do not appear in the L^2 norm: we require no control of $\boldsymbol{u}_0^\varepsilon$ near the boundaries. In particular, there is no need to impose $\boldsymbol{u}_0^\varepsilon$ to behave like Ekman–Hartmann layers as described in section 4.2 near $z = 0$ and $z = 1$. Using the decay result of section 5.1, condition (21) can be replaced by a condition on the initial data $\boldsymbol{u}_0^{int}$, which gives the following.

Theorem 3.4. *Let $\boldsymbol{u}_0^{int}(0, x, y)$ be a given $H^s(\mathbb{R}^2)$ function, with $s > 5$. Let $\boldsymbol{u}_0^{int}(t, x, y)$ be the global solution of (15), (16) with initial data $\boldsymbol{u}_0^{int}(0, x, y)$. Let $\boldsymbol{u}_0^\varepsilon$ and $\boldsymbol{b}_0^\varepsilon$ be given sequences of respectively $L^2(\Omega)$ and $L^2(\mathbb{R}^3)$ functions such that*

$$\|\boldsymbol{u}_0^\varepsilon(x, y, z) - \boldsymbol{u}_0^{int}(0, x, y)\|_{L^2(\Omega)}^2 + \frac{\Lambda\theta}{\varepsilon}\|\boldsymbol{b}_0^\varepsilon(x, y, z)\|_{L^2(\mathbb{R}^3)}^2 \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

and let $\boldsymbol{u}^\varepsilon$, $\boldsymbol{b}^\varepsilon$ be global weak solutions of (10), (11) with initial data $\boldsymbol{u}_0^\varepsilon$ and $\boldsymbol{b}_0^\varepsilon$. Then

$$\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}_0^{int}\|_{L^\infty([0,+\infty[,L^2(\mathbb{R}^2))} + \frac{\Lambda\theta}{\varepsilon}\|\boldsymbol{b}^\varepsilon\|_{L^\infty([0,+\infty[,L^2(\mathbb{R}^2))} \to 0$$

provided

$$\Gamma_{s-1}(|\boldsymbol{u}_0^{int}(0, ., .)|_{H^s(\mathbb{R}^2)})\frac{\varepsilon}{\sqrt{E}} < Re_s(\Lambda). \tag{22}$$

Physically, this ensures the global stability of the solution using the exponential decreasing of the maximum norm of the limit velocities.

We complete this study by proving in the spirit of [4, 9] (in the particular case $b = 0$ to shorten the proof) that weak solutions of (10), (11) are in fact strong and unique for ε small enough.

Theorem 3.5. *Let $s > 5$. Let $\boldsymbol{u}_0^{int}(0, x, y)$ be a given $H^s(\mathbb{R}^2)$ function satisfying the stability criterion (22). Let $\boldsymbol{u}_0^{int}(t, x, y)$ be the corresponding global solution of (15), (16) and let us construct the sequence of approximate solutions $\boldsymbol{u}_\varepsilon^{app}$ as in section 4.3 up to order ε^2. There exists $\varepsilon_0 > 0$ and $C_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and if $\boldsymbol{u}_0$ satisfies*

$$|\nabla(\boldsymbol{u}_0 - \boldsymbol{u}_\varepsilon^{app})(0, x, y, z)|_{L^2(\Omega)} + \frac{|\left(\boldsymbol{u}_0 - \boldsymbol{u}_\varepsilon^{app}\right)(0, x, y, z)|_{L^2(\Omega)}}{\varepsilon^2} \leqslant C_0 \tag{23}$$

then denoting by $\boldsymbol{u}$ the global weak solutions of the rotating Navier–Stokes equations with initial data $\boldsymbol{u}_0$, we have $D^2\boldsymbol{u} \in L^2((0, +\infty) \times \Omega)$ and $\nabla\boldsymbol{u} \in L^\infty(0, +\infty; L^2(\Omega))$. Moreover, $\boldsymbol{u}$ is unique.

This has important physical consequences, as it guarantees that if at $t = 0$ the velocity field is smooth, it remains smooth for all time.

A similar result (weak solutions are in fact strong) in the case of the quasigeostrophic system with periodic boundary conditions and for ill-prepared initial data has been proved by Chemin in [4]. Notice, however, that here we deal with vanishing viscosity and that the initial conditions have large gradients in the boundary layers. In particular, the condition of smallness of ε which arises in [4] is never fulfilled in our case and we have to replace it by a condition of type (23).

4. Asymptotic behaviour

4.1. The limit system

As usual when we have two parameters which go to 0 (namely ε and $\varepsilon\theta$), the asymptotic expansion depends on particular links between these two quantities, and leads to many cases. In order to avoid technicalities and to get a simple asymptotic expansion we will restricit ourselves to one particular case and assume that $\theta = \varepsilon$. The general case (no link between θ and ε) can be treated in a similar way, and leads to similar first order and similar boundary-layer behaviour.

As usual in boundary-layer theory, we look for approximate solutions of the form

$$\boldsymbol{u}^{\varepsilon,app} = \sum_{k\geqslant 0} \varepsilon^k \left(\boldsymbol{u}_k^{int}(t,x,y,z) + \boldsymbol{u}_k^{BL}\left(t,x,y,\frac{z}{\lambda}\right) + \boldsymbol{u}_k^{BL,upper}\left(t,x,y,\frac{1-z}{\lambda}\right)\right), \tag{24}$$

and similarly we introduce $\boldsymbol{b}_k^{int}, \boldsymbol{b}_k^{BL}$ and $\boldsymbol{b}_k^{BL,upper}, \boldsymbol{j}_k^{int}$ defined by $\boldsymbol{j}_k^{int} = \mathrm{curl}\ \boldsymbol{b}_k^{int}$ and similarly for $\boldsymbol{j}_k^{BL}$ and $\boldsymbol{j}_k^{BL,upper}$, where λ denotes the size of the boundary layer that will be precised later on. Let $\boldsymbol{u}_k^{int} = (u_k^{int}, v_k^{int}, w_k^{int})$ and similarly for $\boldsymbol{u}_k^{BL}$ and $\boldsymbol{u}_k^{BL,upper}$. In (24) we enforce $\boldsymbol{u}_k^{BL}$ and $\boldsymbol{u}_k^{BL,upper}$ to be rapidly decreasing in their last argument. Putting (24) in (10), (11) and identifying the terms in ε^{-1} and θ^{-1} in the interior of the domain we get

$$e_3 \times (\boldsymbol{u}_0^{int} + \Lambda \boldsymbol{j}_0^{int}) + \nabla p_0^{int} = 0, \tag{25}$$

$$\partial_z \boldsymbol{u}_0^{int} + \mathrm{curl}\ \boldsymbol{j}_0^{int} = 0, \tag{26}$$

which leads to

$$\boldsymbol{j}_0^{int} = 0 \tag{27}$$

and

$$w_0^{int} = 0, \qquad \boldsymbol{u}_0^{int} \quad \text{depends only on } x, y \tag{28}$$

('magnetostrophic flow'). Now equalling the terms in ε^{-1} and θ^{-1} in the boundary layers gives that

$$p_0^{BL} = 0, \qquad b_{3,0}^{BL} = 0 \tag{29}$$

as usual in fluid boundary layers (the pressure does not vary much in the layer).

Equalling the terms of order ε^0 in (10) we get

$$\partial_t \boldsymbol{u}_0^{int} + (\boldsymbol{u}_0^{int} \cdot \nabla)\boldsymbol{u}_0^{int} + e_3 \times (\boldsymbol{u}_1^{int} + \Lambda \boldsymbol{j}_1^{int}) + \nabla p_1^{int} = 0$$

and taking the 2D curl of it, with $\omega_0^{int} = \partial_1 v_0^{int} - \partial_2 u_0^{int}$ (which only depends on x and y), we have

$$\partial_t \omega_0^{int} + (\boldsymbol{u}_0^{int} \cdot \nabla)\omega_0^{int} = \partial_z w_1^{int} + \Lambda \partial_z j_{3,1}^{int} \tag{30}$$

which after a vertical integration, since ω_0^{int} and u_0 do not depend on z, gives the two-dimensional limit equation

$$\partial_t \omega_0^{int} + (\boldsymbol{u}_0^{int}.\nabla)\omega_0^{int} = w_1^{int}(x,y,1) - w_1^{int}(x,y,0) + \Lambda j_{3,1}^{int}(x,y,1) - \Lambda j_{3,1}^{int}(x,y,0). \tag{31}$$

For $\Lambda = 0$ we recover the case of Ekman layers [12], as studied in [14]. We have now to compute w_1^{int} and $j_{3,1}^{int}$ on $z = 0$ and $z = 1$. For this we will study the boundary layers which appear near $z = 0$ and $z = 1$. We will prove in the next section that

$$w_1^{int}(x,y,1) - w_1^{int}(x,y,0) + \Lambda j_{3,1}^{int}(x,y,1) - \Lambda j_{3,1}^{int}(x,y,0) = -\beta \omega_0^{int}$$

with

$$\beta = \sqrt{\frac{2E}{\varepsilon^2 \tan\frac{\tau}{2}}}$$

where

$$\tan\frac{\tau}{2} = \frac{1}{\Lambda + \sqrt{1+\Lambda^2}}.$$

This will lead to the limit system on $\omega_0^{int}(t, x, y)$

$$\partial_t \omega_0^{int} + (\boldsymbol{u}_0^{int} \cdot \nabla)\omega_0^{int} + \beta\omega_0^{int} = 0 \tag{32}$$

with div $\boldsymbol{u}_0^{int} = 0$, $w_0^{int} = 0$, $\omega_0^{int} = \partial_1 v_0^{int} - \partial_2 u_0^{int}$ and $\boldsymbol{j}_0^{int} = 0$. Notice that (32) has global strong solutions for smooth initial data.

4.2. *Boundary layers*

Let us focus on the boundary layer near $z = 0$. Let $\zeta = z/\lambda$ and let $\boldsymbol{u}_0^{BL} = (u_0^{BL}, v_0^{BL}, w_0^{BL})$, $\boldsymbol{u}_1^{BL} = (u_1^{BL}, v_1^{BL}, w_1^{BL})$, $\boldsymbol{b}_0^{BL} = (b_{1,0}^{BL}, b_{2,0}^{BL}, b_{3,0}^{BL})$, $\boldsymbol{b}_1^{BL} = (b_{1,1}^{BL}, b_{2,1}^{BL}, b_{3,1}^{BL})$, $\boldsymbol{j}_1^{BL} = (j_{1,1}^{BL}, j_{2,1}^{BL}, j_{3,1}^{BL})$ with

$$\lim_{\zeta=+\infty} \boldsymbol{u}_0^{BL} = \lim_{\zeta=+\infty} \boldsymbol{u}_1^{BL} = \lim_{\zeta=+\infty} \boldsymbol{b}_0^{BL} = \lim_{\zeta=+\infty} \boldsymbol{b}_1^{BL} = \lim_{\zeta=+\infty} \boldsymbol{j}_1^{BL} = 0.$$

By incompressibility condition, $\partial_\zeta w_0^{BL} = 0$ hence

$$w_0^{BL} = 0.$$

Moreover, in the boundary layer, (10) and (11) give

$$-v_0^{BL} - \frac{E}{\lambda^2}\partial_\zeta^2 u_0^{BL} = \frac{\Lambda}{\lambda}\partial_\zeta b_{1,0}^{BL}, \qquad \partial_\zeta^2 b_{1,0}^{BL} + \lambda\partial_\zeta u_0^{BL} = 0, \tag{33}$$

and

$$u_0^{BL} - \frac{E}{\lambda^2}\partial_\zeta^2 v_0^{BL} = \frac{\Lambda}{\lambda}\partial_\zeta b_{2,0}^{BL}, \qquad \partial_\zeta^2 b_{2,0}^{BL} + \lambda\partial_\zeta v_0^{BL} = 0, \tag{34}$$

hence, eliminating $b_{1,0}^{BL}$ and $b_{2,0}^{BL}$, we obtain

$$-\partial_\zeta v_0^{BL} - \frac{E}{\lambda^2}\partial_\zeta^3 u_0^{BL} = -\Lambda\partial_\zeta u_0^{BL}, \tag{35}$$

$$\partial_\zeta u_0^{BL} - \frac{E}{\lambda^2}\partial_\zeta^3 v_0^{BL} = -\Lambda\partial_\zeta v_0^{BL}. \tag{36}$$

It follows $A = \partial_\zeta u_0^{BL} + \mathrm{i}\partial_\zeta v_0^{BL} \in \mathbb{C}$ is solution of

$$\partial_\zeta^2 A = A\frac{\lambda^2}{E}(\Lambda + \mathrm{i}). \tag{37}$$

Defining τ by

$$\cos\tau = \frac{\Lambda}{\sqrt{1+\Lambda^2}}, \qquad \text{and} \qquad \sin\tau = \frac{1}{\sqrt{1+\Lambda^2}},$$

recalling that $A \to 0$ when $\zeta \to +\infty$ and choosing

$$\lambda = \left(\frac{E}{\Lambda}\right)^{\frac{1}{2}} \frac{\sqrt{\cos\tau}}{\cos\frac{\tau}{2}} = \sqrt{2E\tan\frac{\tau}{2}}, \tag{38}$$

we obtain

$$A(t,x,y,\zeta) = A(t,x,y,0)\exp\left(-\zeta\left(1+\mathrm{i}\tan\frac{\tau}{2}\right)\right),$$
$$\tan\frac{\tau}{2} = \frac{1}{\Lambda+\sqrt{1+\Lambda^2}}, \tag{39}$$

and

$$u_0^{BL}(t,x,y,\zeta)+\mathrm{i}v_0^{BL}(t,x,y,\zeta)$$
$$= (u_0^{BL}(t,x,y,0)+\mathrm{i}v_0^{BL}(t,x,y,0))\exp\left(-\zeta\left(1+\mathrm{i}\tan\frac{\tau}{2}\right)\right). \tag{40}$$

As a result, using $\boldsymbol{u}_0^{BL}+\boldsymbol{u}_0^{int}=0$ at $\zeta=0$, we finally write

$$u_0^{BL}(t,x,y,\zeta) = \exp(-\zeta)\left\{-u_0^{int}(t,x,y,0)\cos\left(\zeta\tan\frac{\tau}{2}\right) - v_0^{int}(t,x,y,0)\sin\left(\zeta\tan\frac{\tau}{2}\right)\right\} \tag{41}$$

$$v_0^{BL}(t,x,y,\zeta) = \exp(-\zeta)\left\{u_0^{int}(t,x,y,0)\sin\left(\zeta\tan\frac{\tau}{2}\right) - v_0^{int}(t,x,y,0)\cos\left(\zeta\tan\frac{\tau}{2}\right)\right\}. \tag{42}$$

Using the incompressibility condition

$$\partial_x u_0^{BL}+\partial_y v_0^{BL}+\frac{\varepsilon}{\lambda}\partial_\zeta w_1^{BL}=0$$

we deduce

$$w_1^{BL}(t,x,y,\zeta) = -\exp(-\zeta)\omega_0^{int}(t,x,y)\sin\left(\zeta\tan\frac{\tau}{2}+\frac{\tau}{2}\right)\sqrt{E\varepsilon^{-2}\sin\tau}. \tag{43}$$

As $w_1^{int}+w_1^{BL}=0$ at $\zeta=0$, we get the succion expression

$$w_1^{int}(x,y,0) = \omega_0^{int}(t,x,y)\sin\left(\frac{\tau}{2}\right)\sqrt{E\varepsilon^{-2}\sin\tau}. \tag{44}$$

Next, using (11), $\partial^2_{\zeta\zeta}\boldsymbol{b}_0^{BL}=0$ and hence

$$\boldsymbol{b}_0^{BL}=0.$$

Moreover, $D=b_{1,1}^{BL}+\mathrm{i}b_{2,1}^{BL}$ satisfies

$$\partial_\zeta^2 D = -\frac{\lambda}{\varepsilon}A,$$

hence

$$D = \frac{\lambda}{\varepsilon}\frac{(u_0^{BL}+\mathrm{i}v_0^{BL})}{1+\mathrm{i}\tan\frac{\tau}{2}}, \tag{45}$$

and

$$b_{1,1}^{BL} = \left(u_0^{BL}\cos\frac{\tau}{2}+v_0^{BL}\sin\frac{\tau}{2}\right)\sqrt{E\varepsilon^{-2}\sin\tau}, \tag{46}$$
$$b_{2,1}^{BL} = \left(-u_0^{BL}\sin\frac{\tau}{2}+v_0^{BL}\cos\frac{\tau}{2}\right)\sqrt{E\varepsilon^{-2}\sin\tau}. \tag{47}$$

In other words, we have

$$b_{1,1}^{BL}(t,x,y,\zeta) = \exp(-\zeta)\left\{-u_0^{int}(t,x,y,0)\cos\left(\frac{\tau}{2}+\zeta\tan\frac{\tau}{2}\right)\right.$$
$$\left. - v_0^{int}(t,x,y,0)\sin\left(\frac{\tau}{2}+\zeta\tan\frac{\tau}{2}\right)\right\}\sqrt{E\varepsilon^{-2}\sin\tau} \tag{48}$$

$$b_{2,1}^{BL}(t,x,y,\zeta) = \exp(-\zeta)\left\{u_0^{int}(t,x,y,0)\sin\left(\frac{\tau}{2}+\zeta\tan\frac{\tau}{2}\right)\right.$$
$$\left. - v_0^{int}(t,x,y,0)\cos\left(\frac{\tau}{2}+\zeta\tan\frac{\tau}{2}\right)\right\}\sqrt{E\varepsilon^{-2}\sin\tau}. \tag{49}$$

From (48), (49), we deduce that

$$j_{3,1}^{BL}(t,x,y,\zeta) = -\exp(-\zeta)\omega_0^{int}(t,x,y)\cos\left(\frac{\tau}{2}+\zeta\tan\frac{\tau}{2}\right)\sqrt{E\varepsilon^{-2}\sin\tau}, \tag{50}$$

and using $j_{3,1}^{BL} + j_{3,1}^{int} = 0$ for $\zeta = 0$, the currents entering the layer write

$$j_{3,1}^{int}(t,x,y,0) = \omega_0^{int}(t,x,y)\cos\left(\frac{\tau}{2}\right)\sqrt{E\varepsilon^{-2}\sin\tau}. \tag{51}$$

It follows that

$$(w_1^{int} + \Lambda j_{3,1}^{int})(t,x,y,0) = \omega_0^{int}(t,x,y)\sqrt{\frac{E}{2\varepsilon^2\tan\frac{\tau}{2}}}. \tag{52}$$

Let us observe that in the limit $\Lambda \to 0$, $\lambda = \sqrt{2E}$, $\beta = \sqrt{2E/\varepsilon^2}$ and we recover classical Ekman layers:

$$\boldsymbol{b}_1^{BL} \equiv 0,$$
$$u_0^{BL}(t,x,y,\zeta) = \exp(-\zeta)\{u_0^{BL}(t,x,y,0)\cos\zeta + v_0^{BL}(t,x,y,0)\sin\zeta\}$$
$$v_0^{BL}(t,x,y,\zeta) = \exp(-\zeta)\{-u_0^{BL}(t,x,y,0)\sin\zeta - v_0^{BL}(t,x,y,0)\cos\zeta\}.$$

On the other hand, in the limit when $\Lambda \to +\infty$, $\lambda^2 \sim E/\Lambda$ and one obtains Hartmann-type layers

$$u_0^{BL}(t,x,y,\zeta) = u_0^{BL}(t,x,y,0)\exp(-\zeta), \tag{53}$$
$$v_0^{BL}(t,x,y,\zeta) = v_0^{BL}(t,x,y,0)\exp(-\zeta), \tag{54}$$
$$b_{1,1}^{BL} = \frac{\lambda}{\varepsilon}u_0^{BL}, \qquad b_{2,1}^{BL} = \frac{\lambda}{\varepsilon}v_0^{BL}. \tag{55}$$

Observe that the Ekman succion (44) vanishes in this case, whereas the magnetic damping (51) tends to infinity.

4.3. *Construction of approximate solutions*

It is now routine work to construct an approximate solution u_ε^{app}, j_ε^{app}, b_ε^{app} starting from u_0^{int} and the boundary-layer terms constructed in the previous section (see for instance [14] for details in the case of pure Ekman layer, and [6]). By approximate solutions, we mean functions which match the boundary conditions, which satisfy the divergence free conditions, and which satisfy (10) up to small error terms $R_{1,\varepsilon}$, and (11) up to $R_{2,\varepsilon}$, and moreover for every $t \geqslant 0$, $s > 5$,

$$|R_{1,\varepsilon}|_{L^2(\Omega)} \leqslant C\varepsilon^{\frac{1}{2}}\|u_0^{int}\|_{H^s(\mathbb{R}^2)}, \tag{56}$$
$$|R_{2,\varepsilon}|_{L^2(\Omega)} \leqslant C\varepsilon^{\frac{3}{2}}\|u_0^{int}\|_{H^s(\mathbb{R}^2)}, \tag{57}$$
$$|\boldsymbol{u}_\varepsilon^{app}|_{L^\infty(\Omega)} + \varepsilon^{-1}|\boldsymbol{b}_\varepsilon^{app}|_{L^\infty(\Omega)} \leqslant C\|u_0^{int}\|_{L^\infty(\mathbb{R}^2)}, \tag{58}$$
$$|\partial_x \boldsymbol{u}_\varepsilon^{app}|_{L^\infty(\Omega)} + |\partial_y \boldsymbol{u}_\varepsilon^{app}|_{L^\infty(\Omega)} + |\partial_z w_\varepsilon^{app}|_{L^\infty(\Omega)} \leqslant C\|u_0^{int}\|_{L^\infty(\mathbb{R}^2)}. \tag{59}$$

We have to estimate

$$g(\Lambda) = \sup_{x,y}\left|\int_0^{1/2} z|\partial_z u_0^{BL}|\mathrm{d}z\right|$$

and similar integrals for $\frac{1}{2} \leqslant z \leqslant 1$, and with u_0^{BL} replaced by v_0^{BL}. Using (41) we get

$$\partial_z u_0^{BL} = \lambda^{-1}\left(u_0^{int} - v_0^{int}\tan\frac{\tau}{2}\right)\cos\left(\zeta\tan\frac{\tau}{2}\right)\mathrm{e}^{-\zeta}$$
$$+\lambda^{-1}\left(u_0^{int}\tan\frac{\tau}{2} + v_0^{int}\right)\sin\left(\zeta\tan\frac{\tau}{2}\right)\mathrm{e}^{-\zeta}$$

and therefore

$$g(\Lambda) \leqslant \lambda \left| u_0^{int} - v_0^{int} \tan\frac{\tau}{2} \right| \int_0^{+\infty} \zeta \left| \cos\left(\zeta \tan\frac{\tau}{2}\right) \right| \mathrm{e}^{-\zeta}\, \mathrm{d}\zeta$$
$$+\lambda \left| u_0^{int} \tan\frac{\tau}{2} + v_0^{int} \right| \int_0^{+\infty} \zeta \left| \sin\left(\zeta \tan\frac{\tau}{2}\right) \right| \mathrm{e}^{-\zeta}\, \mathrm{d}\zeta$$
$$\leqslant \sqrt{1 + \tan^2\left(\frac{\tau}{2}\right)} \lambda \| \boldsymbol{u}_0^{int} \|_{L^\infty(\mathbb{R}^2)} \int_0^{+\infty} \zeta \left(\left| \cos\left(\zeta \tan\frac{\tau}{2}\right) \right| + \left| \sin\left(\zeta \tan\frac{\tau}{2}\right) \right| \right) \mathrm{e}^{-\zeta}\, \mathrm{d}\zeta.$$

Hence, as $\lambda = \sqrt{2E\tan(\tau/2)}$,

$$\left| \int_0^{1/2} z |\partial_z u_0^{BL}|\, \mathrm{d}z \right| \leqslant \sqrt{2E} \| \boldsymbol{u}_0^{int} \|_{L^\infty(\mathbb{R}^2)} \Xi\left(\tan\frac{\tau}{2}\right). \tag{60}$$

4.4. Slanted magnetic field and rotation

Let us consider in this section Ekman–Hartmann layers in an half-space, at a colatitude θ_0, with a uniform velocity field at infinity. The angle between the outward normal of the plane and the rotation vector is therefore θ_0. Let ψ be the angle of the magnetic field with the normal of the plane. Provided $\theta_0 \neq \pi/2$ and $\psi \neq \pi/2$, the calculations of the boundary layers can be carried out and the results are very similar to those of section 4.2. Let us present them in the case $\theta \in [0, \pi/2)$: the size of the layer λ is now

$$\lambda = \sqrt{\frac{2E}{\cos\theta_0} \tan\frac{\tau'}{2}},$$

where

$$\tan\frac{\tau'}{2} = \frac{\cos\theta_0}{\Lambda\cos^2\psi + (\Lambda^2\cos^4\psi + \cos^2\theta_0)^{1/2}}.$$

Let

$$Re_s(\Lambda, \theta) = \frac{\sqrt{\cos\theta_0}}{\Xi\left(\tan\frac{\tau'}{2}\right)}.$$

If we assume that the static magnetic field $\boldsymbol{B}_0$ is a pure axial dipole with internal sources, one obtains

$$\cos\psi = \frac{2\cos\theta_0}{(1 + 3\cos^2\theta_0)^{\frac{1}{2}}},$$

since $\boldsymbol{B}_0$ is proportional to $2\cos\theta_0 \mathrm{e}_r + \sin\theta_0 \mathrm{e}_{\theta_0}$ in spherical coordinates. Up to the above parameter changes, the boundary-layer expressions (41), (42), (48), (49) still hold in suitably scaled coordinates.

Those expressions clearly degenerate at the equator. For $\theta_0 = \pi/2$ (see [12]) and $\psi = \pi/2$ (see [22]), we do not study this singularity here, and will restrict our work to the values of θ_0 such that the layer is well-defined by the above expressions (roughly $\theta_0 < \pi/2 - E^{1/3}$).

5. Stability of mixed Ekman–Hartmann boundary layers

5.1. Time decay of limit solutions

The aim of this section is to prove that the maximum norm of the interior velocity $\boldsymbol{u}^{int}$ which is known to exist for all time (by a small modification of Youdovich argument) decays exponentially in time. In terms of the vorticity $\omega_0^{int} = \operatorname{curl} \boldsymbol{u}_0^{int}$, the limit system reads as a damped two-dimensional Euler equation

$$\partial_t \omega_0^{int} + (\boldsymbol{u}_0^{int} \cdot \nabla)\omega_0^{int} + \beta\omega_0^{int} = 0 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}_t^+ \times \mathbb{R}^2) \qquad \text{and} \quad \omega_{0|t=0}^{int} = \omega_{0,0}^{int}.$$

As u_0^{int} is divergence free, we have for all $p \in [1, +\infty]$

$$|\omega_0^{int}(t, .)|_{L^p(\mathbb{R}^2)} = |\omega_{0,0}^{int}|_{L^p(\mathbb{R}^2)} \mathrm{e}^{-\beta t}.$$

Using the following classical estimates on commutators [17]

$$|[D^\alpha, u\nabla]\omega|_{L^2} \leqslant C(|\omega|_{L^\infty}|u|_{H^{s+1}} + |\nabla u|_{L^\infty}|\omega|_{H^s})$$

for $|\alpha| = s \geqslant 2$, and assuming that $\operatorname{div} u = 0$ and $\omega = \operatorname{curl} u$,

$$|\nabla u|_{L^\infty} \leqslant C(|\omega|_{L^\infty} + |\omega|_{L^2}) + C_s|\omega|_{L^\infty} \log_+ \left(\frac{|\omega|_{H^s}}{|\omega|_{L^\infty}}\right)$$

we deduce that for all fixed $s \in \mathbb{N}$, $s \geqslant 2$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\omega_0^{int}(t, .)|_{H^s(\mathbb{R}^2)}^2 + \beta|\omega_0^{int}(t, .)|_{H^s(\mathbb{R}^2)}^2 \leqslant C(|\omega_0^{int}(t, .)|_{L^\infty(\mathbb{R}^2)} + |\nabla \boldsymbol{u}_0^{int}|_{L^\infty(\mathbb{R}^2)})|\omega_0^{int}(t, .)|_{H^s(\mathbb{R}^2)}^2.$$

It follows that defining α by

$$\alpha(t) = |\omega_0^{int}(t, .)|_{H^s(\mathbb{R}^2)} \mathrm{e}^{\beta t}, \tag{61}$$

we observe that

$$\alpha(t) \leqslant \alpha(0) + C\int_0^t \mathrm{e}^{-\beta s}\alpha(s) \times \left(|\omega_{0,0}^{int}|_{L^\infty(\mathbb{R}^2)} \log_+ \left(\frac{\alpha(s)}{|\omega_{0,0}^{int}|_{L^\infty(\mathbb{R}^2)}}\right) + |\omega_{0,0}^{int}|_{L^\infty(\mathbb{R}^2)} + |\omega_{0,0}^{int}|_{L^2(\mathbb{R}^2)}\right) \mathrm{d}s. \tag{62}$$

Let $A = |\omega_{0,0}^{int}|_{L^2(\mathbb{R}^2)}$, let $B = |\omega_{0,0}^{int}|_{L^\infty} + |\omega_{0,0}^{int}|_{L^2(\mathbb{R}^2)}$, and let $\psi(t)$ be the right-hand side of (62). We have

$$\dot{\psi}(t) \leqslant C \exp(-\beta t)\psi(t) \left(A \log_+ \left(\frac{\psi(t)}{A}\right) + B\right)$$

therefore

$$\int_{\psi(0)/A}^{\psi(t)/A} \frac{\mathrm{d}u}{u(\log_+ u + BA^{-1})} \leqslant \int_0^t CA\exp(-\beta s) \leqslant CA\beta^{-1}. \tag{63}$$

But $u \log u$ is not integrable near $+\infty$, therefore (63) bounds $\psi(t)$ and therefore $\alpha(t)$ using (62) in terms of $|\omega_{0,0}^{int}|_{L^2(\mathbb{R}^2)}$ and $|\omega_{0,0}^{int}|_{L^\infty}$ which gives theorem 3.2.

5.2. Proof of the stability result

We will only prove theorem 3.4, the proof of theorem 3.3 being similar and easier. Denoting $\boldsymbol{v} = \boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^{app}$, $\mathrm{e} = \boldsymbol{E}_\varepsilon - \boldsymbol{E}_\varepsilon^{app}$, and $\boldsymbol{m} = \boldsymbol{b}_\varepsilon - \boldsymbol{b}_\varepsilon^{app}$, we obtain

$$\partial_t \boldsymbol{v} + \boldsymbol{u}_\varepsilon \cdot \nabla\boldsymbol{v} + \boldsymbol{v} \cdot \nabla\boldsymbol{u}_\varepsilon^{app} - \frac{E}{\varepsilon}\Delta\boldsymbol{v} + \nabla\pi + \frac{e_3 \times \boldsymbol{v}}{\varepsilon} = \frac{\Lambda}{\varepsilon}\operatorname{curl} \boldsymbol{m} \times e_3 + \frac{\Lambda\theta}{\varepsilon}\left(\boldsymbol{b}_\varepsilon.\nabla\boldsymbol{m} + \boldsymbol{m}.\nabla\boldsymbol{b}_\varepsilon^{app}\right) - R_{1,\varepsilon}, \tag{64}$$

$$\operatorname{div} \boldsymbol{v} = 0, \qquad \operatorname{div} \boldsymbol{m} = 0, \qquad \text{and} \qquad \boldsymbol{v}_{|\partial\Omega} \equiv 0, \tag{65}$$

and

$$\partial_t \boldsymbol{m} + \boldsymbol{u}_\varepsilon.\nabla \boldsymbol{m} + \boldsymbol{v}\cdot\nabla \boldsymbol{b}_\varepsilon^{app} - \boldsymbol{b}_\varepsilon\cdot\nabla\boldsymbol{v} - \boldsymbol{m}\cdot\nabla\boldsymbol{u}_\varepsilon^{app} = \frac{1}{\theta}\Delta\boldsymbol{m} - \frac{1}{\theta}\text{curl }(\boldsymbol{v}\times e_3) - R_{2,\varepsilon}, \quad (66)$$

whereas in Ω^c, we simply have

$$\text{curl }\boldsymbol{m} = 0, \quad \text{curl }\boldsymbol{e} = -\theta\partial_t\boldsymbol{m}, \quad \text{div }\boldsymbol{m} = 0 \quad \text{and} \quad \text{div }\boldsymbol{e} = 0. \quad (67)$$

We recall estimates (56)–(60) for all time $t \geqslant 0$. In order to obtain energy bounds, we multiply (64) by $\boldsymbol{v}$ and (66) by $\boldsymbol{m}\Lambda\theta/\varepsilon$ to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\left(|\boldsymbol{v}|^2 + \frac{\Lambda\theta}{\varepsilon}|\boldsymbol{m}|^2\right)\mathrm{d}x + \frac{E}{\varepsilon}\int_\Omega|\nabla\boldsymbol{v}|^2\,\mathrm{d}x + \frac{\Lambda}{\varepsilon}\int_\Omega|\text{curl }\boldsymbol{m}|^2\,\mathrm{d}x&\\ +\frac{\Lambda}{\varepsilon}\int_{\partial\Omega}(\text{curl }\boldsymbol{m}\times\boldsymbol{m})\cdot\boldsymbol{n} \leqslant |R_{1,\varepsilon}|_{L^2(\Omega)}|\boldsymbol{v}|_{L^2(\Omega)} + \frac{\Lambda\theta}{\varepsilon}|R_{2,\varepsilon}|_{L^2(\Omega)}|\boldsymbol{m}|_{L^2(\Omega)}&\\ +\left|\int_\Omega\left(\frac{\Lambda\theta}{\varepsilon}\boldsymbol{m}_i\boldsymbol{m}_k + \boldsymbol{v}_i\boldsymbol{v}_k\right)\partial_i\boldsymbol{u}_{k,\varepsilon}^{app}\,\mathrm{d}x\right| + \left|\int_\Omega\frac{\Lambda\theta}{\varepsilon}\left(\boldsymbol{m}_i\boldsymbol{v}_k + \boldsymbol{m}_k\boldsymbol{v}_i\right)\partial_i\boldsymbol{b}_{k,\varepsilon}^{app}\,\mathrm{d}x\right|,&\end{aligned}$$

where $\boldsymbol{n}$ denotes the outward normal to Ω. Next, we observe using the fact that $\boldsymbol{v}$ vanishes on $\partial\Omega$

$$\int_\Omega \boldsymbol{m}_i\boldsymbol{m}_k\partial_i\boldsymbol{u}_{k,\varepsilon}^{app}\,\mathrm{d}x = -\int_\Omega \boldsymbol{u}_{k,\varepsilon}^{app}\boldsymbol{m}_i(\partial_i\boldsymbol{m}_k - \partial_k\boldsymbol{m}_i)\,\mathrm{d}x,$$

hence

$$\begin{aligned}\left|\int_\Omega\frac{\Lambda\theta}{\varepsilon}\boldsymbol{m}_i\boldsymbol{m}_k\partial_i\boldsymbol{u}_{k,\varepsilon}^{app}\,\mathrm{d}x\right| &\leqslant C\frac{\Lambda\theta}{\varepsilon}|\text{curl }\boldsymbol{m}|_{L^2(\Omega)}|\boldsymbol{m}|_{L^2(\Omega)}\Gamma_0\exp(-\beta t),\\ &\leqslant \kappa\frac{\Lambda}{\varepsilon}|\text{curl }\boldsymbol{m}|^2_{L^2(\Omega)} + C_\kappa\Gamma_0^2\frac{\Lambda\theta^2}{\varepsilon}|\boldsymbol{m}|^2_{L^2(\Omega)}\exp(-2\beta t).\end{aligned}$$

Similarly, we have

$$\int_\Omega(\boldsymbol{m}_i\boldsymbol{v}_k + \boldsymbol{m}_k\boldsymbol{v}_i)\partial_i\boldsymbol{b}_{k,\varepsilon}^{app}\,\mathrm{d}x = \int_\Omega\boldsymbol{b}_{k,\varepsilon}^{app}(\boldsymbol{m}_i(\partial_k\boldsymbol{v}_i - \partial_i\boldsymbol{v}_k) + \boldsymbol{v}_i(\partial_k\boldsymbol{m}_i - \partial_i\boldsymbol{m}_k))\,\mathrm{d}x$$

hence

$$\begin{aligned}&\left|\int_\Omega\frac{\Lambda\theta}{\varepsilon}(\boldsymbol{m}_i\boldsymbol{v}_k + \boldsymbol{m}_k\boldsymbol{v}_i)\partial_i\boldsymbol{b}_{k,\varepsilon}^{app}\,\mathrm{d}x\right|\\ &\quad\leqslant \Lambda\theta(|\boldsymbol{m}|_{L^2(\Omega)}|\nabla\boldsymbol{v}|_{L^2(\Omega)} + |\boldsymbol{v}|_{L^2(\Omega)}|\text{curl }\boldsymbol{m}|_{L^2(\Omega)})\exp(-\beta t),\\ &\quad\leqslant \frac{\kappa E}{\varepsilon}|\nabla\boldsymbol{v}|^2_{L^2(\Omega)} + \kappa\frac{\Lambda}{\varepsilon}|\text{curl }\boldsymbol{m}|^2_{L^2(\Omega)} + C_\kappa\frac{\Lambda\theta}{\varepsilon}|\boldsymbol{m}|^2_{L^2(\Omega)}\Lambda\theta\frac{\varepsilon^2}{E}\exp(-2\beta t)\\ &\quad\quad + C_\kappa\varepsilon\Lambda\theta^2|\boldsymbol{v}|^2_{L^2(\Omega)}\exp(-2\beta t).\end{aligned}$$

The last term involving the velocity $\boldsymbol{v}$ is estimated as in [6, 14]. Namely, for $i = x, y$ and arbitrary k, and for $i = k = z$, using (58), (59),

$$\left|\int_\Omega\boldsymbol{v}_i\boldsymbol{v}_k\partial_i\boldsymbol{u}_{k,\varepsilon}^{app}\,\mathrm{d}x\right| \leqslant C\Gamma_s|\boldsymbol{v}|^2_{L^2(\Omega)}\exp(-\beta t).$$

It remains to handle the case $i = z$ and $k = x, y$. For this, we first remark that

$$\left|\int_\Omega\boldsymbol{v}_i\boldsymbol{v}_k\partial_i(\boldsymbol{u}_\varepsilon^{app} - \boldsymbol{u}_\varepsilon^{BL})\right| \leqslant C\Lambda_s\exp(-\beta t)|\boldsymbol{v}|^2_{L^2}.$$

Moreover,

$$\begin{aligned}\left|\int_{\mathbb{R}^2\times[0,1/2]} v_3 v_1 \partial_z u_\varepsilon^{BL}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| &= \left|\int_{\mathbb{R}^2\times[0,1/2]}\left(\int_0^z \partial_z v_3(t,x,y,z')\,\mathrm{d}z'\right)\right.\\ &\quad\left.\times\left(\int_0^z \partial_z v_1(t,x,y,z')\,\mathrm{d}z'\right)\partial_z u_\varepsilon^{BL}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\\ &\leqslant \sqrt{\int_0^{1/2}\int_{\mathbb{R}^2}|\partial_z v_3|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}\sqrt{\int_0^{1/2}\int_{\mathbb{R}^2}|\partial_z v_1|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}\sup_{(x,y)\in\mathbb{R}^2}\\ &\quad\times\left|\int_0^{1/2} z|\partial_z u_\varepsilon^{BL}|\,\mathrm{d}z\right|\\ &\leqslant \|\partial_z v_3\|_{L^2(\mathbb{R}^2\times[0,1/2])}\|\partial_z v_1\|_{L^2(\mathbb{R}^2\times[0,1/2])}\|\boldsymbol{u}_0^{int}\|_{L^\infty(\mathbb{R}^2)}\sqrt{2E}\,\Xi\left(\tan\frac{\tau}{2}\right)\\ &\leqslant \left(\frac{1}{2\sqrt{2}}\|\partial_z v_3\|^2_{L^2(\mathbb{R}^2\times[0,1/2])}+\frac{1}{\sqrt{2}}\|\partial_z v_1\|^2_{L^2(\mathbb{R}^2\times[0,1/2])}\right)\\ &\quad\times\|\boldsymbol{u}_0^{int}\|_{L^\infty(\mathbb{R}^2)}\sqrt{2E}\,\Xi\left(\tan\frac{\tau}{2}\right)\end{aligned}$$

and similarly for $i=z$ and $k=y$. Therefore,

$$\begin{aligned}\left|\int_{\mathbb{R}^2\times[0,1]} v_3 v_1 \partial_z u_\varepsilon^{BL}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| + \left|\int_{\mathbb{R}^2\times[0,1]} v_3 v_2 \partial_z v_\varepsilon^{BL}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\\ \leqslant \|\nabla v\|^2_{L^2(\Omega)}\|\boldsymbol{u}_0^{int}\|_{L^\infty(\mathbb{R}^2)}\sqrt{E}\,\Xi\left(\tan\frac{\tau}{2}\right)\\ \leqslant \sqrt{E}\,\Xi\left(\tan\frac{\tau}{2}\right)\|\nabla v\|^2_{L^2(\Omega)}\Gamma_s\exp(-\beta t)\end{aligned}$$

where we used theorem 3.2. Thus, finally using the equations (67) in Ω^c, we obtain

$$\begin{aligned}\frac{\Lambda\theta}{\varepsilon}\int_{\partial\Omega}(\mathrm{curl}\,\boldsymbol{m}\times\boldsymbol{m})\cdot\boldsymbol{n} &= \frac{\Lambda\theta}{\varepsilon}\int_{\partial\Omega}(\boldsymbol{e}\times\boldsymbol{m})\cdot\boldsymbol{n} = -\frac{\Lambda\theta}{\varepsilon}\int_{\Omega^c}\mathrm{div}\,(\boldsymbol{e}\times\boldsymbol{m})\,\mathrm{d}x\\ &= \frac{\Lambda\theta^2}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^c}|\boldsymbol{m}|^2\,\mathrm{d}x,\end{aligned}$$

so that we can estimate the energy in the whole space $\mathbb{R}^3$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\int_\Omega\frac{1}{2}|\boldsymbol{v}|^2+\frac{\Lambda\theta}{2\varepsilon}\int_\Omega|\boldsymbol{m}|^2\,\mathrm{d}x+\frac{\Lambda\theta^2}{2\varepsilon}\int_{\Omega^c}|\boldsymbol{m}|^2\,\mathrm{d}x\right)+\frac{E}{\varepsilon}\int_\Omega|\nabla\boldsymbol{v}|^2\,\mathrm{d}x\\ &\quad+\frac{\Lambda}{\varepsilon}\int_\Omega|\mathrm{curl}\,\boldsymbol{m}|^2\,\mathrm{d}x\\ &\leqslant C\left(|\boldsymbol{v}|^2_{L^2(\Omega)}(1+\varepsilon\Lambda\theta^2)+\frac{\Lambda\theta}{\varepsilon}|\boldsymbol{m}|^2_{L^2(\Omega)}\theta\left(1+\Lambda\frac{\varepsilon^2}{E}\right)+\varepsilon\right)\exp(-\beta t)\\ &\quad+\left(\Gamma_s\sqrt{E}\,\Xi\left(\tan\frac{\tau}{2}\right)+2\kappa\frac{E}{\varepsilon}\right)|\nabla\boldsymbol{v}|^2_{L^2(\Omega)}+2\kappa\frac{\Lambda}{\varepsilon}|\mathrm{curl}\,\boldsymbol{m}|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore, if (22) is satisfied, and for κ small enough we have the global estimate

$$\begin{aligned}\sup_{t\geqslant 0}&\left(|\boldsymbol{v}(t,.)|^2_{L^2(\Omega)}+\frac{\Lambda\theta}{\varepsilon}|\boldsymbol{m}(t,.)|^2_{L^2(\Omega)}+\frac{\Lambda\theta^2}{\varepsilon}|\boldsymbol{m}(t,.)|^2_{L^2(\Omega^c)}\right)\\ &\quad+\kappa'\frac{E}{\varepsilon}\int_0^{+\infty}|\nabla\boldsymbol{v}|^2_{L^2(\Omega)}\,\mathrm{d}s+\kappa'\frac{\Lambda}{\varepsilon}\int_0^{+\infty}|\mathrm{curl}\,\boldsymbol{m}|^2_{L^2(\Omega)}\,\mathrm{d}s\\ &\leqslant C_\beta\left(|\boldsymbol{v}(0,.)|^2_{L^2(\Omega)}+\frac{\Lambda\theta}{\varepsilon}|\boldsymbol{m}(0,.)|^2_{L^2(\Omega)}+\frac{\Lambda\theta^2}{\varepsilon}|\boldsymbol{m}(0,.)|^2_{L^2(\Omega^c)}+\varepsilon\right)\end{aligned}$$

for some constant depending on β, and for κ' arbitrary close to 1.

5.3. *Global strong solutions for the Navier–Stokes equations for small enough* ε

Hereafter, we focus on the case when $\boldsymbol{b} \equiv 0$ which correspond to the rotating incompressible Navier–Stokes equations between two parallel plates. As proven in [2, 4] in the three-dimensional periodic case, large enough Rossby numbers ε yield global classical solutions for the Navier–Stokes equations for suitable initial conditions. We want here to prove that this result holds for the boundary-value problem.

As in the above section, we define $\boldsymbol{v} = \boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^{app}$ solution of

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \pi - \frac{1}{Re} \Delta \boldsymbol{v} = L_\varepsilon, \qquad \mathrm{div}\, \boldsymbol{v} = 0, \tag{68}$$

where

$$L_\varepsilon = -\frac{e_3 \times \boldsymbol{v}}{\varepsilon} - \boldsymbol{v} \cdot \nabla \boldsymbol{u}_\varepsilon^{app} - \boldsymbol{u}_\varepsilon^{app} \cdot \nabla \boldsymbol{v} + R_\varepsilon^N \tag{69}$$

with

$$|R_\varepsilon^N|_{L^2(\Omega)} \leqslant C \varepsilon^{N+\frac{1}{2}} \mathrm{e}^{-\beta t}, \tag{70}$$

where N is a given integer such that $N \geqslant 3$. If the stability criterion is satisfied, we obtain when $Re = \varepsilon^{-1} C_{Re}$ for some constant C_{Re},

$$\sup_{t \geqslant 0} |\boldsymbol{v}(t,.)|^2_{L^2(\Omega)} + \frac{1}{Re} \int_0^{+\infty} |\nabla \boldsymbol{v}|^2_{L^2(\Omega)}\, \mathrm{d}s \leqslant C |\boldsymbol{v}(0,.)|^2_{L^2(\Omega)} + C \varepsilon^{2N+1} = K_\varepsilon^0. \tag{71}$$

In order to prove that $\boldsymbol{u}_\varepsilon$ is smooth whenever ε is small enough, we proceed as in [9] in the context of two-dimensional multiphase MHD flows. First, we multiply (68) by $\partial_t \boldsymbol{v}$ and integrate by parts as follows

$$\begin{aligned} \int_0^t |\partial_t \boldsymbol{v}|^2_{L^2(\Omega)}\, \mathrm{d}s + \frac{1}{2Re} |\nabla \boldsymbol{v}(t,.)|^2_{L^2(\Omega)} &\leqslant \frac{1}{2Re} |\nabla \boldsymbol{v}(0,.)|^2_{L^2(\Omega)} + \frac{1}{2} \int_0^t |\partial_t \boldsymbol{v}|^2_{L^2}\, \mathrm{d}s \\ &\quad + C \int_0^t (|L_\varepsilon|^2_{L^2(\Omega)} + |\boldsymbol{v} \nabla \boldsymbol{v}|^2_{L^2})\, \mathrm{d}s \end{aligned}$$

hence

$$\begin{aligned} \int_0^t |\partial_t \boldsymbol{v}|^2_{L^2(\Omega)}\, \mathrm{d}s + \frac{1}{Re} |\nabla \boldsymbol{v}(t,.)|^2_{L^2(\Omega)} &\leqslant \frac{1}{Re} |\nabla \boldsymbol{v}(0,.)|^2_{L^2(\Omega)} + C \int_0^t |L_\varepsilon|^2_{L^2(\Omega)}\, \mathrm{d}s \\ &\quad + C \int_0^t |\boldsymbol{v}|^2_{L^4(\Omega)} |\nabla \boldsymbol{v}|^2_{L^4(\Omega)}\, \mathrm{d}s. \end{aligned} \tag{72}$$

Rewriting (68) as a Stokes equation

$$-\frac{1}{Re} \Delta \boldsymbol{v} + \nabla \pi = L_\varepsilon - \boldsymbol{v}.\nabla \boldsymbol{v} - \partial_t \boldsymbol{v}, \quad \mathrm{div}\, \boldsymbol{v} = 0, \quad \text{and} \quad \boldsymbol{v} = 0 \quad \text{on} \quad \partial\Omega,$$

classical estimates yield

$$\frac{1}{Re^2} |D^2 \boldsymbol{v}|^2_{L^2(\Omega)} \leqslant C (|\partial_t \boldsymbol{v}|^2_{L^2(\Omega)} + |L_\varepsilon|^2_{L^2(\Omega)} + |\boldsymbol{v} \cdot \nabla \boldsymbol{v}|^2_{L^2(\Omega)}).$$

Combining with (72) we get

$$\begin{aligned} \int_0^t \left(|\partial_t \boldsymbol{v}|^2_{L^2} + \frac{|D^2 \boldsymbol{v}|^2_{L^2}}{Re^2} \right) \mathrm{d}s + \frac{1}{Re} |\nabla \boldsymbol{v}(t,.)|^2_{L^2} &\leqslant \frac{C}{Re} |\nabla \boldsymbol{v}(0,.)|^2_{L^2} \\ &\quad + C \int_0^t (|\boldsymbol{v}|^2_{L^4} |\nabla \boldsymbol{v}|^2_{L^4} + |L_\varepsilon|^2_{L^2})\, \mathrm{d}s. \end{aligned} \tag{73}$$

Combining the following two Gagliardo–Nirenberg inequalities,

$$|\boldsymbol{v}|_{L^4}^2 \leqslant C|\boldsymbol{v}|_{L^2}^{1/2}|\nabla\boldsymbol{v}|_{L^2}^{3/2}$$

and

$$|\nabla\boldsymbol{v}|_{L^4}^2 \leqslant C|\nabla\boldsymbol{v}|_{L^2}^{1/2}(|\nabla\boldsymbol{v}|_{L^2} + |D^2\boldsymbol{v}|_{L^2})^{3/2}$$

we deduce that for all $\kappa > 0$

$$|\boldsymbol{v}|_{L^4(\Omega)}^2|\nabla\boldsymbol{v}|_{L^4(\Omega)}^2 \leqslant \frac{\kappa}{Re^2}(|D^2\boldsymbol{v}|_{L^2(\Omega)}^2 + |\nabla\boldsymbol{v}|_{L^2(\Omega)}^2) + C_\kappa Re^6|\boldsymbol{v}|_{L^2(\Omega)}^2|\nabla\boldsymbol{v}|_{L^2(\Omega)}^8.$$

Next, we observe that arguing as in the preceding section, we can estimate

$$|\boldsymbol{v}\cdot\nabla\boldsymbol{u}_\varepsilon^{app}|_{L^2(\Omega)}^2 + |\boldsymbol{u}_\varepsilon^{app}.\nabla\boldsymbol{v}|_{L^2(\Omega)}^2 \leqslant C|\nabla\boldsymbol{v}|_{L^2(\Omega)}^2\exp(-2\beta t)$$

which leads to

$$|L_\varepsilon|_{L^2}^2 \leqslant C|\nabla\boldsymbol{v}|_{L^2}^2\exp(-2\beta t) + C\frac{|\boldsymbol{v}|_{L^2}^2}{\varepsilon^2} + C\varepsilon^{2N+1}\exp(-2\beta t).$$

As a result, we obtain using the Poincaré lemma, for $\varepsilon \leqslant 1$,

$$\int_0^t\left(|\partial_t\boldsymbol{v}|_{L^2(\Omega)}^2 + \frac{1}{Re^2}|D^2\boldsymbol{v}|_{L^2(\Omega)}^2\right)\mathrm{d}s + \frac{1}{Re}|\nabla\boldsymbol{v}(t,.)|_{L^2(\Omega)}^2 \leqslant \frac{C}{Re}|\nabla\boldsymbol{v}(0,.)|_{L^2(\Omega)}^2$$
$$+\frac{C}{Re^6}\int_0^t|\boldsymbol{v}|_{L^2}^2|\nabla\boldsymbol{v}|_{L^2}^8\,\mathrm{d}s + C\int_0^t\frac{|\nabla\boldsymbol{v}|_{L^2}^2}{\varepsilon^2}\,\mathrm{d}s + C\varepsilon^{2N+1}.$$

Using the Poincaré lemma and $|\boldsymbol{v}(t,.)|_{L^2}^2 \leqslant K_\varepsilon^0$ we get

$$\int_0^t\left(|\partial_t\boldsymbol{v}|_{L^2(\Omega)}^2 + \frac{1}{Re^2}|D^2\boldsymbol{v}|_{L^2(\Omega)}^2\right)\mathrm{d}s + \frac{1}{Re}|\nabla\boldsymbol{v}(t,.)|_{L^2(\Omega)}^2 \leqslant \frac{C}{Re}|\nabla\boldsymbol{v}(0,.)|_{L^2}^2$$
$$+CRe^{10}K_0^\varepsilon\int_0^t\left(\frac{|\nabla\boldsymbol{v}|_{L^2}^2}{Re}\right)^4\mathrm{d}s + C\frac{Re}{\varepsilon^2}K_0^\varepsilon + C\varepsilon^{2N+1}$$
$$\leqslant C\frac{|\nabla\boldsymbol{v}(0,.)|_{L^2}^2}{Re} + \frac{CK_0^\varepsilon}{\varepsilon^{10}}\int_0^t\left(\frac{|\nabla\boldsymbol{v}|_{L^2}^2}{Re}\right)^4\mathrm{d}s + \frac{CK_0^\varepsilon}{\varepsilon^3}.$$

Let

$$\alpha(t) = \frac{|\nabla\boldsymbol{v}(t,.)|_{L^2}^2}{Re},$$
$$C_0 = C\frac{|\nabla\boldsymbol{v}(0,.)|_{L^2}^2}{Re} + \frac{CK_0^\varepsilon}{\varepsilon^3},$$
$$C_1 = \frac{CK_0^\varepsilon}{\varepsilon^{10}}$$

and

$$\psi(t) = C_0 + C_1\int_0^t\alpha^4\,\mathrm{d}s.$$

We have

$$\alpha \leqslant C_0 + C_1\int_0^t\alpha^4\,\mathrm{d}s$$

therefore,

$$\dot\psi \leqslant C_1\alpha\psi^3$$

hence

$$\psi(t) \leqslant \psi(0)\left(1 - 2\psi(0)^2 C_1 \int_0^{+\infty} \alpha \, \mathrm{d}s\right)^{-1/2}$$

which bounds $\psi(t)$ in terms of $\psi(0)$ provided

$$2\psi(0)^2 C_1 \int_0^{+\infty} \alpha \, \mathrm{d}s \leqslant 1/2$$

that is provided

$$\left(\frac{|\nabla \boldsymbol{v}(0,.)|_{L^2}^2}{Re} + \frac{K_0^\varepsilon}{\varepsilon^3}\right)^2 K_0^\varepsilon \frac{K_0^\varepsilon}{\varepsilon^{10}} \leqslant C_\infty$$

where C_∞ is some universal constant, or equivalently provided

$$|\nabla \boldsymbol{v}(0,.)|_{L^2}^2 + \frac{K_0^\varepsilon}{\varepsilon^4} \leqslant C_\infty$$

which ends the proof. Notice that the uniqueness property is a straightforward consequence of the above regularity. Besides, additional bounds can be obtained by deriving the equation in time and integrating by parts, but no further details will be given here.

5.4. Proof of theorem 3.1

To prove theorem 3.1 just follow section 5.2 and notice that the term $(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_s$ reduces to $v_3 \partial_z \boldsymbol{u}_s$ which can be absorbed in the viscosity exactly as in section 5.2 leading to the fact that $\int |\boldsymbol{v}|^2 + \frac{\Lambda\theta}{\varepsilon} \int |\boldsymbol{m}|^2$ is decreasing.

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