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THE VORTEX METHOD FOR TWO-DIMENSIONAL IDEAL FLOWS IN EXTERIOR DOMAINS*

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Abstract. The vortex method is a common numerical and theoretical approach used to implement the motion of an ideal flow, in which the vorticity is approximated by a sum of point vortices, so that the Euler equations read as a system of ordinary differential equations. Such a method is well justified in the full plane, thanks to the explicit representation formulas of Biot and Savart. In an exterior domain, we also replace the impermeable boundary by a collection of point vortices generating the circulation around the obstacle. The density of these point vortices is chosen in order that the flow remains tangent at midpoints between adjacent vortices and that the total vorticity around the obstacle is conserved. In this work, we provide a rigorous justification of this method for any smooth exterior domain, one of the main mathematical difficulties being that the Biot–Savart kernel defines a singular integral operator when restricted to a curve (here, the boundary of the domain). We also introduce an alternative method—the fluid charge method—which, as we argue, is better conditioned and therefore leads to significant numerical improvements.

Key words. Euler equations, elliptic problems in exterior domains, double layer potential, discretization of singular integral operators, spectral analysis, Poincaré–Bertrand formula, Cauchy integrals

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Numerical methods describing the evolution of a fluid flow have an important practical interest in engineering and applications. Such approximation methods often also provide deeper theoretical insight and physical intuition into the properties of fluids. It is therefore important to justify that given methods provide good approximations of analytic solutions. The goal of this article is to validate mathematically the vortex method in exterior smooth domains for the two-dimensional Euler equations and to further develop other similar refined methods.

1. The Euler equations in exterior domains.

The motion of an incompressible ideal fluid filling a domain $\Omega \subset \mathbb{R}^2$ is governed by the Euler equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad [0,\infty) \times \Omega, \\ u \cdot n = 0 & \text{on} \quad [0,\infty) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

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where $u = (u_1(t,x_1,x_2), u_2(t,x_1,x_2))$ is the velocity, $p = p(t,x_1,x_2)$, the pressure and n the unit inward normal vector.

There is an extensive literature about the study of this difficult system, first on physical motivations and second because it provides elegant mathematical problems at the frontier of elliptic theory, dynamical systems, convex geometry, and evolution partial differential equations. One may argue that the richness of these equations is due to the role of the vorticity:

$$\omega(t,x) := \operatorname{curl} u(t,x) = \partial_1 u_2 - \partial_2 u_1.$$

Indeed, taking the curl of the momentum equation in (1.1), we note that this quantity satisfies a transport equation:

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{in} \quad (0,\infty) \times \Omega. \tag{1.2}$$

From this form, we may deduce several conservation properties (e.g., the conservation of all $L^p(\Omega)$-norms of ω for all $1 \leqslant p \leqslant \infty$) which allow us to establish the wellposedness of the Euler equations in various settings (standard references can be found in [13, 26]). One of the key steps in the analysis of (1.1) consists in reconstructing the velocity u from the vorticity ω by solving the following elliptic problem:

$$\begin{cases} \operatorname{div} u = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u = \omega & \text{in} \quad \Omega, \\ u \cdot n = 0 & \text{on} \quad \partial\Omega, \\ u \to 0 & \text{as} \quad x \to \infty, \end{cases} \tag{1.3}$$

where $\omega \in C_c^{0,\alpha}(\Omega)$ for some $0 < \alpha \leqslant 1$.

In the case of the full plane $\Omega = \mathbb{R}^2$, any solution of

$$\operatorname{div} u = 0 \text{ in } \mathbb{R}^2, \quad \operatorname{curl} u = \omega \text{ in } \mathbb{R}^2, \quad u \to 0 \text{ as } x \to \infty, \tag{1.4}$$

satisfies

$$\Delta u = \nabla^\perp \omega \quad \text{in } \mathbb{R}^2,$$

in the sense of distributions, which easily yields

$$u = K_{\mathbb{R}^2}[\omega] = \mathcal{F}^{-1} \frac{-i\xi^\perp}{|\xi|^2} \mathcal{F}\omega.$$

Here, the superscript $\perp$ denotes the rotation by $\pi/2$, that is, $(x_1,x_2)^\perp = (-x_2,x_1)$. It follows, employing standard results on Fourier multipliers, that $K_{\mathbb{R}^2}$ has bounded extensions from L^p to $\dot{W}^{1,p}$ for any $1 < p < \infty$. Furthermore, writing $\Phi(x) = -\frac{1}{2\pi} \log|x|$ the fundamental solution of the Laplacian in $\mathbb{R}^2$, it holds that (see, e.g., [14])

$$\begin{aligned} u = K_{\mathbb{R}^2}[\omega] &= -\Phi * \left(\nabla^\perp \omega\right) = -\nabla^\perp (\Phi * \omega) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy \in C^1\left(\mathbb{R}^2\right). \end{aligned} \tag{1.5}$$

We refer to [9, p. 249] for a justification of the C^1-regularity of $K_{\mathbb{R}^2}[\omega]$ for any $\omega \in C_c^{0,\alpha}(\Omega)$ (which may also be deduced from the representation formula (7.7), below).

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When $\Omega = \mathbb{R}^2 \setminus \mathcal{C}$ is an exterior domain, with $\mathcal{C}$ a compact, smooth (i.e., its boundary is C^∞), and simply connected set, there are an infinite number of solutions of (1.3), because there exists a unique harmonic vector field $H \in C^1(\Omega) \cap C(\overline{\Omega})$ (see [21, Proposition 2.1], for instance) verifying

$$\begin{aligned} &\operatorname{div} H = 0 \text{ in } \Omega, \quad \operatorname{curl} H = 0 \text{ in } \Omega, \quad \oint_{\partial\Omega} H \cdot \tau ds = 1, \\ &H \cdot n = 0 \text{ on } \partial\Omega, \quad H(x) \to 0 \text{ as } x \to \infty, \end{aligned} \tag{1.6}$$

where $\tau := n^\perp$ is the tangent vector to $\partial\Omega$. (Note that n points out of the obstacle $\mathcal{C}$ so that τ orients $\partial\Omega$ counterclockwise.) In fact, it can be shown that H belongs to $C^\infty(\Omega)$ and that all its derivatives are continuous up to the boundary $\partial\Omega$ (use the representation formula (1.9), below).

Thus, in order to reconstruct uniquely the velocity in terms of the vorticity, the standard idea consists in prescribing the circulation:

$$\oint_{\partial\Omega} u \cdot \tau \, ds = \gamma,$$

where $\gamma \in \mathbb{R}$. This constraint is natural because Kelvin's theorem implies then that the circulation of u around an obstacle is a conserved quantity for the Euler equations. With this additional condition, it now holds true that there exists a unique classical solution $u \in C^1(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} \operatorname{div} u = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u = \omega & \text{in} \quad \Omega, \\ u \cdot n = 0 & \text{on} \quad \partial\Omega, \\ u \to 0 & \text{as} \quad x \to \infty, \\ \oint_{\partial\Omega} u \cdot \tau ds = \gamma, \end{cases} \tag{1.7}$$

where $\omega \in C_c^{0,\alpha}(\Omega)$ for some $0 < \alpha \leqslant 1$, and $\gamma \in \mathbb{R}$.

To solve this elliptic problem, we may introduce (as in [21, Lemma 2.2 and Proposition 2.1]; see also [28, section 1.2]) the Green function with Dirichlet boundary condition $G_\Omega : \Omega \times \Omega \to \mathbb{R}$ as the function verifying the following:

$$\begin{aligned} G_\Omega(x,y) &= G_\Omega(y,x) && \text{for all } (x,y) \in \Omega^2, \\ \Delta_x G_\Omega(x,y) &= \delta(x-y) && \text{for all } (x,y) \in \Omega^2, \\ G_\Omega(x,y) &= 0 && \text{for all } (x,y) \in \partial\Omega \times \Omega, \end{aligned}$$

where δ denotes the Dirac function centered at the origin. Using a conformal C^∞-diffeomorphism $T : \Omega \to \{|x| > 1\}$ such that (this transformation can be constructed through the Riemann mapping theorem; see [21, Lemma 2.1])

- it is bijective from $\overline{\Omega}$ onto $\{|x| \geqslant 1\}$,
- all its derivatives are continuous up to $\partial\Omega$ and are uniformly bounded over Ω,
- all the derivatives of its inverse are continuous up to $\{|x| = 1\}$ and are uniformly bounded over $\{|x| > 1\}$,
- there is $\beta \in \mathbb{R} \setminus \{0\}$ such that $T(x) - \beta x$ and $T^{-1}(x) - \beta^{-1}x$ are uniformly bounded over Ω and $\{|x| > 1\}$, respectively,

one has the formula

$$G_\Omega(x,y) = \frac{1}{2\pi} \log \frac{|T(x) - T(y)|}{|T(x) - T(y)^*| \, |T(y)|}$$

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with the notation $y^* = \frac{y}{|y|^2}$ for any $y \in \mathbb{R}^2 \setminus \{0\}$. This expression allows us to write explicitly the solution of (1.7) (for all details, we refer, e.g., to [21]):

$$
\begin{aligned}
u(x) &= K_\Omega[\omega](x) + \alpha H(x) := \int_\Omega \nabla_x^\perp G_\Omega(x,y)\omega(y)dy + \alpha H(x) \\
&= \frac{1}{2\pi}\int_\Omega \left(\frac{DT^t(x)\,(T(x)-T(y))}{|T(x)-T(y)|^2} - \frac{DT^t(x)\,(T(x)-T(y)^*)}{|T(x)-T(y)^*|^2} \right)^\perp \omega(y)dy \\
&\quad + \frac{\alpha}{2\pi}\frac{(DT^t(x)T(x))^\perp}{|T(x)|^2},
\end{aligned} \tag{1.8}
$$

where we have set

$$\alpha = \gamma + \int_\Omega \omega(y)dy.$$

Note that the total mass of the vorticity is also a conserved quantity of incompressible ideal two-dimensional flows. Note also that (1.8) uses the representation

$$H(x) = \frac{(DT^t(x)T(x))^\perp}{2\pi\,|T(x)|^2} \tag{1.9}$$

for the unique solution H of (1.6). Further employing that (see [9, p. 249], for instance, or use the representation formula (7.7), below)

$$\int_{\{|y|>1\}} \frac{x-y}{|x-y|^2}\omega\left(T^{-1}(y)\right)\left|\det DT^{-1}(y)\right| dy \in C^1\left(\mathbb{R}^2\right), \tag{1.10}$$

it is readily seen from (1.8) that $u \in C^1\left(\overline{\Omega}\right)$, thus yielding the unique classical solution to (1.7).

All in all, the Euler equations around the obstacle $\mathcal{C}$ can be seen as the transport of the vorticity (1.2) by the velocity field u defined by (1.8). This property conveniently allows for the use of various mathematical theories and it is therefore crucial to develop efficient and robust methods to rebuild the velocity field u from the vorticity ω or an approximation of it. In particular, for the sake of applications, we are going to focus on the theoretical and numerical approximation of (1.8).

2. The vortex method.

In the full plane $\mathbb{R}^2$, when the initial vorticity is close to being concentrated at N given points $\{x_i^0\}_{i=1}^N \subset \mathbb{R}^2$, i.e., $\omega(t=0) \sim \sum_{i=1}^N \gamma_i \delta_{x_i^0}$ in some suitable sense, Marchioro and Pulvirenti [27] have shown that the corresponding solution of the Euler equations in the full plane has a vorticity which remains close to a combination of Dirac masses $\omega(t) \sim \sum_{i=1}^N \gamma_i \delta_{x_i(t)}$ (in some suitable sense) where the centers $\{x_i\}_{i=1}^N$ verify a system of ODEs, called the point vortex system:

$$
\begin{cases}
\dot{x}_i(t) = \dfrac{1}{2\pi}\displaystyle\sum_{j\neq i} \gamma_j \frac{(x_i(t)-x_j(t))^\perp}{|x_i(t)-x_j(t)|^2}, \\
x_i(0) = x_i^0.
\end{cases} \tag{2.1}
$$

Here, the point vortex $\gamma_i\delta_{x_i(t)}$ moves under the velocity field produced by the other point vortices. This approximation by point vortices remains valid as long as the solution to (2.1) exists, i.e., as long as no collisions occur.

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It turns out that this Lagrangian formulation is much easier to handle numerically than the Eulerian formulation (1.2). Indeed, standard numerical methods on (1.2) generate an "inherent numerical viscosity" and some quantities which should be conserved instead decrease (see, e.g., [20, 32]). Actually, smoothing the Biot–Savart kernel by mollifying $\frac{x^\perp}{|x|^2}$ in (2.1) gives a more stable system, called the vortex blob method (i.e., an approximation of the vorticity by Dirac masses and a regularization of the kernel). The stability and the convergence as $N \to \infty$ of the vortex blob and point vortex methods have been extensively studied: in [5] for the vortex blob method when the initial vorticity is bounded, in [17] for the point vortex method for smooth initial data, and in [25, 31] for both methods and for a weak solutions as, e.g., a vortex sheet (see also the textbook [8]).

However, all these works use the explicit formula of the Biot–Savart law in the full plane (1.5) where the flow $\frac{(x-x_i)^\perp}{2\pi|x-x_i|^2}$ is identified with $K_{\mathbb{R}^2}[\delta_{x_i}]$. In an exterior domain, the Biot–Savart law is much more complicated. A possible approach could be to use the explicit formula (1.8) in order to adapt the previous vortex methods. But such an approach would yield serious practical difficulties. Indeed, explicit Riemann mappings are only available for a few domains with specific symmetry properties. In general, if we consider that Ω is the exterior of a smooth, compact, simply connected subset of $\mathbb{R}^2$, formula (1.8) only gives an implicit representation, which has some theoretical interest, but remains impractical.

Our alternative strategy consists in modeling the impermeable boundary of the exterior domain by a collection of point vortices $\sum_{i=1}^N \frac{\gamma_i^N(t)}{N}\delta_{x_i^N}$, where the vortex positions $\{x_i^N\}_{i=1}^N$ are fixed on $\partial\Omega$ but the density of points $\{\gamma_i^N\}_{i=1}^N$ now evolves with time and is chosen in order that the resulting velocity field remains tangent at midpoints on the boundary between the x_i^N's. Note that this approach appears sometimes in physics and engineering books (see, e.g., [3, 15]).

2.1. Static convergence of the vortex approximation. We need to explain now how the vortex method is used to replace the obstacle in (1.7) by vortices. To this end, we introduce u_P the solution of (1.4) in the full plane, which is explicitly given by (1.5),

$$u_P := K_{\mathbb{R}^2}[\omega] \in C^1\left(\mathbb{R}^2\right) \subset C^1\left(\overline{\Omega}\right), \tag{2.2}$$

and the remainder velocity field u_R defined by

$$u_R := u - u_P \in C^0\left(\overline{\Omega}\right) \cap C^1(\Omega), \tag{2.3}$$

where u is the unique solution to (1.7). As ω is compactly supported in Ω we get by the Stokes formula that $\oint_{\partial\Omega} u_P \cdot \tau ds = \int_{\overline{\Omega}^c} \operatorname{curl} u_P = \int_{\overline{\Omega}^c} \omega = 0$. Hence, it is readily seen that u_R solves

$$\begin{cases} \operatorname{div} u_R = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u_R = 0 & \text{in} \quad \Omega, \\ u_R \cdot n = -u_P \cdot n & \text{on} \quad \partial\Omega, \\ u_R \to 0 & \text{as} \quad x \to \infty, \\ \oint_{\partial\Omega} u_R \cdot \tau ds = \gamma. \end{cases} \tag{2.4}$$

In particular, u_R is harmonic in Ω and therefore it is smooth in Ω, i.e., $u_R \in C^\infty(\Omega)$ (see [14, Corollary 8.11] or [19]).

The vortex method for the exterior domain Ω is essentially an approximation procedure of u_R by point vortices on $\partial\Omega$. More precisely, let now $\left(x_1^N, x_2^N, \ldots, x_N^N\right)$

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be the positions of N distinct point vortices on the boundary $\partial\Omega$. Given an arc-length parametrization $l : [0, |\partial\Omega|] \to \mathbb{R}^2$ of $\partial\Omega$, oriented counterclockwise so that $l' = \tau$, we consider

$$0 = s_1^N < s_2^N < \cdots < s_N^N < |\partial\Omega| \text{ such that } x_i^N = l\left(s_i^N\right). \tag{2.5}$$

We further introduce some intermediate points on the boundary for each $i = 1, \dots, N$ (setting $s_{N+1}^N = |\partial\Omega|$):

$$\tilde{s}_i^N \in (s_i^N, s_{i+1}^N), \quad \tilde{x}_i^N := l\left(\tilde{s}_i^N\right). \tag{2.6}$$

The method consists in approximating the solution u_R to (2.4) by a suitable flow

$$u_{\rm app}^N(x) := \frac{1}{2\pi}\sum_{j=1}^N \frac{\gamma_j^N}{N}\frac{(x - x_j^N)^\perp}{|x - x_j^N|^2} = K_{\mathbb{R}^2}\Big[\sum_{j=1}^N \frac{\gamma_j^N}{N}\delta_{x_j^N}\Big], \tag{2.7}$$

whose vorticity is precisely made of N point vortices with densities $\{\frac{\gamma_i^N}{N}\}_{i=1}^N$ on the boundary $\partial\Omega$. Observe that the approximate flow $u_{\rm app}^N$ is not defined at each x_i^N on the boundary, which is why we introduce the intermediate points $\tilde{x}_i^N$. These points will be used to compute some values of $u_{\rm app}^N$ on the boundary.

It is to be emphasized that this approximation is consistent with and motivated by the physical idea that the circulation around an obstacle is created by a collection of vortices on the boundary of the obstacle, i.e., a vortex sheet on the boundary.

However, it is a priori not obvious that such a flow $u_{\rm app}^N$ can be made a good approximation of u_R. Nevertheless, note that $u_{\rm app}^N$ already naturally satisfies

$$\begin{cases} \operatorname{div} u_{\rm app}^N = 0 & \text{in } \Omega, \\ \operatorname{curl} u_{\rm app}^N = 0 & \text{in } \Omega, \\ u_{\rm app}^N \to 0 & \text{as } x \to \infty. \end{cases}$$

Therefore, the key idea lies in enforcing that the boundary and circulation conditions be satisfied as $N \to \infty$ by setting $\gamma^N = (\gamma_1^N, \dots, \gamma_N^N) \in \mathbb{R}^N$ to be the solution of the following system of N linear equations:

$$\begin{aligned} &\frac{1}{2\pi}\sum_{j=1}^N \frac{\gamma_j^N}{N}\frac{(\tilde{x}_i^N - x_j^N)^\perp}{|\tilde{x}_i^N - x_j^N|^2}\cdot n(\tilde{x}_i^N) = -[u_P \cdot n](\tilde{x}_i^N) \quad \text{for all } i = 1, \dots, N-1, \\ &\sum_{i=1}^N \frac{\gamma_i^N}{N} = \gamma. \end{aligned} \tag{2.8}$$

In order to emphasize the dependence of $u_{\rm app}^N$ on ω (through u_P) and γ, we will sometimes use the notation $u_{\rm app}^N = u_{\rm app}^N[\omega, \gamma]$. Note that $u_{\rm app}^N$ is linear in (ω, γ).

It is shown below, under suitable hypotheses on the placement of point vortices and provided N is sufficiently large, that the above system always has a unique solution γ^N. The fact that $u_{\rm app}^N$ is a good approximation of u_R is precisely the content of our first main theorem below (see Theorem 2.1). Clearly, it then follows that u is well approximated by $u_{\rm app}^N + K_{\mathbb{R}^2}[\omega]$, which concludes the rigorous justification of the vortex method for the boundary of an obstacle applied to the elliptic system (1.7).

We give now a precise definition of a well distributed mesh $\{x_i^N\}_{1\leqslant i\leqslant N}$ and $\{\tilde{x}_i^N\}_{1\leqslant i\leqslant N}$.

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DEFINITION. *We say that the points* $\{x_i^N\}_{1\leqslant i\leqslant N}$ *and* $\{\tilde{x}_i^N\}_{1\leqslant i\leqslant N}$ *given by* (2.5)–(2.6) *are* well distributed *on* $\partial\Omega$ *if there exists an integer* $\kappa \geqslant 2$ *such that, as* $N \to \infty$,

$$\max_{i=1,\dots,N} \left|s_i^N - \theta_i^N\right| = \mathcal{O}\left(N^{-(\kappa+1)}\right) \text{ and } \max_{i=1,\dots,N} \left|\tilde{s}_i^N - \tilde{\theta}_i^N\right| = \mathcal{O}\left(N^{-(\kappa+1)}\right), \tag{2.9}$$

where

$$\theta_i^N = \frac{(i-1)\,|\partial\Omega|}{N} \quad \text{and} \quad \tilde{\theta}_i^N = \frac{(i-\frac{1}{2})\,|\partial\Omega|}{N} \quad \text{for all } i = 1,\dots,N. \tag{2.10}$$

The points on $\partial\Omega$ *corresponding to* $\{\theta_i^N\}_{1\leqslant i\leqslant N}$ *and* $\{\tilde{\theta}_i^N\}_{1\leqslant i\leqslant N}$ *are said to be* uniformly distributed.

Our first main result states that the approximate flow u^N_{app}, constructed through the procedure (2.8), is a good approximation of u_R provided the vortices are well distributed on $\partial\Omega$.

THEOREM 2.1. *Let* $\omega \in L^1_c(\Omega)$ *and* $\gamma \in \mathbb{R}$ *be given. For any* $N \geqslant 2$, *we consider a well distributed mesh satisfying* (2.9) *and* u_P *defined in* (2.2).

Then, there exists N_0 *(independent of* ω *and* γ*) such that, for all* $N \geqslant N_0$, *the system* (2.8) *admits a unique solution* $\gamma^N \in \mathbb{R}^N$. *Moreover, for any closed set* $K \subset \Omega$ *there exists a constant* $C > 0$ *independent of* N, K, ω, *and* γ *such that*

$$\begin{aligned} &\|u_R - u^N_{\text{app}}\|_{L^\infty(K)} \\ &\qquad \leqslant \frac{C}{N^\kappa}\left(\frac{1}{\operatorname{dist}(K,\partial\Omega)} + \frac{1}{\operatorname{dist}(K,\partial\Omega)^{\kappa+2}}\right) \\ &\qquad \times \left(\left(\frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)} + \frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)^{\kappa+1}}\right)\|\omega\|_{L^1(\mathbb{R}^2)} + |\gamma|\right), \end{aligned}$$

where u^N_{app} *is given by* (2.7) *in terms of* γ^N *and* u_R *is the continuous flow* (2.3).

We refer to [1] for an equivalent theorem in the much simpler case of the unit disk $\mathcal{C} = \overline{B(0,1)}$. This restricted geometry allows for an easier proof based on the circular Hilbert transform.

If we consider the uniformly distributed mesh, then the estimate in Theorem 2.1 holds true for any $\kappa \geqslant 2$ but C depends on κ. This implies that the numerical method converges faster than any fixed order method. It is likely that assuming the boundary is analytic enforces exponential convergence. (See Appendix A for the convergence of Riemann sums.)

The proof of Theorem 2.1 is developed over the course of the next few sections. In section 3, we establish important representation formulas for the solution of (2.4) and we show the link between our problem, Cauchy integrals, and the Poincaré–Bertrand formula. Then, in section 4, we prove that the linear system (2.8) is invertible for sufficiently many point vortices. In section 5, we establish that $(u_R - u^N_{\text{app}}) \cdot n|_{\partial\Omega}$ converges to zero in a weak sense together with other related convergence properties on $\partial\Omega$. Finally, in section 6, we deduce that such a weak convergence implies a stronger form of convergence away from the boundary and thus reach the conclusion of Theorem 2.1.

Remark. Let us already advertise here that, in section 8, we introduce and discuss a novel discretization method of the flow u_R—the fluid charge method—and establish convergence results similar to Theorem 2.1 (see Theorems 8.1 and 8.4 therein), which may potentially improve the efficiency of corresponding numerical methods.

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Remark. Removing the harmonic part $H(x)$ from (1.8) and the circulation condition in (1.7) and (2.4), the above main result can be readily adapted to describe an ideal fluid inside a bounded domain. It is also possible to consider obstacles with several connected components by prescribing a circulation condition for each obstacle. More precisely, assuming that the whole obstacle is such that its boundary can be decomposed into a disjoint union

$$\partial\Omega = \partial\Omega_1 \cup \cdots \cup \partial\Omega_n,$$

where each $\partial\Omega_i$ is a simple closed curve enclosing a connected component of the obstacle, the elliptic problem (1.7) has then to be generalized to

$$\begin{cases} \operatorname{div} u = 0 & \text{in} & \Omega, \\ \operatorname{curl} u = \omega & \text{in} & \Omega, \\ u \cdot n = 0 & \text{on} & \partial\Omega, \\ u \to 0 & \text{as} & x \to \infty, \\ \oint_{\partial\Omega_i} u \cdot \tau ds = \gamma_i & \text{for each} & i = 1, \dots, n, \end{cases}$$

where $\gamma_i \in \mathbb{R}$ is the prescribed circulation around each connected component. The corresponding systems (2.4) and (2.8) undergo similar modifications. Theorem 2.1 can then be adapted to account for multiple circulations γ_i.

Finally, one can also adapt the methods in this work to handle flows with nonzero velocities at infinity. Note, however, that the consideration of nonsmooth domains (with corners and cusps, for instance) would require subtle and nontrivial adaptations (particularly altering the analysis conducted in section 3, below) which we leave for other works.

Numerically, we indeed verify that the system (2.8) is always invertible for large N and that, on any compact set K, the flow u_{app}^N (given in (2.7)) converges in the L^∞-norm faster than any $N^{-\kappa}$ with $\kappa > 0$ for a regular distribution of points. Deviation from a regular grid yields a slower convergence. The rate obtained in Theorem 2.1 may, however, not be optimal in the case of random perturbations (of size $N^{-\kappa-1}$) of the uniform grid, because numerical simulations indicate faster convergence than $N^{-\kappa}$.

2.2. Dynamic convergence of the vortex approximation. We have previously explained how the influence of an obstacle on a flow solving (1.7) can be modeled by a collection of vortices on its boundary. We explain now how this approximation procedure is used to replace the obstacle in the Euler equations (1.1) by an evolving collection of vortices, thereby providing a dynamic picture of the vortex method.

To this end, let $\omega_0 \in C_c^1(\Omega)$ and consider the unique classical solution $\omega \in C_c^1\left([0,t_1] \times \overline{\Omega}\right)$ (note that all natural definitions of C^1 on closed sets are equivalent here, for $\partial\Omega$ is smooth; see, e.g., [34]) constructed in [23], for some fixed but arbitrary time $t_1 > 0$, of

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \omega(t=0) = \omega_0 \end{cases} \tag{2.11}$$

with a velocity flow

$$u = K_\Omega[\omega](x) + \alpha H(x) \in C^1\left([0,t_1] \times \overline{\Omega}\right),$$

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where K_Ω, α, and H are given explicitely in (1.8), for some prescribed circulation $\gamma \in \mathbb{R}$. It is to be emphasized that the main theorem from [23] concerns the Eulerian formulation (1.1). The corresponding proofs, however, are based on the vorticity formulation and indeed establish the above-mentioned wellposedness of (2.11).

We recall that a classical estimate, which we reproduce, later on, in section 7.1, shows that the support of ω in x remains uniformly bounded away from the boundary $\partial\Omega$ (see (7.1)).

Let us also focus on the following vortex approximation of (2.11), for sufficiently large integers N (at least as large as N_0 determined by Theorem 2.1 so that (2.8) is invertible):

$$\begin{cases} \partial_t \omega^N + u^N \cdot \nabla \omega^N = 0, \\ \omega^N(t=0) = \omega_0 \end{cases} \tag{2.12}$$

for the same initial data $\omega_0 \in C_c^1(\Omega)$ extended by zero outside Ω and with a velocity flow

$$u^N = K_{\mathbb{R}^2}[\omega^N] + u_{\rm app}^N[\omega^N, \gamma], \tag{2.13}$$

where $u_{\rm app}^N[\omega^N, \gamma]$ is given by (2.7)–(2.8) for some prescribed $\gamma \in \mathbb{R}$ and where u_P in the right-hand side of (2.8) is now $K_{\mathbb{R}^2}[\omega^N]$.

The difficulty in solving system (2.12) resides in the fact that the velocity flow u^N is singular at the mesh points x_i^N on the boundary $\partial\Omega$. However, we are able to claim the wellposedness of (2.12) in C^1 at least on some finite time interval, which can be arbitrarily large provided N is sufficiently large. More precisely we show the following theorem, whose proof relies crucially on Theorem 2.1 and is deferred to section 7, for clarity.

THEOREM 2.2. *Let $\omega_0 \in C_c^1(\Omega)$, $\gamma \in \mathbb{R}$ and consider any fixed time $t_1 > 0$. Then, for a well distributed mesh on $\partial\Omega$, there exists $N_1 \geqslant N_0$ (N_0 is determined in Theorem 2.1) such that, for any $N \geqslant N_1$, there is a unique classical solution $\omega^N \in C_c^1([0,t_1] \times \Omega)$ to (2.12). Moreover, the sequence of solutions $\{\omega^N\}_{N \geqslant N_1}$ is uniformly bounded in $C_c^1([0,t_1] \times \Omega)$.*

Remark. It is to be emphasized that, for a given fixed N, it may not be possible to prolong the classical solution from the preceding theorem indefinitely due to the interaction of the vortices defining the singular flow $u_{\rm app}^N$ with the vorticity ω^N. This justifies the introduction of N_1 possibly depending on t_1 and ω_0.

Remark. In section 8, we give in Theorem 8.7 a similar wellposedness result for the fluid charge method.

The following main theorem establishes the convergence of system (2.12) toward system (2.11) as $N \to \infty$, thereby completing the mathematical validation of the vortex method for the boundary of an obstacle in the Euler equations (1.1). We emphasize, again, that the practical usefulness of this method lies in the computation of γ^N through (2.8) allowing the construction of an approximate flow $K_{\mathbb{R}^2}[\omega^N] + u_{\rm app}^N[\omega^N, \gamma]$ which only requires the use of the Biot–Savart kernel in the whole plane and does not resort to (1.8).

THEOREM 2.3. *Let $\omega_0 \in C_c^1(\Omega)$, $\gamma \in \mathbb{R}$ and consider any fixed time $t_1 > 0$. Then, for a well distributed mesh on $\partial\Omega$, as $N \to \infty$, the unique classical solution $\omega^N \in C_c^1([0,t_1] \times \Omega)$ to (2.12) converges uniformly toward the unique classical solution $\omega \in$*

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$C_c^1([0,t_1]\times\Omega)$ *to* (2.11). *More precisely, it holds that*

$$\left\|\omega-\omega^N\right\|_{L^\infty([0,t_1]\times\Omega)}=\mathcal{O}\left(N^{-\kappa}\right).$$

The proof of the above theorem is given in section 7. It relies on both Theorems 2.1 and 2.2.

Remark. In section 8, we provide in Theorem 8.8 a similar convergence result for the fluid charge method.

Remark. In the above theorem, our original motivation was to reconstruct classical solutions from the vortex approximation. Observe, however, that the convergence estimate given in Theorem 2.1 only requires that the vorticity ω be in L_c^1, whereas Theorems 2.2 and 2.3 operate at a much higher C^1-regularity for the vorticity.

In fact, in view of the weak assumptions of Theorem 2.1, we believe that, using the vortex approximation studied in this work, it is also possible to recover solutions of the incompressible Euler equations in the much weaker setting of Yudovich's solutions, where the vorticity ω merely belongs to L_c^∞ uniformly in time (see [2, section 7.2] for Yudovich's theorem, as well as other global existence results in the whole plane).

Indeed, such a convergence result would hinge on the fact that all vorticities produced by the vortex approximation have compact supports that remain uniformly bounded away from the boundary of the domain, in order to ensure that all solutions exist on a uniform interval of time. This fact is a consequence of our analysis from section 7.1 and, in particular, from the proof of Proposition 7.5, where the distance from the support of the vorticity to the boundary is controlled by the L^∞-norm and the support of the initial vorticity (and not of its gradient). It seems that the remainder of a convergence proof would follow from standard weak compactness arguments.

For the sake of brevity, however, we will not be going into further detail on this interesting topic.

Remark. Finally, notice that it is also possible to combine the vortex method for the boundary of an exterior domain with the aforementioned vortex method in the whole plane (consisting in an approximation of the vorticity ω by a collection of point vortices) in order to obtain a full and dynamic vortex method for an exterior domain. To this end, we consider an approximation of the initial vorticity ω_0 by a combination of point vortices $\sum_{k=1}^M \alpha_k\delta_{y_k(0)}$. Then, the position $y_k(t)$ of each point vortex evolves under the influence of the vector field created by the remaining vortices $\sum_{p\neq k}\alpha_p\delta_{y_p(t)}$ (with possible regularization of the kernel for the vortex blob method) and the fixed vortices on the boundary $\sum_{i=1}^N \frac{\gamma_i^N(t)}{N}\delta_{x_i^N}$, where the variable vortex density $\gamma_i^N(t)$ is computed through (2.8) with u_P replaced by $K_{\mathbb{R}^2}[\sum_{k=1}^M \alpha_k\delta_{y_k(t)}]$. A rigorous proof of convergence for this full vortex method remains challenging, though.

3. Representation formulas. In this section, we present some representation formulas for the solution u_R of (2.4), which are crucial for the justification of Theorem 2.1 and whose understanding sheds light on the approximation of u_R by point vortices on the boundary $\partial\Omega$.

Here, we are considering some given vorticity $\omega\in C_c^{0,\alpha}(\Omega)$ with $0<\alpha\leqslant 1$ and $\gamma\in\mathbb{R}$ and wish to construct a velocity field $u_R\in C^0(\overline{\Omega})\cap C^1(\Omega)$ solving (2.4). Essentially, we show below that it is possible to express the solution to (2.4) as a vortex sheet on the boundary $\partial\Omega$, which, again, is consistent with the physical idea that the flow around an obstacle is produced by a boundary layer of vortices.

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To this end, we need to introduce the integral operators

$$
\begin{aligned}
A\varphi(x) &= \int_{\partial\Omega} \frac{x-y}{|x-y|^2} \cdot n(x)\varphi(y)dy, \quad x \in \partial\Omega, \\
B\varphi(x) &= \int_{\partial\Omega} \frac{x-y}{|x-y|^2} \cdot \tau(x)\varphi(y)dy, \quad x \in \partial\Omega,
\end{aligned}
\tag{3.1}
$$

and their adjoints

$$
\begin{aligned}
A^*\varphi(x) &= -\int_{\partial\Omega} \frac{x-y}{|x-y|^2} \cdot n(y)\varphi(y)dy, \qquad x \in \mathbb{R}^2, \\
B^*\varphi(x) &= -\int_{\partial\Omega} \frac{x-y}{|x-y|^2} \cdot \tau(y)\varphi(y)dy, \qquad x \in \mathbb{R}^2.
\end{aligned}
$$

These operators are closely related to Cauchy integrals, i.e., complex valued integrals of the type $\int_\Gamma \frac{\varphi(z_1)}{z-z_1}dz_1$, where $\Gamma \subset \mathbb{C}$ is a contour. Indeed, identifying the contour $\Gamma \subset \mathbb{C}$ with the boundary of our domain $\partial\Omega \subset \mathbb{R}^2$, note that

$$
\int_\Gamma \frac{\varphi(y_1,y_2)}{(x_1+ix_2)-(y_1+iy_2)} d(y_1+iy_2) = -\left(B^* + iA^*\right)\varphi(x_1,x_2). \tag{3.2}
$$

Up to a multiplicative constant, the function $A^*\varphi$ is the so-called *double layer potential* associated with the density φ. Such operators have been extensively studied in the context of Dirichlet and Neumann problems for Laplace's equation (see the classical references [9, Chapter IV] and [22], for instance). For the sake of clarity and completeness, we will nevertheless provide below complete justifications of our methods. We also refer to [10] for clear proofs of some functional properties, which are explored in this work, of such operators on smooth domains in any dimension. Finally, we emphasize that the smoothness of the domain, which we assume to hold throughout this paper, is not a mere technical simplification and that the loss of regularity of $\partial\Omega$ brings on subtle and difficult questions about the above operators. We avoid this interesting and important discussion altogether, though, and leave it for subsequent works. We refer to [4, 7, 11, 12, 33] concerning the theory of double layer potentials on Lipschitz domains.

We begin this section by gathering and discussing several properties of the above operators, which will then allow us to establish important representation formulas for u_R.

3.1. Boundedness and adjointness. Expressing $y-x = \tau(x)\left((y-x)\cdot\tau(x)\right) + \mathcal{O}\left(|y-x|^2\right)$, the integrals defining $A\varphi(x)$ and $A^*\varphi(x)$ are always well defined for any $x \in \partial\Omega$ and $\varphi \in C(\partial\Omega)$, and the operators A and A^* are bounded over $L^p(\partial\Omega)$ for all $1 \leqslant p \leqslant \infty$. In particular, for all $\varphi, \psi \in C(\partial\Omega)$, it holds that

$$
\int_{\partial\Omega} \psi(x) A\varphi(x)dx = \int_{\partial\Omega} A^*\psi(y)\varphi(y)dy. \tag{3.3}
$$

In fact, since Ω is smooth, a slightly more refined analysis (employing an explicit Taylor expansion of a smooth parametrization of $\partial\Omega$, for instance) shows that the integral kernels of A and A^* are smooth so that these operators are regularizing. More precisely, one can show that $A\varphi$ and $A^*\varphi$ belong to $C^\infty(\partial\Omega)$ for any $\varphi \in L^1(\partial\Omega)$.

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On the other hand, for any $x \in \partial\Omega$, $B\varphi(x)$ and $B^*\varphi(x)$ only make sense in the sense of Cauchy's principal value:

$$\begin{aligned} B\varphi(x) &= \lim_{\varepsilon\to 0} \int_{\partial\Omega\setminus B(x,\varepsilon)} \frac{x-y}{|x-y|^2} \cdot \tau(x)\varphi(y)dy, \\ B^*\varphi(x) &= -\lim_{\varepsilon\to 0} \int_{\partial\Omega\setminus B(x,\varepsilon)} \frac{x-y}{|x-y|^2} \cdot \tau(y)\varphi(y)dy. \end{aligned} \tag{3.4}$$

Indeed, notice that $\frac{x-y}{|x-y|^2} \cdot \tau(y) = 0$ whenever $y \in \partial B(x,\varepsilon)$ and $x \in \partial\Omega$. It is therefore always possible to replace the integration domain in the above limit defining B^* by $(\partial\Omega \setminus B(x,\varepsilon)) \cup (\partial B(x,\varepsilon) \cap \Omega^c)$, thereby avoiding the singularity at x of the kernel, whence

$$\begin{aligned} B^*1(x) &= -\lim_{\varepsilon\to 0} \int_{\partial\Omega\setminus B(x,\varepsilon)} \frac{x-y}{|x-y|^2} \cdot \tau(y)dy \\ &= -\lim_{\varepsilon\to 0} \int_{\Omega^c\setminus B(x,\varepsilon)} \operatorname{curl} \frac{x-y}{|x-y|^2} dy = 0, \end{aligned} \tag{3.5}$$

by the divergence theorem (which holds for piecewise smooth domains). It follows that B and B^* are given by the formulas, for all $x \in \partial\Omega$,

$$\begin{aligned} B\varphi(x) &= \int_{\partial\Omega} \frac{x-y}{|x-y|^2} \cdot (\tau(x) - \tau(y))\, \varphi(y)dy - B^*\varphi(x), \\ B^*\varphi(x) &= \int_{\partial\Omega} \frac{x-y}{|x-y|^2} \cdot \tau(y)\, (\varphi(x) - \varphi(y))\, dy, \end{aligned}$$

which are clearly well defined for every $\varphi \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha \leqslant 1$, and that the limits in (3.4) are uniform over $\partial\Omega$. In particular, it is now readily verified that, for all $\varphi, \psi \in C^{0,\alpha}(\partial\Omega)$,

$$\begin{aligned} \int_{\partial\Omega} \psi(x) B\varphi(x)dx &= \lim_{\varepsilon\to 0} \int_{\partial\Omega\times\partial\Omega\setminus\{|x-y|\geqslant\varepsilon\}} \frac{x-y}{|x-y|^2} \cdot \tau(x)\psi(x)\varphi(y)dxdy \\ &= \int_{\partial\Omega} B^*\psi(y)\varphi(y)dy. \end{aligned} \tag{3.6}$$

Employing $\frac{x-y}{|x-y|^2} \cdot n(y) = \frac{1}{\varepsilon}$ whenever $y \in \partial B(x,\varepsilon) \cap \Omega^c$, observe that a similar calculation yields

$$\begin{aligned} A^*1(x) &= -\lim_{\varepsilon\to 0} \int_{\partial\Omega\setminus B(x,\varepsilon)} \frac{x-y}{|x-y|^2} \cdot n(y)dy \\ &= \lim_{\varepsilon\to 0} \left(\frac{1}{\varepsilon} |\partial B(x,\varepsilon) \cap \Omega^c| - \int_{\Omega^c\setminus B(x,\varepsilon)} \operatorname{div} \frac{x-y}{|x-y|^2} dy \right) = \pi. \end{aligned} \tag{3.7}$$

By duality, notice that the identities (3.5) and (3.7) establish

$$\int_{\partial\Omega} B\varphi dx = 0 \quad \text{and} \quad \int_{\partial\Omega} A\varphi dx = \pi \int_{\partial\Omega} \varphi dx \tag{3.8}$$

for every $\varphi \in C^{0,\alpha}(\partial\Omega)$.

More generally, the operators B and B^* are bounded over $L^p(\partial\Omega)$ for any $1 < p < \infty$. Perhaps the easiest way to justify such boundedness properties is by considering

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the arc-length parametrization $l : [0, |\partial\Omega|] \to \mathbb{R}^2$ of $\partial\Omega$ (interpreted as a smooth periodic function over $[0, |\partial\Omega|]$) and expressing the kernel, whenever $|s-t| < \frac{2|\partial\Omega|}{3}$, as

$$\frac{l(s)-l(t)}{|l(s)-l(t)|^2} = \frac{\tau(l(t))}{s-t} + r(s,t), \tag{3.9}$$

where, using that $\tau(l(t)) = l'(t)$ and $|l'(t)| = 1$, the function $r(s,t)$ can be written as

$$\begin{aligned} r(s,t) &= \frac{l(s)-l(t)}{|l(s)-l(t)|^2} - \frac{l'(t)}{s-t} \\ &= \left(l'(t) + \frac{l(s)-l(t)}{s-t}\right) \cdot \left(\frac{l(t)+l'(t)(s-t)-l(s)}{(s-t)^2}\right) \frac{\frac{l(s)-l(t)}{s-t}}{\left|\frac{l(s)-l(t)}{s-t}\right|^2} \\ &\quad + \frac{l(s)-l(t)-l'(t)(s-t)}{(s-t)^2}. \end{aligned}$$

By smoothness of the parametrization l, it is then clear that $r(s,t)$ is smooth over $\{|s-t| < \frac{2|\partial\Omega|}{3}\}$ and that (recall that $l'(t) \cdot l''(t) = 0$ because l is an arc-length parametrization)

$$\lim_{s\to t} r(s,t) = \frac{l''(t)}{2}.$$

The periodicity of l and the boundedness of the Hilbert transform over $L^p(\mathbb{R})$ for any $1 < p < \infty$ (see [18, section 4.1]) then yields corresponding bounds on B and B^*. Similarly, the behavior of the Hilbert transform over Hölder spaces allows us to deduce that $B\varphi, B^*\varphi \in C^{0,\alpha}(\partial\Omega)$ as soon as $\varphi \in C^{0,\alpha}$ with $0 < \alpha < 1$ and that $B\varphi, B^*\varphi \in C^{0,1-\varepsilon}(\partial\Omega)$ for all $0 < \varepsilon < 1$, as soon as $\varphi \in C^{0,1}$. This result is known as the Plemelj–Privalov theorem (see [29, Chapter 2, sections 19 and 20]).

3.2. The Plemelj formulas and the Poincaré–Bertrand formula. The theory of double layer potentials (or of Cauchy integrals; see [10, 29]) instructs us that, for a smooth boundary $\partial\Omega$ and for any $\varphi \in C^{0,\alpha}(\partial\Omega)$, with $0 < \alpha \leqslant 1$, the functions $A^*\varphi$ and $B^*\varphi$ are continuous up to the boundary $\partial\Omega$ (see [29, Chapter 2, section 16]) and that the limiting values of $A^*\varphi$ and $B^*\varphi$ on $\partial\Omega$ are given by the Plemelj formulas (see [29, Chapter 2, section 17]):

$$\begin{aligned} &\lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\Omega\cup\overline{\Omega}^c}} B^*\varphi(x) = B^*\varphi(x_0), \\ &\lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\Omega}} A^*\varphi(x) = A^*\varphi(x_0) - \pi\varphi(x_0), \\ &\lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\overline{\Omega}^c}} A^*\varphi(x) = A^*\varphi(x_0) + \pi\varphi(x_0). \end{aligned} \tag{3.10}$$

These limiting formulas can be used to show the celebrated Poincaré–Bertrand formula (see [29, Chapter 3, section 23]), which we now recall in its simpler version concerning the inversion of Cauchy integrals (see [29, Chapter 3, section 27]): for any smooth contour $\Gamma \subset \mathbb{C}$ and any $\varphi \in C^{0,\alpha}(\Gamma)$ with $0 < \alpha \leqslant 1$, one has that

$$\int_\Gamma \frac{1}{z-z_1} \int_\Gamma \frac{\varphi(z_2)}{z_1-z_2} dz_2 dz_1 = -\pi^2\varphi(z) \quad \text{for all } z \in \Gamma.$$

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Translating this identity into real variables, utilizing (3.2), yields

$$(B^* + iA^*)^2 \varphi(x) = -\pi^2 \varphi(x).$$

Therefore, we deduce that

$$\begin{aligned} \left(A^{*2} - B^{*2}\right) \varphi &= \pi^2 \varphi, \\ \left(A^* B^* + B^* A^*\right) \varphi &= 0. \end{aligned} \tag{3.11}$$

Equivalently, in view of (3.3) and (3.6), we note that the adjoint operators satisfy

$$\begin{aligned} \left(A^{2} - B^{2}\right) \varphi &= \pi^2 \varphi, \\ \left(AB + BA\right) \varphi &= 0. \end{aligned} \tag{3.12}$$

In the case of the disk $\mathcal{C} = \overline{B(0,1)}$, these identities correspond exactly to the inversion of the circular Hilbert transform.

3.3. Boundary vortex sheets. We focus now on velocity flows given as boundary vortex sheets:

$$\begin{aligned} v(x) = K_{\mathbb{R}^2}\left[g\delta_{\partial\Omega}\right] &= \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} g(y) dy \\ &= \frac{1}{2\pi} \left(B^*[ng] - A^*[\tau g]\right)(x) \in C^\infty\left(\mathbb{R}^2 \setminus \partial\Omega\right) \end{aligned} \tag{3.13}$$

for some suitable $g \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha \leqslant 1$. We show, later on, that any flow u_R solving (2.4) can be written as a boundary vortex sheet (3.13).

Note that the formula (3.13) merely defines the flow v away from the boundary $\partial\Omega$. However, using the Plemelj formulas (3.10), one can extend this flow by continuity to $\partial\Omega$ in two different ways, yielding either $v \in C\left(\overline{\Omega}\right)$ or $v \in C\left(\Omega^c\right)$. More precisely, it is understood that the flow $v \in C\left(\overline{\Omega}\right)$ has the limit boundary values

$$\lim_{\substack{x \to x_0 \in \partial\Omega \\ x \in \Omega}} v(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x_0-y)^\perp}{|x_0-y|^2} g(y) dy + \frac{1}{2}\tau(x_0) g(x_0), \tag{3.14}$$

whereas the flow $v \in C\left(\Omega^c\right)$ has the limit boundary values

$$\lim_{\substack{x \to x_0 \in \partial\Omega \\ x \in \overline{\Omega}^c}} v(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x_0-y)^\perp}{|x_0-y|^2} g(y) dy - \frac{1}{2}\tau(x_0) g(x_0), \tag{3.15}$$

where, again, the integrals in the right-hand sides above are defined in the sense of Cauchy's principal value. In other words, the normal component of $v(x)$ is continuous across the boundary $\partial\Omega$, where it takes the value

$$v \cdot n(x) = -\frac{1}{2\pi} Bg(x) \quad \text{for all } x \in \partial\Omega,$$

whereas its tangential component has a jump of size $g(x_0)$ at $x_0 \in \partial\Omega$ and takes the values on the boundary

$$\begin{aligned} v \cdot \tau(x) &= \frac{1}{2\pi} Ag(x) + \frac{1}{2} g(x) \quad \text{for all } x \in \partial\Omega \text{ from within } \Omega, \\ v \cdot \tau(x) &= \frac{1}{2\pi} Ag(x) - \frac{1}{2} g(x) \quad \text{for all } x \in \partial\Omega \text{ from within } \overline{\Omega}^c. \end{aligned} \tag{3.16}$$

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Further employing (3.8), it follows that the flow $v(x)$ solves uniquely the systems

$$\begin{cases} \operatorname{div} v = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} v = 0 & \text{in} \quad \Omega, \\ v \cdot n = -\frac{1}{2\pi} Bg(x) & \text{on} \quad \partial\Omega, \\ v \to 0 & \text{as} \quad x \to \infty, \\ \oint_{\partial\Omega} v \cdot \tau ds = \oint_{\partial\Omega} g ds \end{cases} \tag{3.17}$$

and

$$\begin{cases} \operatorname{div} v = 0 & \text{in} \quad \overline{\Omega}^c, \\ \operatorname{curl} v = 0 & \text{in} \quad \overline{\Omega}^c, \\ v \cdot n = -\frac{1}{2\pi} Bg(x) & \text{on} \quad \partial\Omega. \end{cases} \tag{3.18}$$

3.4. Invertibility of B and $A^2 - \pi^2$. The operators B and $A^2 - \pi^2$ over $L^2(\partial\Omega)$ are not invertible, for their image is a proper subset of $L^2(\partial\Omega)$, by (3.8). However, as we are about to show, they are invertible when their action is restricted to functions of zero mean value.

More precisely, introducing the space

$$L_0^2(\partial\Omega) = \left\{ h \in L^2(\partial\Omega) \, : \, \oint_{\partial\Omega} h \, ds = 0 \right\},$$

we are now looking for an inverse of the bounded operator $B : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega)$. We also consider the bounded operator $A : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega)$, which is well defined by (3.8).

Notice that A is a Hilbert–Schmidt operator, for the kernel of A is smooth. It therefore follows that A^2 is compact and that the Fredholm alternative (see [30, Theorem VI.14]) applies to $A^2 - \pi^2$. More precisely, either $(A^2 - \pi^2)^{-1} : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega)$ exists or $A^2\varphi = \pi^2\varphi$ (i.e., $B^2\varphi = 0$ by (3.12)) has a nontrivial solution in $L_0^2(\partial\Omega)$. It turns out that the latter alternative never holds.

Indeed, suppose that there is some $g \in L_0^2(\partial\Omega)$ such that $Bg = 0$. By (3.12), it holds that $\pi^2 g = A^2 g$ so that g is smooth, for A is a regularizing operator. Then, plugging g into (3.13) yields a velocity field $v(x)$ solving the system

$$\begin{cases} \operatorname{div} v = 0 & \text{in} \quad \Omega \cup \overline{\Omega}^c, \\ \operatorname{curl} v = 0 & \text{in} \quad \Omega \cup \overline{\Omega}^c, \\ v \cdot n = 0 & \text{on} \quad \partial\Omega, \\ v \to 0 & \text{as} \quad x \to \infty, \\ \oint_{\partial\Omega} v \cdot \tau ds = 0. \end{cases}$$

By uniqueness, we find that $v \equiv 0$ on $\Omega \cup \overline{\Omega}^c$, whence $g = 0$ by (3.14) and (3.15).

In virtue of the Fredholm alternative, this establishes that $A^2 - \pi^2 : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega)$ always has an inverse. Using (3.12), it is now possible to produce an inverse for $B : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega)$, too. Indeed, noticing that $(A^2 - \pi^2)^{-1}$ commutes with B because A^2 commutes with B by virtue of (3.12), one verifies that

$$B^{-1} = \left(A^2 - \pi^2\right)^{-1} B : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega). \tag{3.19}$$

Observe, finally, that an inverse for $A - \pi : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega)$ is readily given by

$$(A - \pi)^{-1} = \left(A^2 - \pi^2\right)^{-1} (A + \pi) : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega).$$

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3.5. Representation of u_R as a boundary vortex sheet. We show now that u_R can be expressed as a boundary vortex sheet (3.13).

Since the flow $v(x)$ defined by (3.13) is the unique solution to (3.17), we conclude that $v(x)$ coincides with the unique solution $u_R(x) \in C^0\left(\overline{\Omega}\right) \cap C^1(\Omega)$ of (2.4) if and only if $g \in C^{0,\alpha}(\partial\Omega)$ satisfies

$$-\frac{1}{2\pi} Bg(x) = u_R \cdot n(x) = -u_P \cdot n(x) \quad \text{for every } x \in \partial\Omega \tag{3.20}$$

and

$$\int_{\partial\Omega} g(x)dx = \gamma. \tag{3.21}$$

By (3.8), all functions in the image of B have zero mean over $\partial\Omega$. Note, in particular, that the right-hand side of (3.20) has indeed zero mean because u_P is solenoidal in $\mathbb{R}^2$. Moreover, by linearity, considering $g - \frac{\gamma}{|\partial\Omega|}$ instead of g, inverting (3.20)–(3.21) easily reduces to finding an inverse for B over functions with zero mean value, which we have already shown to exist in (3.19).

The system (3.20)–(3.21) is then solved by

$$g = B^{-1}\left[2\pi u_P \cdot n - B\frac{\gamma}{|\partial\Omega|}\right] + \frac{\gamma}{|\partial\Omega|} \in L^2(\partial\Omega). \tag{3.22}$$

It is to be emphasized that $B^{-1}B1 \neq 1$ for $B^{-1}B1$ has mean zero.

Remark. When the obstacle is the unit disk $\mathcal{C} = \overline{B(0,1)}$, the system (3.20)–(3.21) is related to the inversion of the circular Hilbert transform. This restricted geometry leads to more explicit representation formulas for g, because B becomes a Hilbert transform and A is an averaging operator (so that Bg and Ag are fully determined by (3.20) and (3.21), respectively). We refer to [1] for full details on this setting. In general, condition (3.21) is not easily expressed in terms of A and B unless $\partial\Omega$ is a circle.

Observe that the above formula a priori only places the density g in the space $L^2(\partial\Omega)$. Nonetheless, by (3.20) and (3.12), it is readily seen that

$$\pi^2 g = A^2 g - 2\pi B[u_P \cdot n],$$

which implies, by the aforementioned regularity properties of the operators A and B, since $g \in L^1(\partial\Omega)$ and $u_P \cdot n \in C^\infty(\partial\Omega)$, that $g \in C^{0,\alpha}(\partial\Omega)$ for all $0 < \alpha \leqslant 1$ (in fact, g is even smoother than this).

On the whole, we have shown, for any given $0 < \alpha \leqslant 1$, that there exists a unique $g \in C^{0,\alpha}(\partial\Omega)$ (given by (3.22)) such that u_R is expressed as a boundary vortex sheet (3.13). Thus, combining (3.13) with (3.22), we find that

$$\begin{aligned} u_R(x) = &\int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} B^{-1}\left[u_P \cdot n\right](y)dy \\ &+ \frac{\gamma}{2\pi|\partial\Omega|}\int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2}\left(1 - B^{-1}B1\right)(y)dy. \end{aligned}$$

Considering this representation formula for the unique harmonic vector field $H(x)$ in Ω defined by (1.6), i.e., setting $u_P \cdot n = 0$ and $\gamma = 1$ above, we further obtain that

$$H(x) = \frac{1}{2\pi|\partial\Omega|}\int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2}\left(1 - B^{-1}B1\right)(y)dy \quad \text{in } \Omega. \tag{3.23}$$

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It follows that

$$
\begin{aligned}
u_R(x) &= \int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} B^{-1}\left[u_P\cdot n\right](y)dy + \gamma H(x) \\
&= \int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} \left(A^2-\pi^2\right)^{-1} B\left[u_P\cdot n\right](y)dy + \gamma H(x).
\end{aligned} \tag{3.24}
$$

(We strongly advise the reader to compare the above representation formula for u_R on the exterior of any smooth obstacle with the corresponding much simpler representation formula (2.10) in [1] for the exterior of the unit disk. The operator H therein represents the circular Hilbert transform whereas $\frac{x^\perp}{2\pi|x|^2}$ is precisely the harmonic vector field for the unit disk.)

The existence of the density $g \in C^{0,\alpha}(\partial\Omega)$ satisfying conditions (3.20) and (3.21) for any suitable given data is nontrivial and at the heart of the present work, for (2.7) is essentially a discretization of (3.13). The abstract construction of the inverse of B in section 3.4 through the Fredlhom alternative is not suitable for a discretization procedure, though. In order to use the invertibility (3.20)–(3.21) to solve system (2.8), we need now to refine our understanding of the operators A and B and their respective spectra.

3.6. Kernels of B and $A-\pi$. Further observe, by (3.18), that the right-hand side of (3.23) also defines the unique solution (which is trivially zero) to

$$
\begin{cases}
\operatorname{div} v = 0 & \text{in} \quad \overline{\Omega}^c, \\
\operatorname{curl} v = 0 & \text{in} \quad \overline{\Omega}^c, \\
v\cdot n = 0 & \text{on} \quad \partial\Omega,
\end{cases}
$$

whereby, by (3.16), we find the relations

$$
\begin{aligned}
|\partial\Omega|\, H\cdot\tau(x) &= \frac{1}{2\pi} A\left[1-B^{-1}B1\right](x) + \frac{1}{2}\left(1-B^{-1}B1\right)(x), \\
0 &= \frac{1}{2\pi} A\left[1-B^{-1}B1\right](x) - \frac{1}{2}\left(1-B^{-1}B1\right)(x)
\end{aligned}
$$

for all $x \in \partial\Omega$ (we emphasize here that the values of H on $\partial\Omega$ are given by its limiting values from Ω), which are equivalent to

$$
\begin{aligned}
\left(1-B^{-1}B1\right) &= |\partial\Omega|\, H\cdot\tau, \\
A[H\cdot\tau] &= \pi H\cdot\tau,
\end{aligned} \tag{3.25}
$$

on $\partial\Omega$.

We conclude that $H\cdot\tau$ lies in the kernels of B and $A-\pi$ and that one has the representations (using (3.23) and then (3.14), again)

$$
\begin{aligned}
H(x) &= \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} H\cdot\tau(y)dy & \text{in } \Omega, \\
H(x) &= \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} H\cdot\tau(y)dy + \frac{1}{2}\tau(x)H\cdot\tau(x) & \text{on } \partial\Omega.
\end{aligned} \tag{3.26}
$$

Finally, since any $g \in L^2(\partial\Omega)$ satisfies $g-\gamma H\cdot\tau \in L^2_0(\partial\Omega)$ for some appropriate $\gamma\in\mathbb{R}$ and both operators $A-\pi$ and B are invertible over $L^2_0(\partial\Omega)$, we deduce that the kernels of $A-\pi : L^2(\partial\Omega) \to L^2_0(\partial\Omega)$ and $B : L^2(\partial\Omega) \to L^2_0(\partial\Omega)$ coincide exactly with the span of $H\cdot\tau$.

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3.7. Spectrum of A. It is now possible to deduce some simple spectral properties for $A : L^2(\partial\Omega) \to L^2(\partial\Omega)$ from the preceding developments. First, since A is compact, by the classical Riesz–Schauder theorem (see [30, Theorem VI.15] or [35, Chapter X, section 5]), we know that its spectrum $\sigma(A)$ is at most countable with no limit points except, possibly, at zero. Moreover, $\sigma(A)\setminus\{0\}$ is composed solely of eigenvalues with finite multiplicity (i.e., corresponding eigenspaces are finite dimensional).

We also consider the spectrum of $A : L^2_0(\partial\Omega) \to L^2_0(\partial\Omega)$, which we distinguish from $\sigma(A)$ by denoting it by $\sigma_0(A)$. Clearly, if $\lambda \in \mathbb{C}\setminus\sigma(A)$, then $\lambda \neq \pi$ (for $\pi - A$ has a nontrivial kernel; see (3.25)), the operator $\lambda - A$ is invertible and, by (3.8), its inverse leaves $L^2_0(\partial\Omega)$ invariant, whereby $\lambda \in \mathbb{C}\setminus(\sigma_0(A)\cup\{\pi\})$. It follows that $\sigma_0(A)\cup\{\pi\} \subset \sigma(A)$. On the other hand, if $\lambda \in \mathbb{C}\setminus(\sigma_0(A)\cup\{\pi\})$, then $\lambda - A$ has a bounded inverse over functions with mean zero. It is then possible to extend this inverse to functions with nonzero average with the definition

$$(\lambda - A)^{-1}\varphi = (\lambda - A)^{-1}(\varphi - \gamma H\cdot\tau) + \frac{\gamma}{\lambda - \pi} H\cdot\tau, \quad \text{where } \gamma = \oint_{\partial\Omega} \varphi dx,$$

and one verifies that this produces a well defined inverse which is bounded over $L^2(\partial\Omega)$, whereby $\lambda \in \mathbb{C}\setminus\sigma(A)$ and therefore $\sigma(A) \subset \sigma_0(A)\cup\{\pi\}$. On the whole, we conclude that $\sigma(A) = \sigma_0(A)\cup\{\pi\}$.

We have already identified, in section 3.6, the span of $H\cdot\tau$ as the eigenspace corresponding to the eigenvalue $\pi \in \sigma(A)$. In particular, since $H\cdot\tau$ does not have mean zero over $\partial\Omega$, we see that $\pi \notin \sigma_0(A)$.

Suppose now that $\varphi \in L^2(\partial\Omega)$ satisfies $A\varphi = -\pi\varphi$. Then, by (3.12), it holds that $B^2\varphi = 0$, whence $B\varphi$ both belongs to the kernel of B and has mean zero by (3.8), which implies that $B\varphi = 0$ (recall that the mean of $H\cdot\tau$ is nonzero). We conclude that φ also belongs to the kernel of B and that it is therefore a constant multiple of $H\cdot\tau$. Since we have already shown that $A[H\cdot\tau] = \pi H\cdot\tau$, this establishes that $-\pi$ is not an eigenvalue of A.

Finally, by (3.12), if $\varphi \in L^2_0(\partial\Omega)$ is an eigenvector of A for some eigenvalue $\lambda \in \mathbb{C}\setminus\{\pm\pi\}$, we find that $B\varphi \in L^2_0(\partial\Omega)$ is an eigenvector of A for the eigenvalue $-\lambda$, whence $\sigma_0(A) = -\sigma_0(A)$.

In fact, it is well-known that the spectral radius of A is no larger than π (see [22, Chapter XI, section 11]). For the convenience of the reader, though, we provide here a short argument showing that any eigenvalue $\lambda \in \mathbb{C}$ is actually real and has modulus bounded by π. To this end, consider an eigenvector $g \in L^2(\partial\Omega)$ of A corresponding to some eigenvalue $\lambda \in \mathbb{C}\setminus\{0,\pi\}$. (Note that, since A is regularizing, g is actually smooth and that, by (3.8), g has mean zero.) Then, considering the velocity field given by (3.13) and defining $h(x) = \frac{1}{2\pi}\int_{\partial\Omega}\log(|x-y|)g(y)dy$, we compute that, employing (3.14), (3.15) and that $v(x) = v(x) - \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\int_{\partial\Omega} g dx = \mathcal{O}(|x|^{-2})$ for densities with mean zero,

$$\begin{aligned}
&(\lambda-\pi)\int_\Omega |v(x)|^2 dx + (\lambda+\pi)\int_{\overline{\Omega}^c} |v(x)|^2 dx \\
&= (\lambda-\pi)\int_\Omega v(x)\cdot\nabla^\perp\overline{h(x)}dx + (\lambda+\pi)\int_{\overline{\Omega}^c} v(x)\cdot\nabla^\perp\overline{h(x)}dx \\
&= (\lambda-\pi)\int_\Omega \operatorname{curl}\left(v(x)\overline{h(x)}\right)dx + (\lambda+\pi)\int_{\overline{\Omega}^c}\operatorname{curl}\left(v(x)\overline{h(x)}\right)dx \\
&= \frac{\pi-\lambda}{2\pi}\int_{\partial\Omega}(Ag(x)+\pi g(x))\,\overline{h(x)}dx + \frac{\pi+\lambda}{2\pi}\int_{\partial\Omega}(Ag(x)-\pi g(x))\,\overline{h(x)}dx \\
&\quad + \lim_{R\to\infty}(\lambda-\pi)\int_{\partial B(0,R)} v(x)\cdot\tau(x)\overline{h(x)}dx = 0,
\end{aligned}$$

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where the tangent vector $\tau(x)$ on $\partial B(0,R)$ points in the counterclockwise direction. Thus, since $v \neq 0$ (otherwise $g = 0$ by (3.14)–(3.15)), we conclude that the origin $0 \in \mathbb{C}$ can be expressed as a convex combination of $\lambda - \pi$ and $\lambda + \pi$. Some elementary geometry implies then that $\lambda \in [-\pi, \pi] \subset \mathbb{C}$.

On the whole, we conclude that

$$\sigma_0(A) = -\sigma_0(A) \subset (-\pi,\pi) \quad \text{and} \quad \sigma(A) = \sigma_0(A) \cup \{\pi\} \subset (-\pi,\pi]\,.$$

In particular, by Gelfand's formula for the spectral radius (see [35, Chapter VIII, section 2], for instance), we obtain that

$$\begin{aligned} &\lim_{k\to\infty} \|A^k\|_{\mathcal{L}(L^2)}^{\frac{1}{k}} = \inf_{k\geqslant 1} \|A^k\|_{\mathcal{L}(L^2)}^{\frac{1}{k}} = \pi, \\ &\lim_{k\to\infty} \|A^k\|_{\mathcal{L}(L^2_0)}^{\frac{1}{k}} = \inf_{k\geqslant 1} \|A^k\|_{\mathcal{L}(L^2_0)}^{\frac{1}{k}} < \pi, \end{aligned} \tag{3.27}$$

and the inverse of $A^2 - \pi^2 : L^2_0(\partial\Omega) \to L^2_0(\partial\Omega)$ is therefore given by the Neumann series

$$\left(A^2 - \pi^2\right)^{-1} = -\pi^{-2} \sum_{n=0}^{\infty} \left(\frac{A}{\pi}\right)^{2n},$$

which is absolutely convergent in $\mathcal{L}(L^2_0)$ for, by (3.27), there is some $\varepsilon > 0$ such that $\|(\frac{A}{\pi})^k\|_{\mathcal{L}(L^2_0)} \leqslant (1-\varepsilon)^k$ for large k.

Contrary to the abstract method of contruction of inverses based on the Fredholm alternative from section 3.4, the present spectral approach allows us to deduce precise bounds on the inverses by quantifying the spectral gap of A at $\pm\pi$. The ensuing estimates are robust and well adapted for discretization procedures, which will be crucial in the remainder of this work.

3.8. Other representations of u_R. It turns out that there is yet another convenient representation formula for the flow u_R, which is a variant of the boundary vortex sheet (3.13).

More precisely, we claim now that in the exterior of a given obstacle, u_R can also be expressed as

$$\begin{aligned} w(x) &= \frac{1}{2\pi} \int_{\partial\Omega} \frac{x-y}{|x-y|^2} h(y)dy + \gamma H(x) \\ &= -\frac{1}{2\pi} \left(A^*[nh] + B^*[\tau h]\right)(x) + \gamma H(x) \in C^\infty\left(\mathbb{R}^2 \setminus \partial\Omega\right) \end{aligned} \tag{3.28}$$

for some suitable $h \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha \leqslant 1$. Recall that $H(x)$ is the harmonic vector field uniquely defined in Ω by (1.6), which we extend into $\overline{\Omega}^c$ by zero so that $H(x)$ is represented by (3.23) in $\Omega \cup \overline{\Omega}^c$.

As before, the theory of Cauchy integrals instructs us that, for a smooth boundary $\partial\Omega$ and for any $h \in C^{0,\alpha}(\partial\Omega)$, the flow w is continuous up to the boundary $\partial\Omega$, that is, $w \in C\left(\overline{\Omega}\right) \cup C\left(\Omega^c\right)$, and that the limiting values of w on $\partial\Omega$ are given by the Plemelj formulas (3.10). Hence, we deduce that

$$\lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\Omega}} w(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{x_0-y}{|x_0-y|^2} h(y)dy + \frac{1}{2} n(x_0)h(x_0) + \gamma H(x_0)$$

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and

$$\lim_{\substack{x\to x_0\in\partial\Omega\\ x\in\overline{\Omega}^c}} w(x)=\frac{1}{2\pi}\int_{\partial\Omega}\frac{x_0-y}{|x_0-y|^2}h(y)dy-\frac{1}{2}n(x_0)h(x_0).$$

Again, we emphasize that the values of H on $\partial\Omega$ are given here by its limiting values from Ω so that the representation formulas (3.26) are valid.

Therefore, we conclude that the flow $w(x)$ given by (3.28) defines the unique solution $u_R(x)\in C^0\left(\overline{\Omega}\right)\cap C^1\left(\Omega\right)$ of (2.4) if and only if $h\in C^{0,\alpha}\left(\partial\Omega\right)$ satisfies

$$\begin{aligned}\frac{1}{2\pi}\left(A+\pi\right)h(x)&=\frac{1}{2\pi}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}\cdot n(x)h(y)dy+\frac{1}{2}h(x)\\&=u_R\cdot n(x)=-u_P\cdot n(x)\quad\text{for every } x\in\partial\Omega.\end{aligned}\tag{3.29}$$

Provided (3.29) is verified and using (3.8), note that it necessarily holds that

$$\int_{\partial\Omega}h(x)dx=\frac{1}{2\pi}\int_{\partial\Omega}\left(A+\pi\right)h(x)dx=-\int_{\partial\Omega}u_P\cdot n(x)dx=0$$

and that the circulation condition

$$\begin{aligned}\int_{\partial\Omega}u_R\cdot\tau(x)dx&=\int_{\partial\Omega}\left(\frac{1}{2\pi}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}h(y)dy+\frac{1}{2}n(x)h(x)+\gamma H(x)\right)\cdot\tau(x)dx\\&=\frac{1}{2\pi}\int_{\partial\Omega}Bh(x)dx+\gamma=\gamma\end{aligned}$$

is automatically satisfied.

The existence of such a density $h\in C^{0,\alpha}\left(\partial\Omega\right)$ satisfying (3.29) for any suitable given data is nontrivial. (Again, we refer to [1] for a treatment of the simpler case of the unit disk.) However, in view of the above spectral analysis of the operator A, it is readily seen that $A+\pi: L^2_0\left(\partial\Omega\right)\to L^2_0\left(\partial\Omega\right)$ has an inverse given by the Neumann series

$$\left(A+\pi\right)^{-1}=\pi^{-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{A}{\pi}\right)^n,$$

which is absolutely convergent in $\mathcal{L}(L^2_0)$. In fact, observing that the spectrum of $A-\pi$ is contained in $(-2\pi,0]$, i.e., $\sigma\left(A-\pi\right)\subset(-2\pi,0]$, yields, by Gelfand's formula again, the precise estimate

$$\lim_{k\to\infty}\left\|\left(A-\pi\right)^k\right\|_{\mathcal{L}(L^2)}^{\frac{1}{k}}=\inf_{k\geqslant 1}\left\|\left(A-\pi\right)^k\right\|_{\mathcal{L}(L^2)}^{\frac{1}{k}}<2\pi,\tag{3.30}$$

which implies that $A+\pi: L^2\left(\partial\Omega\right)\to L^2\left(\partial\Omega\right)$ also has a bounded inverse given by the Neumann series

$$\left(A+\pi\right)^{-1}=\frac{1}{2\pi}\sum_{n=0}^{\infty}\left(\frac{\pi-A}{2\pi}\right)^n,$$

which is absolutely convergent in $\mathcal{L}(L^2)$ for, by (3.30), there is some $\varepsilon>0$ such that $\|(\frac{\pi-A}{2\pi})^k\|_{\mathcal{L}(L^2)}\leqslant(1-\varepsilon)^k$ for large k.

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Therefore, it is now readily seen that (3.29) is uniquely solved by

$$h = -2\pi (A+\pi)^{-1} [u_P \cdot n] \in C^\infty (\partial\Omega),$$

whereby, in view of (3.28), we obtain the following representation formula on the exterior of a smooth obstacle:

$$u_R(x) = -\int_{\partial\Omega} \frac{x-y}{|x-y|^2} (A+\pi)^{-1} [u_P \cdot n](y)dy + \gamma H(x). \tag{3.31}$$

Remark. For any velocity field H_* satisfying

$$\begin{cases} \operatorname{div} H_* = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} H_* = 0 & \text{in} \quad \Omega, \\ H_* \to 0 & \text{as} \quad x \to \infty, \\ \oint_{\partial\Omega} H_* \cdot \tau ds = 1, \end{cases} \tag{3.32}$$

we note, by the Plemelj formulas (3.10) and by (3.8), that

$$\tilde{H}(x) := H_*(x) - \int_{\partial\Omega} \frac{x-y}{|x-y|^2} (A+\pi)^{-1} [H_* \cdot n](y)dy$$

is divergence and curl free in Ω, goes to 0 when $x \to \infty$, and verifies

$$\tilde{H} \cdot n = H_* \cdot n - (A+\pi)(A+\pi)^{-1}[H_* \cdot n] = 0 \text{ on } \partial\Omega,$$
$$\oint_{\partial\Omega} \tilde{H} \cdot \tau ds = \oint_{\partial\Omega} H_* \cdot \tau ds - \oint_{\partial\Omega} B(A+\pi)^{-1}[H_* \cdot n] = 1.$$

By uniqueness in (1.6), we deduce $\tilde{H} = H$. This can be used to replace the harmonic vector field $H(x)$ in (3.31) with more convenient expressions, thereby yielding a variant formula:

$$u_R(x) = -\int_{\partial\Omega} \frac{x-y}{|x-y|^2} (A+\pi)^{-1} [(u_P + \gamma H_*) \cdot n] (y)dy + \gamma H_*(x). \tag{3.33}$$

For instance, one may consider the velocity field $H_*(x) = \frac{(x-x_*)^\perp}{2\pi|x-x_*|^2} = K_{\mathbb{R}^2}[\delta_{x_*}]$ for any given $x_* \in \overline{\Omega}^c$.

It then follows, by comparing (3.31) with (3.24) and by the uniqueness of solutions to system (2.4), that

$$\begin{aligned} -\int_{\partial\Omega} &\frac{x-y}{|x-y|^2} (A+\pi)^{-1} [u_P \cdot n](y)dy \\ &= \int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} \left(A^2-\pi^2\right)^{-1} B [u_P \cdot n] (y)dy \quad \text{for every } x \in \Omega, \end{aligned}$$

whence we infer that, replacing $u_P \cdot n$ by $B(A-\pi)\varphi$ in view of the arbitrariness of zero-mean boundary data in (2.4) and using the Poincaré–Bertrand identities (3.12),

$$\begin{aligned} \int_{\partial\Omega} \frac{x-y}{|x-y|^2} B\varphi(y)dy &= -\int_{\partial\Omega} \frac{x-y}{|x-y|^2} (A+\pi)^{-1} B (A-\pi) \varphi(y)dy \\ &= \int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} \left(A^2-\pi^2\right)^{-1} B^2 (A-\pi) \varphi(y)dy \\ &= \int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} (A-\pi) \varphi(y)dy \quad \text{for every } x \in \Omega. \end{aligned} \tag{3.34}$$

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By adjointness (see (3.3) and (3.6)), we further obtain that

$$\int_{\partial\Omega}\int_{\partial\Omega}\frac{y-z}{|y-z|^2}\cdot\tau(z)\frac{x-z}{|x-z|^2}dz\varphi(y)dy$$
$$=\int_{\partial\Omega}\int_{\partial\Omega}\frac{y-z}{|y-z|^2}\cdot n(z)\frac{(x-z)^\perp}{|x-z|^2}dz\varphi(y)dy+\pi\int_{\partial\Omega}\frac{(x-y)^\perp}{|x-y|^2}\varphi(y)dy,$$

and, thus, by the arbitrariness of φ, we deduce the identity

$$\begin{aligned}\pi\frac{(x-y)^\perp}{|x-y|^2}=&\int_{\partial\Omega}\frac{y-z}{|y-z|^2}\cdot\tau(z)\frac{x-z}{|x-z|^2}dz\\&-\int_{\partial\Omega}\frac{y-z}{|y-z|^2}\cdot n(z)\frac{(x-z)^\perp}{|x-z|^2}dz\quad(x,y)\in\Omega\times\partial\Omega,\end{aligned}\tag{3.35}$$

which will be useful later on.

Finally, note that, combining (3.35) (or (3.34)) with (3.24), we obtain yet another convenient representation formula

$$\begin{aligned}u_R(x)=&\frac{1}{\pi}\int_{\partial\Omega}\frac{(x-y)^\perp}{|x-y|^2}AB^{-1}\left[u_P\cdot n\right](y)dy\\&-\frac{1}{\pi}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}\left[u_P\cdot n\right](y)dy+\gamma H(x),\end{aligned}\tag{3.36}$$

whereas combining (3.35) (or (3.34)) with (3.31) yields

$$\begin{aligned}u_R(x)=&-\frac{1}{\pi}\int_{\partial\Omega}\frac{(x-z)^\perp}{|x-z|^2}B\left(A+\pi\right)^{-1}\left[u_P\cdot n\right](z)dz\\&-\frac{1}{\pi}\int_{\partial\Omega}\frac{x-z}{|x-z|^2}A\left(A+\pi\right)^{-1}\left[u_P\cdot n\right](z)dz+\gamma H(x).\end{aligned}$$

Remark. A variant representation formula is obtained by combining (3.35) (or (3.34)) with (3.33) instead of (3.31):

$$\begin{aligned}u_R(x)=&-\frac{1}{\pi}\int_{\partial\Omega}\frac{(x-z)^\perp}{|x-z|^2}B\left(A+\pi\right)^{-1}\left[(u_P+\gamma H_*)\cdot n\right](z)dz\\&-\frac{1}{\pi}\int_{\partial\Omega}\frac{x-z}{|x-z|^2}A\left(A+\pi\right)^{-1}\left[(u_P+\gamma H_*)\cdot n\right](z)dz+\gamma H_*(x).\end{aligned}\tag{3.37}$$

Remark. As previously explained, our goal is to justify that u^N_{app}, defined by (2.7), is a good discretization of the formulation (3.24). In fact, it is also possible to discretize (3.31) (or (3.33)), which provides another approximation of u_R. We explore this alternative approach in section 8.

4. Solving system (2.8) and the discrete Poincaré–Bertrand formula. In this section, we explain how system (2.8) can be uniquely solved as soon as N is sufficiently large provided $\{x_i^N\}$ and $\{\tilde{x}_i^N\}$ are well distributed. This will be achieved by employing a strategy inspired by the inversion of system (3.20)–(3.21).

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Considering the parameters $\{s_i^N\}$ and $\{\tilde{s}_i^N\}$ associated to $\{x_i^N\}$ and $\{\tilde{x}_i^N\}$ (see (2.5)–(2.6)), the system (2.8) of N equations can be recast as

$$\begin{aligned} &\frac{1}{N}\sum_{j=1}^{N}\gamma_j^N \frac{l\left(\tilde{s}_i^N\right)-l\left(s_j^N\right)}{\left|l\left(\tilde{s}_i^N\right)-l\left(s_j^N\right)\right|^2}\cdot\tau\left(l\left(\tilde{s}_i^N\right)\right)=f\left(\tilde{s}_i^N\right) \text{ for all } i=1,\dots,N-1,\\ &\frac{1}{N}\sum_{i=1}^{N}\gamma_i^N=\gamma, \end{aligned} \tag{4.1}$$

where $\gamma^N=(\gamma_1^N,\dots,\gamma_N^N)\in\mathbb{R}^N$ is the unknown and $f(s)=2\pi[u_P\cdot n]\,(l(s))$ for all $s\in[0,|\partial\Omega|]$. Loosely speaking, solving system (4.1) amounts to inverting a discrete version of the operator B introduced in (3.1). Indeed, (4.1) clearly is a discretization of (3.20)–(3.21).

From now on, we will also conveniently denote the matrices:

$$\begin{aligned} A_N &:= \left(\frac{l\left(\tilde{s}_i^N\right)-l\left(s_j^N\right)}{\left|l\left(\tilde{s}_i^N\right)-l\left(s_j^N\right)\right|^2}\cdot n\left(l\left(\tilde{s}_i^N\right)\right)\right)_{1\leqslant i,j\leqslant N},\\ \tilde{A}_N &:= \left(\frac{l\left(s_i^N\right)-l\left(\tilde{s}_j^N\right)}{\left|l\left(s_i^N\right)-l\left(\tilde{s}_j^N\right)\right|^2}\cdot n\left(l\left(s_i^N\right)\right)\right)_{1\leqslant i,j\leqslant N},\\ B_N &:= \left(\frac{l\left(\tilde{s}_i^N\right)-l\left(s_j^N\right)}{\left|l\left(\tilde{s}_i^N\right)-l\left(s_j^N\right)\right|^2}\cdot \tau\left(l\left(\tilde{s}_i^N\right)\right)\right)_{1\leqslant i,j\leqslant N},\\ \tilde{B}_N &:= \left(\frac{l\left(s_i^N\right)-l\left(\tilde{s}_j^N\right)}{\left|l\left(s_i^N\right)-l\left(\tilde{s}_j^N\right)\right|^2}\cdot \tau\left(l\left(s_i^N\right)\right)\right)_{1\leqslant i,j\leqslant N}, \end{aligned}$$

and we will make use of the following notation for $z\in\mathbb{R}^N$:

$$\begin{aligned} \|z\|_{\ell^p} &:= \left(\frac{1}{N}\sum_{i=1}^{N}|z_i|^p\right)^{1/p} \quad \text{for any } p\in[1,\infty),\\ \|z\|_{\ell^\infty} &:= \max_{i=1,\dots,N}|z_i|,\\ \langle z\rangle &:= \frac{1}{N}\sum_{i=1}^{N} z_i. \end{aligned}$$

Note that, with this normalization of the norms, we have:

$$\|z\|_{\ell^p}\leqslant\|z\|_{\ell^q} \text{ for any } 1\leqslant p\leqslant q\leqslant\infty.$$

4.1. Boundedness of discretized operators. For a uniformly distributed mesh (2.10), notice that, by odd symmetry of the cotangent function,

$$\sum_{1\leqslant j\leqslant N}\cot\left(\frac{\left(\tilde{\theta}_i^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)=0 \quad\text{and}\quad \sum_{1\leqslant j\leqslant N}\cot\left(\frac{\left(\tilde{\theta}_j^N-\theta_i^N\right)\pi}{|\partial\Omega|}\right)=0 \tag{4.2}$$

and

$$\sum_{\substack{1\leqslant j\leqslant N\\ j\neq i}}\cot\left(\frac{\left(\theta_i^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)=0 \tag{4.3}$$

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for each $i = 1, \dots, N$. In fact, it can be shown that the only possible mesh satisfying (4.2) and $\theta_1^N = 0$ is necessarily given by (2.10). Indeed, suppose that some other given mesh $\{\phi_i^N\}$ and $\{\tilde\phi_i^N\}$ with $\phi_1^N = 0$ also satisfies (4.2). Then, suitable linear combinations of (4.2) yield that

$$\sum_{1 \leqslant i,j \leqslant N} \left(\left(\tilde\phi_i^N - \phi_j^N \right) - \left(\tilde\theta_i^N - \theta_j^N \right) \right) \times \left(\cot\left(\frac{\left(\tilde\theta_i^N - \theta_j^N \right)\pi}{|\partial\Omega|} \right) - \cot\left(\frac{\left(\tilde\phi_i^N - \phi_j^N \right)\pi}{|\partial\Omega|} \right) \right) = 0,$$

whence, using that $(b-a)(\cot a - \cot b) = (b-a)\int_a^b \frac{1}{\sin^2 x} dx \geqslant (b-a)^2$, for any $0 < a, b < \pi$ (or $-\pi < a, b < 0$),

$$\tilde\phi_i^N - \phi_j^N = \tilde\theta_i^N - \theta_j^N \quad \text{for all } 1 \leqslant i,j \leqslant N.$$

Further using that $\theta_1^N = \phi_1^N$, we conclude that $\theta_i^N = \phi_i^N$ and $\tilde\theta_i^N = \tilde\phi_i^N$ for all $1 \leqslant i \leqslant N$.

The cancellations embodied in identities (4.2) and (4.3) are related with their continuous counterpart $\int_0^\pi \cot(\theta - \tilde\theta) d\theta = 0$ for any $\tilde\theta \in \mathbb{R}$. As the oddness of the cotangent function plays a crucial role in defining Cauchy's principal value, the symmetry of the points $(\theta_i^N, \tilde\theta_i^N)$ is important to make sure that singular integrals defined in the sense of Cauchy's principal value are suitably approximated by their corresponding discretization.

The first result in this section is technical and shows that a well distributed mesh retains sufficient approximate symmetry to satisfy an approximation of (4.2). This property will be important to ensure that singular integrals are well approximated by discretizations corresponding to well distributed meshes.

Lemma 4.1. *For any $N \geqslant 2$, consider a well distributed mesh $(s_1^N, \dots, s_N^N) \in [0, |\partial\Omega|)^N$, $(\tilde s_1^N, \dots, \tilde s_N^N) \in [0, |\partial\Omega|)^N$. Then, as $N \to \infty$,*

$$\max_{1 \leqslant i \leqslant N} \left| \sum_{1 \leqslant j \leqslant N} \cot\left(\frac{\left(\tilde s_i^N - s_j^N \right)\pi}{|\partial\Omega|} \right) \right| = \mathcal{O}\left(N^{-\kappa+1} \right)$$
$$\textit{and} \max_{1 \leqslant i \leqslant N} \left| \sum_{1 \leqslant j \leqslant N} \cot\left(\frac{\left(\tilde s_j^N - s_i^N \right)\pi}{|\partial\Omega|} \right) \right| = \mathcal{O}\left(N^{-\kappa+1} \right).$$

Proof. Note first that, for all $1 \leqslant i, j \leqslant N$, and large enough N,

$$\begin{aligned} \left| \tilde s_i^N - s_j^N \right| &\geqslant \left| \tilde\theta_i^N - \theta_j^N \right| - \left| \tilde s_i^N - \tilde\theta_i^N \right| - \left| s_j^N - \theta_j^N \right| = \frac{\left| i - j + \frac12 \right| |\partial\Omega|}{N} - \mathcal{O}\left(N^{-3} \right) \\ &\geqslant \frac{\left| i - j + \frac12 \right| |\partial\Omega|}{N} - \frac{|\partial\Omega|}{4N} \geqslant \frac{\left| i - j + \frac12 \right| |\partial\Omega|}{2N} \end{aligned}$$

and

$$\begin{aligned} \left| \tilde s_i^N - s_j^N \right| &\leqslant \left| \tilde\theta_i^N - \theta_j^N \right| + \left| \tilde s_i^N - \tilde\theta_i^N \right| + \left| s_j^N - \theta_j^N \right| = \frac{\left| i - j + \frac12 \right| |\partial\Omega|}{N} + \mathcal{O}\left(N^{-3} \right) \\ &\leqslant \frac{\left| i - j + \frac12 \right| |\partial\Omega|}{N} + \frac{|\partial\Omega|}{4N} \leqslant \frac{|\partial\Omega|}{2} + \frac{\left| i - j + \frac12 \right| |\partial\Omega|}{2N}. \end{aligned}$$

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Therefore, by (2.9) and the mean value theorem, defining the open interval

$$I_{ij} = \left(\pi \frac{\left|i-j+\frac{1}{2}\right|}{2N}, \pi \left(\frac{1}{2} + \frac{\left|i-j+\frac{1}{2}\right|}{2N} \right) \right),$$

we find that

$$\begin{aligned}
&\left| \cot\left(\frac{\left(\tilde{s}_i^N - s_j^N\right)\pi}{|\partial\Omega|} \right) - \cot\left(\frac{\left(\tilde{\theta}_i^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) \right| \\
&\qquad \leqslant \frac{\pi}{|\partial\Omega|} \left| \tilde{s}_i^N - s_j^N - \tilde{\theta}_i^N + \theta_j^N \right| \sup_{x \in I_{ij}} \frac{1}{\sin^2 x} \\
&\qquad \leqslant \mathcal{O}\left(N^{-(\kappa+1)}\right) \times \max\left\{ \frac{N^2}{\left|i-j+\frac{1}{2}\right|^2}, \frac{N^2}{\left(N - \left|i-j+\frac{1}{2}\right|\right)^2} \right\} \\
&\qquad = \mathcal{O}\left(N^{-(\kappa+1)}\right) \times \mathcal{O}(N^2) = \mathcal{O}\left(N^{-\kappa+1}\right).
\end{aligned} \tag{4.4}$$

Then, summing over $1 \leqslant i \leqslant N$ or $1 \leqslant j \leqslant N$ yields

$$\begin{aligned}
&\sum_{\substack{i=1 \\ \text{or} \\ j=1}}^{N} \left| \cot\left(\frac{\left(\tilde{s}_i^N - s_j^N\right)\pi}{|\partial\Omega|} \right) - \cot\left(\frac{\left(\tilde{\theta}_i^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) \right| \\
&\qquad \leqslant \mathcal{O}\left(N^{-\kappa+1}\right) \sum_{k=1-N}^{N-1} \max\left\{ \frac{1}{\left|k+\frac{1}{2}\right|^2}, \frac{1}{\left(N - \left|k+\frac{1}{2}\right|\right)^2} \right\} \\
&\qquad \leqslant \mathcal{O}\left(N^{-\kappa+1}\right) \sum_{k=1-N}^{N-1} \frac{1}{\left|k+\frac{1}{2}\right|^2} = \mathcal{O}\left(N^{-\kappa+1}\right),
\end{aligned} \tag{4.5}$$

which, when combined with the identities (4.2), is sufficient to conclude the proof of the lemma. ∎

Remark. The preceding lemma essentially asserts that approximate Riemann sums of $\int_0^\pi \cot(\theta - \tilde{\theta}) d\theta$ for any given $\tilde{\theta} \in \mathbb{R}$ on a well distributed mesh satisfying (2.9) vanish with a convergence rate $\mathcal{O}(N^{-\kappa})$. This is crucial if one aims at obtaining a convergence rate $\mathcal{O}(N^{-\kappa})$ in Theorem 2.1. In other words, the consideration of a better mesh produces a faster convergence rate in Lemma 4.1 which, in turn, results in a faster rate in Theorem 2.1.

The following lemma is a precise ℓ^2-estimate for the uniformly distributed mesh (2.10). The first part of this result was already featured in [1] for the unit disk.

Lemma 4.2. *Consider the uniformly distributed mesh* $(\theta_1^N, \ldots, \theta_N^N) \in [0, |\partial\Omega|)^N$, $(\tilde{\theta}_1^N, \ldots, \tilde{\theta}_N^N) \in [0, |\partial\Omega|)^N$ *defined by* (2.10). *Then, for any* $z \in \mathbb{R}^N$, *we have that*

$$\frac{1}{N} \left\| \left\{ \sum_{1 \leqslant j \leqslant N} \cot\left(\frac{\left(\tilde{\theta}_k^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) z_j \right\}_{1 \leqslant k \leqslant N} \right\|_{\ell^2} = \|z - \langle z \rangle \mathbf{1}\|_{\ell^2} \tag{4.6}$$

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and

$$\frac{1}{N}\left\|\left\{\sum_{\substack{1\leqslant j\leqslant N\\ j\neq k}}\cot\left(\frac{\left(\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)z_j\right\}_{1\leqslant k\leqslant N}\right\|_{\ell^2}\leqslant\|z-\langle z\rangle\mathbf{1}\|_{\ell^2}, \tag{4.7}$$

where $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^N$.

Proof. First, we compute, utilizing (4.2),

$$\begin{aligned}
&N\left\|\left\{\sum_{1\leqslant j\leqslant N}\cot\left(\frac{\left(\tilde\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)z_j\right\}_{1\leqslant k\leqslant N}\right\|_{\ell^2}^2\\
&\quad=\sum_{1\leqslant k\leqslant N}\left|\sum_{1\leqslant j\leqslant N}\cot\left(\frac{\left(\tilde\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)z_j\right|^2\\
&\quad=\sum_{1\leqslant k\leqslant N}\sum_{1\leqslant i,j\leqslant N}\cot\left(\frac{\left(\tilde\theta_k^N-\theta_i^N\right)\pi}{|\partial\Omega|}\right)\cot\left(\frac{\left(\tilde\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)z_iz_j\\
&\quad=-\frac{1}{2}\sum_{1\leqslant i,j\leqslant N}(z_i-z_j)^2\sum_{1\leqslant k\leqslant N}\cot\left(\frac{\left(\tilde\theta_k^N-\theta_i^N\right)\pi}{|\partial\Omega|}\right)\cot\left(\frac{\left(\tilde\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)\\
&\qquad+\sum_{1\leqslant i,k\leqslant N}|z_i|^2\cot\left(\frac{\left(\tilde\theta_k^N-\theta_i^N\right)\pi}{|\partial\Omega|}\right)\sum_{1\leqslant j\leqslant N}\cot\left(\frac{\left(\tilde\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)\\
&\quad=-\frac{1}{2}\sum_{1\leqslant i,j\leqslant N}(z_i-z_j)^2\sum_{1\leqslant k\leqslant N}\cot\left(\frac{\left(\tilde\theta_k^N-\theta_i^N\right)\pi}{|\partial\Omega|}\right)\cot\left(\frac{\left(\tilde\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right).
\end{aligned}$$

Similarly, employing (4.3) instead of (4.2), we find

$$\begin{aligned}
&N\left\|\left\{\sum_{\substack{1\leqslant j\leqslant N\\ j\neq k}}\cot\left(\frac{\left(\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)z_j\right\}_{1\leqslant k\leqslant N}\right\|_{\ell^2}^2\\
&\quad=-\frac{1}{2}\sum_{1\leqslant i,j\leqslant N}(z_i-z_j)^2\sum_{\substack{1\leqslant k\leqslant N\\ k\neq i,j}}\cot\left(\frac{\left(\theta_k^N-\theta_i^N\right)\pi}{|\partial\Omega|}\right)\cot\left(\frac{\left(\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right).
\end{aligned}$$

Next, we use the following elementary relation, valid for any a,b such that $a,b,a-b\notin\pi\mathbb{Z}$,

$$\cot a\cot b=\cot(b-a)[\cot a-\cot b]-1,$$

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to write, using (4.2),

$$\begin{aligned}
N \left\| \left\{ \sum_{1 \leqslant j \leqslant N} \cot\left(\frac{\left(\tilde{\theta}_k^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) z_j \right\}_{1 \leqslant k \leqslant N} \right\|_{\ell^2}^2 \\
&= -\frac{1}{2} \sum_{\substack{1 \leqslant i,j \leqslant N \\ i \neq j}} (z_i - z_j)^2 \cot\left(\frac{\left(\theta_i^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) \\
&\quad \times \sum_{1 \leqslant k \leqslant N} \left[\cot\left(\frac{\left(\tilde{\theta}_k^N - \theta_i^N\right)\pi}{|\partial\Omega|} \right) - \cot\left(\frac{\left(\tilde{\theta}_k^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) \right] \\
&\quad + \frac{N}{2} \sum_{\substack{1 \leqslant i,j \leqslant N \\ i \neq j}} (z_i - z_j)^2 \\
&= \frac{N}{2} \sum_{1 \leqslant i,j \leqslant N} (z_i - z_j)^2,
\end{aligned}$$

and, similarly, using (4.3),

$$\begin{aligned}
N \left\| \left\{ \sum_{\substack{1 \leqslant j \leqslant N \\ j \neq k}} \cot\left(\frac{\left(\theta_k^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) z_j \right\}_{1 \leqslant k \leqslant N} \right\|_{\ell^2}^2 \\
&= -\frac{1}{2} \sum_{\substack{1 \leqslant i,j \leqslant N \\ i \neq j}} (z_i - z_j)^2 \cot\left(\frac{\left(\theta_i^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) \\
&\quad \times \sum_{\substack{1 \leqslant k \leqslant N \\ k \neq i,j}} \left[\cot\left(\frac{\left(\theta_k^N - \theta_i^N\right)\pi}{|\partial\Omega|} \right) - \cot\left(\frac{\left(\theta_k^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) \right] \\
&\quad + \frac{N-2}{2} \sum_{\substack{1 \leqslant i,j \leqslant N \\ i \neq j}} (z_i - z_j)^2 \\
&= \frac{N-2}{2} \sum_{1 \leqslant i,j \leqslant N} (z_i - z_j)^2 - \sum_{\substack{1 \leqslant i,j \leqslant N \\ i \neq j}} (z_i - z_j)^2 \cot^2\left(\frac{\left(\theta_i^N - \theta_j^N\right)\pi}{|\partial\Omega|} \right) \\
&\leqslant \frac{N-2}{2} \sum_{1 \leqslant i,j \leqslant N} (z_i - z_j)^2.
\end{aligned}$$

Finally, the remaining sums are easily recast as

$$\begin{aligned}
\frac{N}{2} \sum_{1 \leqslant i,j \leqslant N} (z_i - z_j)^2 &= N \sum_{1 \leqslant i,j \leqslant N} (z_i - \langle z \rangle)^2 - N \sum_{1 \leqslant i,j \leqslant N} (z_i - \langle z \rangle)(z_j - \langle z \rangle) \\
&= N^2 \sum_{1 \leqslant i \leqslant N} (z_i - \langle z \rangle)^2 = N^3 \| z - \langle z \rangle \mathbf{1} \|_{\ell^2}^2.
\end{aligned}$$

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We have therefore obtained that

$$N\left\|\left\{\sum_{1\leqslant j\leqslant N}\cot\left(\frac{\left(\tilde{\theta}_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)z_j\right\}_{1\leqslant k\leqslant N}\right\|_{\ell^2}^2=N^3\|z-\langle z\rangle\mathbf{1}\|_{\ell^2}^2,$$

$$N\left\|\left\{\sum_{\substack{1\leqslant j\leqslant N\\ j\neq k}}\cot\left(\frac{\left(\theta_k^N-\theta_j^N\right)\pi}{|\partial\Omega|}\right)z_j\right\}_{1\leqslant k\leqslant N}\right\|_{\ell^2}^2\leqslant(N-2)N^2\|z-\langle z\rangle\mathbf{1}\|_{\ell^2}^2,$$

which ends the proof of the lemma. ∎

Combining (4.4) (or a slight variant of it without tildes) with (4.6) and (4.7), it is readily seen that a well distributed mesh enjoys sufficient approximate symmetry to satisfy suitable approximations of (4.6) and (4.7), which we record in precise terms in the corollary below.

COROLLARY 4.3. *For any $N\geqslant 2$, consider a well distributed mesh $(s_1^N,\dots,s_N^N)\in[0,|\partial\Omega|)^N$, $(\tilde{s}_1^N,\dots,\tilde{s}_N^N)\in[0,|\partial\Omega|)^N$. Then, for any $z\in\mathbb{R}^N$, we have that*

$$\left|\frac{1}{N}\left\|\left\{\sum_{1\leqslant j\leqslant N}\cot\left(\frac{\left(\tilde{s}_k^N-s_j^N\right)\pi}{|\partial\Omega|}\right)z_j\right\}_{1\leqslant k\leqslant N}\right\|_{\ell^2}-\|z-\langle z\rangle\mathbf{1}\|_{\ell^2}\right|\leqslant\frac{C}{N^{\kappa-1}}\|z\|_{\ell^1},$$

$$\frac{1}{N}\left\|\left\{\sum_{\substack{1\leqslant j\leqslant N\\ j\neq k}}\cot\left(\frac{\left(s_k^N-s_j^N\right)\pi}{|\partial\Omega|}\right)z_j\right\}_{1\leqslant k\leqslant N}\right\|_{\ell^2}\leqslant\|z-\langle z\rangle\mathbf{1}\|_{\ell^2}+\frac{C}{N^{\kappa-1}}\|z\|_{\ell^1},$$

where $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^N$ and the constant $C>0$ is independent of N and z.

A direct consequence of the preceding estimates on well distributed meshes concerns the uniform boundedness of the operators defined above.

COROLLARY 4.4. *For any $N\geqslant 2$, consider a well distributed mesh $(s_1^N,\dots,s_N^N)\in[0,|\partial\Omega|)^N$, $(\tilde{s}_1^N,\dots,\tilde{s}_N^N)\in[0,|\partial\Omega|)^N$. Then, there exists a constant $C>0$ independent of N such that, for each $1\leqslant p\leqslant\infty$,*

$$\frac{1}{N}\left(\|A_Nz\|_{\ell^p}+\left\|\tilde{A}_Nz\right\|_{\ell^p}\right)\leqslant C\|z\|_{\ell^1}$$

and

$$\frac{1}{N}\left(\|B_Nz\|_{\ell^2}+\left\|\tilde{B}_Nz\right\|_{\ell^2}\right)\leqslant C\|z\|_{\ell^2}$$

for all $z\in\mathbb{R}^N$.

Proof. The boudedness of A_N and $\tilde{A}_N$ easily follows from the uniform boundedness of each component of the corresponding matrices. (Recall that $\frac{l(s)-l(s_*)}{|l(s)-l(s_*)|^2}\cdot n(l(s))$ is a continuous bounded function, which follows directly from (3.9).)

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As for the boundedness of B_N and $\tilde{B}_N$, it follows from Corollary 4.3 combined with the fact that $\frac{l(s)-l(s_*)}{|l(s)-l(s_*)|^2}\cdot\tau(l(s))-\frac{\pi}{|\partial\Omega|}\cot(\frac{(s-s_*)\pi}{|\partial\Omega|})$ is a continuous bounded function (which is a direct consequence of (3.9) and the fact that $\frac{1}{s}-\frac{\pi}{|\partial\Omega|}\cot(\frac{s\pi}{|\partial\Omega|})$ is smooth near $s=0$). ∎

4.2. Approximation of the Poincaré–Bertrand identities. The next proposition provides a crucial discretization of (3.8) and the Poincaré–Bertrand identities (3.12).

PROPOSITION 4.5. *For any $N\geqslant 2$, consider a well distributed mesh $(s_1^N,\dots,s_N^N)\in[0,|\partial\Omega|)^N$, $(\tilde{s}_1^N,\dots,\tilde{s}_N^N)\in[0,|\partial\Omega|)^N$. Then, for all $z\in\mathbb{R}^N$, as $N\to\infty$,*

$$\begin{aligned}\left|\left\langle\frac{|\partial\Omega|}{N}B_Nz\right\rangle\right|+\left|\left\langle\frac{|\partial\Omega|}{N}A_Nz-\pi z\right\rangle\right|&\leqslant\frac{C}{N^\kappa}\|z\|_{\ell^1},\\ \left|\left\langle\frac{|\partial\Omega|}{N}\tilde{B}_Nz\right\rangle\right|+\left|\left\langle\frac{|\partial\Omega|}{N}\tilde{A}_Nz-\pi z\right\rangle\right|&\leqslant\frac{C}{N^\kappa}\|z\|_{\ell^1}\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}&\left\|\frac{|\partial\Omega|^2}{N^2}\left(B_N\tilde{B}_N-A_N\tilde{A}_N\right)z+\pi^2z\right\|_{\ell^2}\\&\qquad+\left\|\frac{|\partial\Omega|^2}{N^2}\left(A_N\tilde{B}_N+B_N\tilde{A}_N\right)z\right\|_{\ell^2}\leqslant\frac{C}{N}\|z\|_{\ell^2},\\&\left\|\frac{|\partial\Omega|^2}{N^2}\left(\tilde{B}_NB_N-\tilde{A}_NA_N\right)z+\pi^2z\right\|_{\ell^2}\\&\qquad+\left\|\frac{|\partial\Omega|^2}{N^2}\left(\tilde{A}_NB_N+\tilde{B}_NA_N\right)z\right\|_{\ell^2}\leqslant\frac{C}{N}\|z\|_{\ell^2},\end{aligned}\tag{4.9}$$

where $C>0$ is an independent constant.

Proof. Recall that $l:[0,|\partial\Omega|]\to\mathbb{R}^2$ denotes a given arc-length parametrization of $\partial\Omega$. For clarity, we also introduce the smooth path $L:[0,|\partial\Omega|]\to\mathbb{C}$ defining the contour $\Gamma\subset\mathbb{C}$ matching $\partial\Omega$ through the usual identification of $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, so that $L(s)$ is identified with $l(s)$ for each $s\in[0,|\partial\Omega|]$.

We claim that

$$\max_{1\leqslant k\leqslant N}\left|\frac{|\partial\Omega|}{N}\sum_{1\leqslant j\leqslant N}\frac{L'\left(\tilde{s}_j^N\right)}{L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)}-i\pi\right|=\mathcal{O}\left(N^{-\kappa}\right),\tag{4.10}$$

$$\max_{1\leqslant k\leqslant N}\left|\frac{|\partial\Omega|^2}{N^2}\sum_{1\leqslant j\leqslant N}\frac{L'\left(\tilde{s}_j^N\right)L'\left(s_k^N\right)}{\left(L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)\right)^2}-\pi^2\right|=\mathcal{O}\left(N^{-1}\right),\tag{4.11}$$

and, for any $z\in\mathbb{R}^N$, that

$$\frac{1}{N}\left\|\left\{\sum_{\substack{1\leqslant j\leqslant N\\ j\neq k}}\frac{L'\left(s_j^N\right)}{L\left(s_k^N\right)-L\left(s_j^N\right)}z_j\right\}_{1\leqslant k\leqslant N}\right\|_{\ell^2}\leqslant C\|z\|_{\ell^2},\tag{4.12}$$

where the constant $C>0$ only depends on fixed parameters.

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Indeed, the first claim (4.10) is obtained by writing

$$\sum_{1\leqslant j\leqslant N}\frac{L'\left(\tilde{s}_j^N\right)}{L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)}=\sum_{1\leqslant j\leqslant N} f\left(\tilde{s}_j^N,s_k^N\right)+\frac{\pi}{|\partial\Omega|}\sum_{1\leqslant j\leqslant N}\cot\left(\frac{\left(\tilde{s}_j^N-s_k^N\right)\pi}{|\partial\Omega|}\right),$$

where

$$f(s,s_*)=\frac{L'(s)}{L(s)-L(s_*)}-\frac{\pi}{|\partial\Omega|}\cot\left(\frac{(s-s_*)\pi}{|\partial\Omega|}\right)\quad\text{with } s,s_*\in[0,|\partial\Omega|]$$

is a smooth function, which is a direct consequence of (3.9) and the fact that $\frac{1}{s}-\frac{\pi}{|\partial\Omega|}\cot(\frac{s\pi}{|\partial\Omega|})$ is smooth near $s=0$. Then, by virtue of Lemma 4.1 and the uniform convergence of Riemann sums for smooth functions (see Corollary A.2 from Appendix A, if necessary), we deduce that

$$\max_{1\leqslant k\leqslant N}\left|\frac{|\partial\Omega|}{N}\sum_{1\leqslant j\leqslant N}\frac{L'\left(\tilde{s}_j^N\right)}{L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)}-\int_0^{|\partial\Omega|}f\left(s,s_k^N\right)ds\right|=\mathcal{O}\left(N^{-\kappa}\right).$$

The justification of (4.10) is then completed upon noticing that, for any $s_*\in[0,|\partial\Omega|]$,

$$\int_0^{|\partial\Omega|}f(s,s_*)ds=\int_\Gamma\frac{1}{z-L(s_*)}dz=i\pi,$$

where the last integral above is defined in the sense of Cauchy's principal value and is easily evaluated employing Plemelj's formulas (3.10) with the representation of Cauchy integrals (3.2).

We turn now to the justification of (4.11). To this end, we use, again, that $f(s,s_*)$ is smooth to deduce that

$$\frac{\partial f}{\partial s_*}(s,s_*)+\frac{\pi^2}{|\partial\Omega|^2}=\frac{L'(s)L'(s_*)}{(L(s)-L(s_*))^2}-\frac{\pi^2}{|\partial\Omega|^2}\cot^2\left(\frac{(s-s_*)\pi}{|\partial\Omega|}\right)$$

is smooth, as well. Therefore, by Corollary A.2 from Appendix A and employing (2.9)–(2.10), we find that

$$\left|\sum_{1\leqslant j\leqslant N}\left(\frac{L'\left(\tilde{s}_j^N\right)L'\left(s_k^N\right)}{\left(L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)\right)^2}-\frac{\pi^2}{|\partial\Omega|^2}\cot^2\left(\frac{\left(\tilde{s}_j^N-s_k^N\right)\pi}{|\partial\Omega|}\right)\right)\right|=\mathcal{O}(N).$$

By (4.5), we further find that

$$\begin{aligned}\sum_{1\leqslant j\leqslant N}&\left|\cot^2\left(\frac{\left(\tilde{s}_j^N-s_k^N\right)\pi}{|\partial\Omega|}\right)-\cot^2\left(\frac{\left(\tilde{\theta}_j^N-\theta_k^N\right)\pi}{|\partial\Omega|}\right)\right|\\&\leqslant CN\sum_{1\leqslant j\leqslant N}\left|\cot\left(\frac{\left(\tilde{s}_j^N-s_k^N\right)\pi}{|\partial\Omega|}\right)-\cot\left(\frac{\left(\tilde{\theta}_j^N-\theta_k^N\right)\pi}{|\partial\Omega|}\right)\right|=\mathcal{O}(N^{-\kappa+2})\end{aligned}$$

for some independent constant $C>0$, which implies that

$$\left|\sum_{1\leqslant j\leqslant N}\left(\frac{L'\left(\tilde{s}_j^N\right)L'\left(s_k^N\right)}{\left(L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)\right)^2}-\frac{\pi^2}{|\partial\Omega|^2}\cot^2\left(\frac{\left(\tilde{\theta}_j^N-\theta_k^N\right)\pi}{|\partial\Omega|}\right)\right)\right|=\mathcal{O}(N).$$

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The justification of (4.11) is then completed upon noticing that

$$\begin{aligned}
&\frac{\pi^2}{N}+\frac{\pi^2}{N^2}\sum_{1\leqslant j\leqslant N}\cot^2\left(\frac{\left(\tilde{\theta}_j^N-\theta_k^N\right)\pi}{|\partial\Omega|}\right)\\
&\quad=\frac{\pi^2}{N^2}\sum_{1-k\leqslant j\leqslant N-k}\frac{1}{\sin^2\left(\frac{(j+\frac{1}{2})\pi}{N}\right)}\quad=\frac{\pi^2}{N^2}\sum_{1-\left[\frac{N}{2}\right]\leqslant j\leqslant N-\left[\frac{N}{2}\right]}\frac{1}{\sin^2\left(\frac{(j+\frac{1}{2})\pi}{N}\right)}\\
&\quad=\frac{\pi^2}{N^2}\sum_{1-\left[\frac{N}{2}\right]\leqslant j\leqslant N-\left[\frac{N}{2}\right]}\left(\frac{1}{\sin^2\left(\frac{(j+\frac{1}{2})\pi}{N}\right)}-\frac{1}{\left(\frac{(j+\frac{1}{2})\pi}{N}\right)^2}\right)\\
&\qquad+\sum_{1-\left[\frac{N}{2}\right]\leqslant j\leqslant N-\left[\frac{N}{2}\right]}\frac{4}{(2j+1)^2},
\end{aligned}$$

whereby, by the convergence of Riemann sums (see Corollary A.2 from Appendix A, if necessary),

$$\begin{aligned}
\frac{\pi^2}{N^2}\sum_{1\leqslant j\leqslant N}\cot^2\left(\frac{\left(\tilde{\theta}_j^N-\theta_k^N\right)\pi}{|\partial\Omega|}\right)&=\mathcal{O}\left(N^{-1}\right)+\sum_{0\leqslant j\leqslant\left[\frac{N}{2}\right]}\frac{8}{(2j+1)^2}\\
&=\mathcal{O}\left(N^{-1}\right)+\sum_{0\leqslant j<\infty}\frac{8}{(2j+1)^2}\\
&=\mathcal{O}\left(N^{-1}\right)+8\left(\sum_{1\leqslant j<\infty}\frac{1}{j^2}-\sum_{1\leqslant j<\infty}\frac{1}{(2j)^2}\right)\\
&=\mathcal{O}\left(N^{-1}\right)+6\sum_{1\leqslant j<\infty}\frac{1}{j^2}=\mathcal{O}\left(N^{-1}\right)+\pi^2.
\end{aligned}$$

As for our third claim (4.12), it directly follows from Corollary 4.3 using that $f(s,s_*)$ is a continuous—and therefore bounded—function.

Now that (4.10), (4.11), and (4.12) are established, we move on to the actual justification of (4.9). To this end, we decompose, for any $z\in\mathbb{R}^N$ and each $1\leqslant k\leqslant N$,

$$\begin{aligned}
&\sum_{1\leqslant i,j\leqslant N}\frac{L'\left(\tilde{s}_j^N\right)}{L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)}\frac{L'\left(s_i^N\right)}{L\left(\tilde{s}_j^N\right)-L\left(s_i^N\right)}z_i\\
&\qquad=\sum_{\substack{1\leqslant i\leqslant N\\ i\neq k}}\frac{L'\left(s_i^N\right)}{L\left(s_k^N\right)-L\left(s_i^N\right)}\\
&\qquad\quad\times\sum_{1\leqslant j\leqslant N}L'\left(\tilde{s}_j^N\right)\left[\frac{1}{L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)}-\frac{1}{L\left(\tilde{s}_j^N\right)-L\left(s_i^N\right)}\right]z_i\\
&\qquad\quad+\sum_{1\leqslant j\leqslant N}\frac{L'\left(\tilde{s}_j^N\right)L'\left(s_k^N\right)}{\left(L\left(\tilde{s}_j^N\right)-L\left(s_k^N\right)\right)^2}z_k
\end{aligned}$$

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$$= \left[\sum_{1\leqslant j\leqslant N} \frac{L'\left(\tilde{s}_j^N\right)}{L\left(\tilde{s}_j^N\right) - L\left(s_k^N\right)} - \frac{i\pi N}{|\partial\Omega|} \right] \sum_{\substack{1\leqslant i\leqslant N\\ i\neq k}} \frac{L'\left(s_i^N\right)}{L\left(s_k^N\right) - L\left(s_i^N\right)} z_i$$

$$- \sum_{\substack{1\leqslant i\leqslant N\\ i\neq k}} \frac{L'\left(s_i^N\right)}{L\left(s_k^N\right) - L\left(s_i^N\right)} \left[\sum_{1\leqslant j\leqslant N} \frac{L'\left(\tilde{s}_j^N\right)}{L\left(\tilde{s}_j^N\right) - L\left(s_i^N\right)} - \frac{i\pi N}{|\partial\Omega|} \right] z_i$$

$$+ \left[\sum_{1\leqslant j\leqslant N} \frac{L'\left(\tilde{s}_j^N\right) L'\left(s_k^N\right)}{\left(L\left(\tilde{s}_j^N\right) - L\left(s_k^N\right)\right)^2} - \frac{\pi^2 N^2}{|\partial\Omega|^2} \right] z_k + \frac{\pi^2 N^2}{|\partial\Omega|^2} z_k.$$

Therefore, employing (4.10), (4.11), and (4.12), it follows that

$$\left\| \left\{ \frac{|\partial\Omega|^2}{N^2} \sum_{1\leqslant i,j\leqslant N} \frac{L'\left(\tilde{s}_j^N\right)}{L\left(\tilde{s}_j^N\right) - L\left(s_k^N\right)} \frac{L'\left(s_i^N\right)}{L\left(\tilde{s}_j^N\right) - L\left(s_i^N\right)} z_i - \pi^2 z_k \right\}_{1\leqslant k\leqslant N} \right\|_{\ell^2} \leqslant \frac{C}{N} \|z\|_{\ell^2}.$$

Clearly, exchanging the symmetric roles of $(s_1^N, \dots, s_N^N)$ and $(\tilde{s}_1^N, \dots, \tilde{s}_N^N)$ yields the equivalent estimate

$$\left\| \left\{ \frac{|\partial\Omega|^2}{N^2} \sum_{1\leqslant i,j\leqslant N} \frac{L'\left(s_j^N\right)}{L\left(s_j^N\right) - L\left(\tilde{s}_k^N\right)} \frac{L'\left(\tilde{s}_i^N\right)}{L\left(s_j^N\right) - L\left(\tilde{s}_i^N\right)} z_i - \pi^2 z_k \right\}_{1\leqslant k\leqslant N} \right\|_{\ell^2} \leqslant \frac{C}{N} \|z\|_{\ell^2}.$$

Finally, noticing that, for all $s, s_* \in [0, |\partial\Omega|]$,

$$\frac{L'(s)}{L(s) - L(s_*)} = \frac{l(s) - l(s_*)}{|l(s) - l(s_*)|^2} \cdot \tau\left(l(s)\right) + i \frac{l(s) - l(s_*)}{|l(s) - l(s_*)|^2} \cdot n\left(l(s)\right), \tag{4.13}$$

it is readily seen that (we suggest the reader compare these identities with (3.2))

$$\left\{ \sum_{1\leqslant i,j\leqslant N} \frac{L'\left(\tilde{s}_j^N\right)}{L\left(\tilde{s}_j^N\right) - L\left(s_k^N\right)} \frac{L'\left(s_i^N\right)}{L\left(\tilde{s}_j^N\right) - L\left(s_i^N\right)} z_i \right\}_{1\leqslant k\leqslant N} = -\left(B_N^* + iA_N^*\right) \left(\tilde{B}_N^* + i\tilde{A}_N^*\right) z$$

and

$$\left\{ \sum_{1\leqslant i,j\leqslant N} \frac{L'\left(s_j^N\right)}{L\left(s_j^N\right) - L\left(\tilde{s}_k^N\right)} \frac{L'\left(\tilde{s}_i^N\right)}{L\left(s_j^N\right) - L\left(\tilde{s}_i^N\right)} z_i \right\}_{1\leqslant k\leqslant N} = -\left(\tilde{B}_N^* + i\tilde{A}_N^*\right) \left(B_N^* + iA_N^*\right) z.$$

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Therefore, identifying the real and imaginary parts in the preceding estimates yields

$$\begin{aligned}\left\| \frac{|\partial\Omega|^2}{N^2}\left(B_N^*\tilde{B}_N^* - A_N^*\tilde{A}_N^*\right)z + \pi^2 z\right\|_{\ell^2} \\ + \left\| \frac{|\partial\Omega|^2}{N^2}\left(A_N^*\tilde{B}_N^* + B_N^*\tilde{A}_N^*\right)z\right\|_{\ell^2} &\leqslant \frac{C}{N}\|z\|_{\ell^2},\\ \left\| \frac{|\partial\Omega|^2}{N^2}\left(\tilde{B}_N^*B_N^* - \tilde{A}_N^*A_N^*\right)z + \pi^2 z\right\|_{\ell^2} \\ + \left\| \frac{|\partial\Omega|^2}{N^2}\left(\tilde{A}_N^*B_N^* + \tilde{B}_N^*A_N^*\right)z\right\|_{\ell^2} &\leqslant \frac{C}{N}\|z\|_{\ell^2}.\end{aligned}$$

A standard duality argument concludes the proof of (4.9).

As for (4.8), by combining (4.10) with (4.13) and then exchanging the symmetric roles of $(s_1^N, \dots, s_N^N)$ and $(\tilde{s}_1^N, \dots, \tilde{s}_N^N)$, we easily obtain

$$\begin{aligned}\left\| \frac{|\partial\Omega|}{N}\left(B_N^* + iA_N^*\right)\mathbf{1} - i\pi\mathbf{1}\right\|_{\ell^\infty} &= \mathcal{O}\left(N^{-\kappa}\right),\\ \left\| \frac{|\partial\Omega|}{N}\left(\tilde{B}_N^* + i\tilde{A}_N^*\right)\mathbf{1} - i\pi\mathbf{1}\right\|_{\ell^\infty} &= \mathcal{O}\left(N^{-\kappa}\right),\end{aligned} \tag{4.14}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$. Consequently, by duality, we find that

$$\begin{aligned}\left|\left\langle \frac{|\partial\Omega|}{N}\left(B_N + iA_N\right)z - i\pi z\right\rangle\right| &\leqslant \left\| \frac{|\partial\Omega|}{N}\left(B_N^* + iA_N^*\right)\mathbf{1} - i\pi\mathbf{1}\right\|_{\ell^\infty}\|z\|_{\ell^1}\\ &\leqslant \mathcal{O}\left(N^{-\kappa}\right)\|z\|_{\ell^1},\\ \left|\left\langle \frac{|\partial\Omega|}{N}\left(\tilde{B}_N + i\tilde{A}_N\right)z - i\pi z\right\rangle\right| &\leqslant \left\| \frac{|\partial\Omega|}{N}\left(\tilde{B}_N^* + i\tilde{A}_N^*\right)\mathbf{1} - i\pi\mathbf{1}\right\|_{\ell^\infty}\|z\|_{\ell^1}\\ &\leqslant \mathcal{O}\left(N^{-\kappa}\right)\|z\|_{\ell^1},\end{aligned}$$

which completes the proof of the proposition. ∎

4.3. Approximation of spectral properties. We will also need a discretized version of the estimate (3.27) on the spectral radius of A acting on zero mean elements, which is precisely the content of the coming lemma. To this end, recall from (3.8) that the operator A preserves the mean value of its argument. In order that the relevant discretized operators enjoy similar properties, we introduce a correction to $\tilde{A}_N A_N$ and $A_N\tilde{A}_N$. More precisely, we introduce the operators $D_N, \tilde{D}_N : \mathbb{R}^N \to \mathbb{R}^N$ defined by, for all $z \in \mathbb{R}^N$,

$$\begin{aligned}D_N z &= \tilde{A}_N A_N z - \left\langle \tilde{A}_N A_N z - \frac{\pi^2 N^2}{|\partial\Omega|^2} z\right\rangle \mathbf{1}\\ &= \tilde{A}_N A_N z - \left\langle \tilde{A}_N A_N z - \frac{\pi N}{|\partial\Omega|} A_N z\right\rangle \mathbf{1} - \left\langle \frac{\pi N}{|\partial\Omega|} A_N z - \frac{\pi^2 N^2}{|\partial\Omega|^2} z\right\rangle \mathbf{1},\\ \tilde{D}_N z &= A_N\tilde{A}_N z - \left\langle A_N\tilde{A}_N z - \frac{\pi^2 N^2}{|\partial\Omega|^2} z\right\rangle \mathbf{1}\\ &= A_N\tilde{A}_N z - \left\langle A_N\tilde{A}_N z - \frac{\pi N}{|\partial\Omega|}\tilde{A}_N z\right\rangle \mathbf{1} - \left\langle \frac{\pi N}{|\partial\Omega|}\tilde{A}_N z - \frac{\pi^2 N^2}{|\partial\Omega|^2} z\right\rangle \mathbf{1},\end{aligned}$$

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where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^N$. Observe that $\frac{|\partial\Omega|^2}{N^2} \langle D_N z \rangle = \frac{|\partial\Omega|^2}{N^2} \langle \tilde{D}_N z \rangle = \pi^2 \langle z \rangle$. We also define the subspace $\ell_0^p \subset \ell^p$ for any $1 \leqslant p \leqslant \infty$, by

$$\ell_0^p = \{ z \in \ell^p \, : \, \langle z \rangle = 0 \} .$$

LEMMA 4.6. *For any $N \geqslant 2$, consider a well distributed mesh $(s_1^N, \ldots, s_N^N) \in [0, |\partial\Omega|)^N$, $(\tilde{s}_1^N, \ldots, \tilde{s}_N^N) \in [0, |\partial\Omega|)^N$. Then, there exist $N_*, k_* \geqslant 1$ and $\delta > 0$ such that*

$$\left\| \left(\frac{|\partial\Omega|^2}{N^2} D_N \right)^k \right\|_{\mathcal{L}(\ell_0^2)}^{\frac{1}{k}} \leqslant (\pi - \delta)^2 \quad \textit{and} \quad \left\| \left(\frac{|\partial\Omega|^2}{N^2} \tilde{D}_N \right)^k \right\|_{\mathcal{L}(\ell_0^2)}^{\frac{1}{k}} \leqslant (\pi - \delta)^2$$

for all $k \geqslant k_*$ and $N \geqslant N_*$.

In particular, provided N is sufficiently large, the Neumann series

$$\begin{aligned} \left(\frac{|\partial\Omega|^2}{N^2} D_N - \pi^2 \right)^{-1} &= -\pi^{-2} \sum_{k=0}^{\infty} \left(\frac{|\partial\Omega|^2}{\pi^2 N^2} D_N \right)^k , \\ \left(\frac{|\partial\Omega|^2}{N^2} \tilde{D}_N - \pi^2 \right)^{-1} &= -\pi^{-2} \sum_{k=0}^{\infty} \left(\frac{|\partial\Omega|^2}{\pi^2 N^2} \tilde{D}_N \right)^k \end{aligned}$$

are absolutely convergent in $\mathcal{L}\left(\ell_0^2\right)$ and the inverse operators they define are bounded in $\mathcal{L}\left(\ell_0^2\right)$ uniformly in N.

Proof. For each $k \geqslant 1$, we denote by $K_k(x, y)$ the kernel of A^k. Note that K_k is smooth and satisfies, for all $x, y \in \partial\Omega$,

$$\begin{aligned} K_k(x, y) = \int_{\partial\Omega \times \cdots \times \partial\Omega} & \frac{x - y_1}{|x - y_1|^2} \cdot n(x) \\ & \times \left(\prod_{j=1}^{k-2} \frac{y_j - y_{j+1}}{|y_j - y_{j+1}|^2} \cdot n(y_j) \right) \frac{y_{k-1} - y}{|y_{k-1} - y|^2} \cdot n(y_{k-1}) dy_1 \ldots dy_{k-1}. \end{aligned}$$

In particular, for even indices, we may write

$$\begin{aligned} & K_{2k}(x, y) \\ & = \int_{\partial\Omega \times \cdots \times \partial\Omega} \left(\int_{\partial\Omega} \frac{x - y_1}{|x - y_1|^2} \cdot n(x) \frac{y_1 - y_2}{|y_1 - y_2|^2} \cdot n(y_1) dy_1 \right) \\ & \quad \times \prod_{j=1}^{k-2} \left(\int_{\partial\Omega} \frac{y_{2j} - y_{2j+1}}{|y_{2j} - y_{2j+1}|^2} \cdot n(y_{2j}) \frac{y_{2j+1} - y_{2j+2}}{|y_{2j+1} - y_{2j+2}|^2} \cdot n(y_{2j+1}) dy_{2j+1} \right) \\ & \quad \times \left(\int_{\partial\Omega} \frac{y_{2k-2} - y_{2k-1}}{|y_{2k-2} - y_{2k-1}|^2} \cdot n(y_{2k-2}) \frac{y_{2k-1} - y}{|y_{2k-1} - y|^2} \cdot n(y_{2k-1}) dy_{2k-1} \right) \prod_{j=1}^{k-1} dy_{2j}. \end{aligned}$$

Therefore, by smoothness of the kernel $\frac{x-y}{|x-y|^2} \cdot n(x)$, approximating the above integrals by their Riemann sums yields, as $N \to \infty$, that $K_{2k}(x, y)$ is arbitrarily close (which

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is symbolized here by $\sim$) in $L^\infty_{x,y}(\partial\Omega\times\partial\Omega)$ to

$$\begin{aligned}
&K_{2k}(x,y)\\
&\sim\int_{\partial\Omega\times\cdots\times\partial\Omega}\left(\frac{|\partial\Omega|}{N}\sum_{i=1}^N\frac{x-l\left(\tilde{s}_i^N\right)}{\left|x-l\left(\tilde{s}_i^N\right)\right|^2}\cdot n(x)\frac{l\left(\tilde{s}_i^N\right)-y_2}{\left|l\left(\tilde{s}_i^N\right)-y_2\right|^2}\cdot n\left(l\left(\tilde{s}_i^N\right)\right)\right)\\
&\quad\times\prod_{j=1}^{k-2}\left(\frac{|\partial\Omega|}{N}\sum_{i=1}^N\frac{y_{2j}-l\left(\tilde{s}_i^N\right)}{\left|y_{2j}-l\left(\tilde{s}_i^N\right)\right|^2}\cdot n\left(y_{2j}\right)\frac{l\left(\tilde{s}_i^N\right)-y_{2j+2}}{\left|l\left(\tilde{s}_i^N\right)-y_{2j+2}\right|^2}\cdot n\left(l\left(\tilde{s}_i^N\right)\right)\right)\\
&\quad\times\left(\frac{|\partial\Omega|}{N}\sum_{i=1}^N\frac{y_{2k-2}-l\left(\tilde{s}_i^N\right)}{\left|y_{2k-2}-l\left(\tilde{s}_i^N\right)\right|^2}\cdot n\left(y_{2k-2}\right)\frac{l\left(\tilde{s}_i^N\right)-y}{\left|l\left(\tilde{s}_i^N\right)-y\right|^2}\cdot n\left(l\left(\tilde{s}_i^N\right)\right)\right)\prod_{j=1}^{k-1}dy_{2j}\\
&\sim\left(\frac{|\partial\Omega|}{N}\right)^{2k-1}\\
&\quad\times\sum_{j_1,\dots,j_{k-1}=1}^N\left(\sum_{i=1}^N\frac{x-l\left(\tilde{s}_i^N\right)}{\left|x-l\left(\tilde{s}_i^N\right)\right|^2}\cdot n(x)\frac{l\left(\tilde{s}_i^N\right)-l\left(s_{j_1}^N\right)}{\left|l\left(\tilde{s}_i^N\right)-l\left(s_{j_1}^N\right)\right|^2}\cdot n\left(l\left(\tilde{s}_i^N\right)\right)\right)\\
&\quad\times\prod_{n=1}^{k-2}\left(\sum_{i=1}^N\frac{l\left(s_{j_n}^N\right)-l\left(\tilde{s}_i^N\right)}{\left|l\left(s_{j_n}^N\right)-l\left(\tilde{s}_i^N\right)\right|^2}\cdot n\left(l\left(s_{j_n}^N\right)\right)\frac{l\left(\tilde{s}_i^N\right)-l\left(s_{j_{n+1}}^N\right)}{\left|l\left(\tilde{s}_i^N\right)-l\left(s_{j_{n+1}}^N\right)\right|^2}\cdot n\left(l\left(\tilde{s}_i^N\right)\right)\right)\\
&\quad\times\left(\sum_{i=1}^N\frac{l\left(s_{j_{k-1}}^N\right)-l\left(\tilde{s}_i^N\right)}{\left|l\left(s_{j_{k-1}}^N\right)-l\left(\tilde{s}_i^N\right)\right|^2}\cdot n\left(l\left(s_{j_{k-1}}^N\right)\right)\frac{l\left(\tilde{s}_i^N\right)-y}{\left|l\left(\tilde{s}_i^N\right)-y\right|^2}\cdot n\left(l\left(\tilde{s}_i^N\right)\right)\right).
\end{aligned}$$

Further discretizing in $x=l(s)$ and $y=l(s_*)$, with $s,s_*\in[0,|\partial\Omega|]$, we deduce that, as $N\to\infty$,

(4.15)

$$\left(K_{2k}(x,y)-\left(\frac{|\partial\Omega|}{N}\right)^{2k-1}\sum_{i,j=1}^N\mathbb{1}_{\left[\theta_i^N,\theta_{i+1}^N\right)}(s)\left(\left(\tilde{A}_NA_N\right)^k\right)_{ij}\mathbb{1}_{\left[\theta_j^N,\theta_{j+1}^N\right)}(s_*)\right)\to0$$

in $L^\infty_{x,y}(\partial\Omega\times\partial\Omega)$, where the θ_i^N's are defined in (2.10).

Now, in view of Corollary 4.4 and Proposition 4.5, it holds that, denoting by e^j the jth vector of the canonical basis of $\mathbb{R}^N$ and interpreting matrices in $\mathbb{R}^{N\times N}$ as vectors in $\mathbb{R}^{N^2}$ measured in the ℓ^∞-norm,

$$\begin{aligned}
&\left\|D_N-\tilde{A}_NA_N\right\|_{\ell^\infty}\\
&\qquad=\sup_{1\leqslant j\leqslant N}\left|\left\langle\tilde{A}_NA_Ne^j-\frac{\pi N}{|\partial\Omega|}A_Ne^j\right\rangle+\left\langle\frac{\pi N}{|\partial\Omega|}A_Ne^j-\frac{\pi^2N^2}{|\partial\Omega|^2}e^j\right\rangle\right|\\
&\qquad\leqslant\frac{C}{N}\left\|A_Ne^j\right\|_{\ell^1}+C\left\|e^j\right\|_{\ell^1}=\mathcal{O}\left(N^{-1}\right),\\
&\left\|D_N-\tilde{A}_NA_N\right\|_{\mathcal{L}(\ell^2)}\\
&\qquad=\sup_{\|z\|_{\ell^2}=1}\left|\left\langle\tilde{A}_NA_Nz-\frac{\pi N}{|\partial\Omega|}A_Nz\right\rangle+\left\langle\frac{\pi N}{|\partial\Omega|}A_Nz-\frac{\pi^2N^2}{|\partial\Omega|^2}z\right\rangle\right|\\
&\qquad\leqslant\frac{C}{N}\left\|A_N\right\|_{\mathcal{L}(\ell^2)}+C=\mathcal{O}\left(1\right).
\end{aligned}$$

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Further writing, for each $k \geqslant 1$,

$$D_N^k - \left(\tilde{A}_N A_N\right)^k = \sum_{j=0}^{k-1} D_N^j \left(D_N - \tilde{A}_N A_N\right) \left(\tilde{A}_N A_N\right)^{k-1-j},$$

we obtain

$$\begin{aligned}
&\left\| D_N^k - \left(\tilde{A}_N A_N\right)^k \right\|_{\ell^\infty} \\
&\qquad \leqslant N^{2k-2} \sum_{j=0}^{k-1} N^{-j} \left\|D_N\right\|_{\ell^\infty}^j \left\| D_N - \tilde{A}_N A_N \right\|_{\ell^\infty} \left\|\tilde{A}_N\right\|_{\ell^\infty}^{k-1-j} \left\|A_N\right\|_{\ell^\infty}^{k-1-j} \\
&\qquad = \mathcal{O}\left(N^{2k-3}\right), \\
&\left\| D_N^k - \left(\tilde{A}_N A_N\right)^k \right\|_{\mathcal{L}(\ell^2)} \\
&\qquad \leqslant \sum_{j=0}^{k-1} \left\|D_N\right\|_{\mathcal{L}(\ell^2)}^j \left\| D_N - \tilde{A}_N A_N \right\|_{\mathcal{L}(\ell^2)} \left\|\tilde{A}_N\right\|_{\mathcal{L}(\ell^2)}^{k-1-j} \left\|A_N\right\|_{\mathcal{L}(\ell^2)}^{k-1-j} \\
&\qquad = \mathcal{O}\left(N^{2k-2}\right).
\end{aligned}$$

Therefore, by (4.15), since

$$\begin{aligned}
\sup_{s,s_*\in[0,|\partial\Omega|]} &\left| \sum_{i,j=1}^{N} \mathbb{1}_{\left[\theta_i^N,\theta_{i+1}^N\right)}(s) \left(D_N^k - \left(\tilde{A}_N A_N\right)^k \right)_{ij} \mathbb{1}_{\left[\theta_j^N,\theta_{j+1}^N\right)}(s_*) \right| \\
&\leqslant \left\| D_N^k - \left(\tilde{A}_N A_N\right)^k \right\|_{\ell^\infty},
\end{aligned}$$

we find that, as $N \to \infty$,

$$\left(K_{2k}(x,y) - \left(\frac{|\partial\Omega|}{N}\right)^{2k-1} \sum_{i,j=1}^{N} \mathbb{1}_{\left[\theta_i^N,\theta_{i+1}^N\right)}(s) \left(D_N^k\right)_{ij} \mathbb{1}_{\left[\theta_j^N,\theta_{j+1}^N\right)}(s_*) \right) \to 0$$

in $L^\infty_{x,y}\left(\partial\Omega \times \partial\Omega\right)$.

It follows that, for any fixed $k \geqslant 1$ and $\varepsilon > 0$, provided N is sufficiently large,

$$\begin{aligned}
&\left\|A^{2k}\right\|_{\mathcal{L}\left(L_0^2\right)} + \varepsilon \\
&\geqslant \sup_{\varphi \in L_0^2(\partial\Omega)} \frac{\left\| \left(\frac{|\partial\Omega|}{N}\right)^{2k-1} \sum_{i,j=1}^{N} \mathbb{1}_{\left[\theta_i^N,\theta_{i+1}^N\right)}(s) \left(D_N^k\right)_{ij} \int_{\theta_j^N}^{\theta_{j+1}^N} \varphi(l(s_*)) ds_* \right\|_{L_s^2}}{\left\|\varphi(l(s))\right\|_{L_s^2}} \\
&= \sup_{\varphi \in L_0^2(\partial\Omega)} \frac{\left(\frac{|\partial\Omega|}{N} \sum_{i=1}^{N} \left(\left(\frac{|\partial\Omega|}{N}\right)^{2k-1} \sum_{j=1}^{N} \left(D_N^k\right)_{ij} \int_{\theta_j^N}^{\theta_{j+1}^N} \varphi(l(s_*)) ds_* \right)^2 \right)^{\frac{1}{2}}}{\left\|\varphi(l(s))\right\|_{L_s^2}} \\
&\geqslant \sup_{\substack{z\in\mathbb{R}^N \\ \langle z\rangle = 0}} \frac{|\partial\Omega|^{\frac{1}{2}} \left\| \left(\frac{|\partial\Omega|}{N}\right)^{2k} D_N^k z \right\|_{\ell^2}}{\left\| \sum_{i=1}^{N} z_i \mathbb{1}_{\left[\theta_i^N,\theta_{i+1}^N\right)}(s) \right\|_{L_s^2}} = \sup_{\substack{z\in\mathbb{R}^N \\ \langle z\rangle = 0}} \frac{\left\| \left(\frac{|\partial\Omega|}{N}\right)^{2k} D_N^k z \right\|_{\ell^2}}{\|z\|_{\ell^2}} = \left\| \left(\frac{|\partial\Omega|}{N}\right)^{2k} D_N^k \right\|_{\mathcal{L}\left(\ell_0^2\right)}.
\end{aligned}$$

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Further deducing from estimate (3.27) that there exist $k_0 \geqslant 1$ and $\delta > 0$ such that $\|A^{2k_0}\|_{\mathcal{L}(L^2_0)}^{\frac{1}{2k_0}} \leqslant \pi - 3\delta$, we infer that, setting $\varepsilon > 0$ sufficiently small,

$$\left\| \left(\frac{|\partial\Omega|}{N}\right)^{2k_0} D_N^{k_0} \right\|_{\mathcal{L}(\ell^2_0)} \leqslant \|A^{2k_0}\|_{\mathcal{L}(L^2_0)} + \varepsilon \leqslant (\pi - 3\delta)^{2k_0} + \varepsilon \leqslant (\pi - 2\delta)^{2k_0}$$

for N sufficiently large.

Now, for any $k \geqslant k_0$, we write $k = pk_0 + q$ with positive integral numbers and $0 \leqslant q \leqslant k_0 - 1$. Then, we obtain

$$\begin{aligned}\left\| \left(\left(\frac{|\partial\Omega|}{N}\right)^2 D_N \right)^k \right\|_{\mathcal{L}(\ell^2_0)} &\leqslant \left\| \left(\left(\frac{|\partial\Omega|}{N}\right)^2 D_N \right)^{k_0} \right\|_{\mathcal{L}(\ell^2_0)}^p \left\| \left(\frac{|\partial\Omega|}{N}\right)^2 D_N \right\|_{\mathcal{L}(\ell^2_0)}^q \\ &\leqslant (\pi - 2\delta)^{2(k-q)} \left\| \left(\frac{|\partial\Omega|}{N}\right)^2 D_N \right\|_{\mathcal{L}(\ell^2_0)}^q .\end{aligned}$$

Note that the preceding step could not possibly be performed for $\tilde{A}_N A_N$, for this operator does not in general preserve the subspace $\ell^2_0 \subset \ell^2$, which justifies the introduction of D_N. Further using that $N^{-2} D_N$ is a bounded operator over ℓ^2 uniformly in N, we arrive at, for some fixed constant $C_* > 0$ independent of N and k, and for sufficiently large k,

$$\left\| \left(\left(\frac{|\partial\Omega|}{N}\right)^2 D_N \right)^k \right\|_{\mathcal{L}(\ell^2_0)} \leqslant C_* (\pi - 2\delta)^{2k} \leqslant (\pi - \delta)^{2k},$$

which, upon exchanging the roles of $(s_1^N, \ldots, s_N^N)$ and $(\tilde{s}_1^N, \ldots, \tilde{s}_N^N)$ to obtain an equivalent estimate on $\tilde{D}_N$, concludes the proof of the lemma. ∎

4.4. Solving the discrete system (4.1). Combining the preceding results allows us to get the existence and the uniqueness of the solution to (4.1). For mere convenience of notation, we introduce the matrix

$$B_{N-1,N} := \left(\frac{l(\tilde{s}_i^N) - l(s_j^N)}{\left|l(\tilde{s}_i^N) - l(s_j^N)\right|^2} \cdot \tau\left(l(\tilde{s}_i^N)\right) \right)_{\substack{1 \leqslant i \leqslant N-1 \\ 1 \leqslant j \leqslant N}} .$$

PROPOSITION 4.7. *For any $N \geqslant 2$, consider a well distributed mesh $(s_1^N, \ldots, s_N^N) \in [0, |\partial\Omega|)^N$, $(\tilde{s}_1^N, \ldots, \tilde{s}_N^N) \in [0, |\partial\Omega|)^N$. Then, provided N is sufficiently large, the following problem,*

$$z \in \mathbb{R}^N, \quad \frac{|\partial\Omega|}{N} B_{N-1,N} z = v, \quad \langle z \rangle = \gamma, \tag{4.16}$$

has a unique solution for any given $v \in \mathbb{R}^{N-1}$ and $\gamma \in \mathbb{R}$. Moreover, this solution satisfies

$$\|z\|_{\ell^1} \leqslant \|z\|_{\ell^2} \leqslant C\left(\|v\|_{\ell^2} + |\gamma| + \sqrt{N}\,|\langle v \rangle|\right) \leqslant C\left(\|v\|_{\ell^\infty} + |\gamma| + \sqrt{N}\,|\langle v \rangle|\right) \tag{4.17}$$

for some independent constant $C > 0$.

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Proof. Let the operators $E_N, \tilde{E}_N : \ell^2 \to \ell_0^2$ be defined by

$$E_N z = B_N z - \langle B_N z \rangle \mathbf{1} \quad \text{and} \quad \tilde{E}_N z = \tilde{B}_N z - \left\langle \tilde{B}_N z \right\rangle \mathbf{1}$$

for all $z \in \mathbb{R}^N$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^N$. In view of Corollary 4.4, $N^{-1} E_N$ and $N^{-1} \tilde{E}_N$ are bounded uniformly in both $\mathcal{L}\left(\ell^2\right)$ and $\mathcal{L}\left(\ell_0^2\right)$.

Next, by (4.8) and (4.9) from Proposition 4.5, we obtain that

$$\begin{aligned}
&\left\| \frac{|\partial\Omega|^2}{N^2} \tilde{E}_N E_N z - \left(\frac{|\partial\Omega|^2}{N^2} D_N - \pi^2 \right) z \right\|_{\ell^2} \\
&\quad \leqslant \left\| \frac{|\partial\Omega|^2}{N^2} \tilde{B}_N B_N z - \left(\frac{|\partial\Omega|^2}{N^2} D_N - \pi^2 \right) z \right\|_{\ell^2} \\
&\quad\quad + \frac{|\partial\Omega|^2}{N^2} \left(|\langle B_N z \rangle| \left\| \tilde{B}_N \mathbf{1} \right\|_{\ell^2} + \left| \left\langle \tilde{B}_N B_N z \right\rangle \right| + \left| \left\langle \tilde{B}_N z \right\rangle \right| |\langle B_N z \rangle| \right) \\
&\quad \leqslant \left\| \frac{|\partial\Omega|^2}{N^2} \left(\tilde{B}_N B_N - \tilde{A}_N A_N \right) z + \pi^2 z \right\|_{\ell^2} + \frac{C}{N^2} \|z\|_{\ell^2} \\
&\quad\quad + \frac{|\partial\Omega|}{N} \left| \left\langle \frac{|\partial\Omega|}{N} \tilde{A}_N A_N z - \pi A_N z \right\rangle \right| + \pi \left| \left\langle \frac{|\partial\Omega|}{N} A_N z - \pi z \right\rangle \right| \\
&\quad \leqslant \frac{C}{N} \|z\|_{\ell^2},
\end{aligned}$$

where $C > 0$ denotes various independent constants. Therefore, by Lemma 4.6, we infer that, provided N is sufficiently large,

$$\left\| \left(\frac{|\partial\Omega|^2}{N^2} D_N - \pi^2 \right)^{-1} \frac{|\partial\Omega|^2}{N^2} \tilde{E}_N E_N z - z \right\|_{\ell_0^2} \leqslant \frac{1}{2} \|z\|_{\ell_0^2}$$

for all $z \in \ell_0^2$, which allows us to deduce, using yet another absolutely convergent Neumann series, that the operator $(\frac{|\partial\Omega|^2}{N^2} D_N - \pi^2)^{-1} \frac{|\partial\Omega|^2}{N^2} \tilde{E}_N E_N$ has an inverse in $\mathcal{L}\left(\ell_0^2\right)$, which is uniformly bounded in N and given by

$$\begin{aligned}
&\left(\left(\frac{|\partial\Omega|^2}{N^2} D_N - \pi^2 \right)^{-1} \frac{|\partial\Omega|^2}{N^2} \tilde{E}_N E_N \right)^{-1} \\
&\qquad\qquad = \sum_{k=0}^{\infty} \left(1 - \left(\frac{|\partial\Omega|^2}{N^2} D_N - \pi^2 \right)^{-1} \frac{|\partial\Omega|^2}{N^2} \tilde{E}_N E_N \right)^k .
\end{aligned}$$

It therefore follows that, for large N, the operator $\frac{|\partial\Omega|}{N} E_N \in \mathcal{L}\left(\ell_0^2\right)$ is invertible with

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an inverse uniformly bounded in N (in the operator norm over ℓ_0^2) and given by

$$\left(\frac{|\partial\Omega|}{N}E_N\right)^{-1}$$
$$= \left(\left(\frac{|\partial\Omega|^2}{N^2}D_N - \pi^2\right)^{-1}\frac{|\partial\Omega|^2}{N^2}\tilde{E}_N E_N\right)^{-1}\left(\frac{|\partial\Omega|^2}{N^2}D_N - \pi^2\right)^{-1}\frac{|\partial\Omega|}{N}\tilde{E}_N \in \mathcal{L}\left(\ell_0^2\right).$$

Now, let us define

$$\Phi \ : \quad \mathbb{R}^N \to \mathbb{R}^{N+1}, \qquad z \mapsto \begin{pmatrix} \frac{|\partial\Omega|}{N}B_N z \\ \langle z\rangle \end{pmatrix},$$

and let us suppose that $\Phi z = 0$ for some $z \in \mathbb{R}^N$. In particular, one has that $z \in \ell_0^2$ and $\frac{|\partial\Omega|}{N}E_N z = 0$, whence $z = 0$. Therefore, Φ is an injective linear mapping. In particular, it is bijective from $\mathbb{R}^N$ onto $\operatorname{Im}\Phi$, so that $\dim(\operatorname{Im}\Phi) = N$ and there exist vectors $r_N = (r_N^1, \ldots, r_N^N, r_N^{N+1}) = (r_N', r_N^{N+1}) \in \mathbb{R}^{N+1}$ such that

$$\operatorname{Im}\Phi = \left\{u \in \mathbb{R}^{N+1} \ : \ r_N \cdot u = 0\right\}.$$

Without any loss of generality, we impose that the r_N's satisfy

$$r_N \cdot \begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix} = N\langle r_N'\rangle \geqslant 0 \quad \text{and} \quad \|r_N'\|_{\ell^2} + \left|r_N^{N+1}\right| = 1. \tag{4.18}$$

Observe that there is a unique $z_N \in \mathbb{R}^N$ such that

$$\Phi(z_N) = \begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix} - \frac{r_N' \cdot \mathbf{1}}{r_N \cdot r_N} r_N \in \operatorname{Im}\Phi.$$

On the one hand, since $\frac{|\partial\Omega|}{N}E_N \in \mathcal{L}\left(\ell_0^2\right)$ is invertible, we have the estimate

$$\begin{aligned}
\|z_N\|_{\ell^2} &\leqslant C\left\|\frac{|\partial\Omega|}{N}E_N\left(z_N - \langle z_N\rangle \mathbf{1}\right)\right\|_{\ell_0^2} + |\langle z_N\rangle| \leqslant C\left\|\frac{|\partial\Omega|}{N}E_N\left(z_N\right)\right\|_{\ell_0^2} + C\,|\langle z_N\rangle| \\
&= C\left\|\mathbf{1} - \frac{r_N' \cdot \mathbf{1}}{r_N \cdot r_N}r_N' - \left\langle \mathbf{1} - \frac{r_N' \cdot \mathbf{1}}{r_N \cdot r_N}r_N'\right\rangle \mathbf{1}\right\|_{\ell_0^2} + C\left|\frac{r_N' \cdot \mathbf{1}}{r_N \cdot r_N}r_N^{N+1}\right| \\
&= C\left\|\frac{r_N' \cdot \mathbf{1}}{r_N \cdot r_N}r_N' - \frac{\left(r_N' \cdot \mathbf{1}\right)^2}{N r_N \cdot r_N}\mathbf{1}\right\|_{\ell_0^2} + C\left|\frac{r_N' \cdot \mathbf{1}}{r_N \cdot r_N}r_N^{N+1}\right| \\
&\leqslant C\frac{\|r_N'\|_{\ell^2}^2 + \|r_N'\|_{\ell^2}\left|r_N^{N+1}\right|}{\|r_N\|_{\ell^2}^2} \leqslant \frac{C}{\|r_N\|_{\ell^2}^2},
\end{aligned}$$

where $C > 0$ denotes various constants independent of N, while, on the other hand,

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using (4.8), we find that

$$\frac{1}{\|r_N\|_{\ell^2}^2}\left\|r_N-\langle r'_N\rangle\begin{pmatrix}\mathbf{1}\\0\end{pmatrix}\right\|_{\ell^2}^2=\left|1-\frac{(r'_N\cdot\mathbf{1})^2}{Nr_N\cdot r_N}\right|=\left|\left\langle\mathbf{1}-\frac{r'_N\cdot\mathbf{1}}{r_N\cdot r_N}r'_N\right\rangle\right|$$
$$=\left|\left\langle\frac{|\partial\Omega|}{N}B_N(z_N)\right\rangle\right|\leqslant\frac{C}{N^2}\|z_N\|_{\ell^1}.$$

Combining the preceding estimates yields

$$\left\|r_N-\langle r'_N\rangle\begin{pmatrix}\mathbf{1}\\0\end{pmatrix}\right\|_{\ell^2}^2=\mathcal{O}\left(N^{-2}\right),\tag{4.19}$$

whence, using that $0\leqslant\langle r'_N\rangle\leqslant 1$,

$$\left|\|r_N\|_{\ell^2}-\langle r'_N\rangle\right|=\mathcal{O}\left(N^{-1}\right),\tag{4.20}$$

and, since $\|z\|_{\ell^\infty}\leqslant\sqrt{N}\|z\|_{\ell^2}$, for any $z\in\mathbb{R}^N$,

$$\left\|r_N-\langle r'_N\rangle\begin{pmatrix}\mathbf{1}\\0\end{pmatrix}\right\|_{\ell^\infty}=\mathcal{O}\left(N^{-\frac{1}{2}}\right).$$

In particular, we further deduce that $r_N^{N+1}=\mathcal{O}(N^{-\frac{1}{2}})$. It therefore holds, by (4.18), that $\|r'_N\|_{\ell^2}=1+\mathcal{O}(N^{-\frac{1}{2}})$ and $\|r_N\|_{\ell^2}=1+\mathcal{O}(N^{-\frac{1}{2}})$ so that $\langle r'_N\rangle=1+\mathcal{O}(N^{-\frac{1}{2}})$, as well, by (4.20). On the whole, we conclude that

$$\left\|r_N-\begin{pmatrix}\mathbf{1}\\0\end{pmatrix}\right\|_{\ell^\infty}=\mathcal{O}\left(N^{-\frac{1}{2}}\right).$$

In particular, considering sufficiently large values of N, we henceforth assume that all components of r'_N are uniformly bounded away from zero.

Now, let $v\in\mathbb{R}^{N-1}$ and $\gamma\in\mathbb{R}$ be fixed. There exists a unique v_N such that $\left(\begin{smallmatrix}v\\v_N\\\gamma\end{smallmatrix}\right)\in\operatorname{Im}\Phi$, namely, $v_N=-\frac{1}{r_N^N}\sum_{i=1}^{N-1}r_N^iv_i-\frac{r_N^{N+1}}{r_N^N}\gamma$. With this v_N, we then deduce the existence of $z\in\mathbb{R}^N$ such that $\Phi(z)=\left(\begin{smallmatrix}v\\v_N\\\gamma\end{smallmatrix}\right)$. In particular z is a solution to (4.16) and, by the invertibility of $\frac{|\partial\Omega|}{N}E_N\in\mathcal{L}\left(\ell_0^2\right)$, it holds that

$$\begin{aligned}\|z-\gamma\mathbf{1}\|_{\ell_0^2}&=\|z-\langle z\rangle\mathbf{1}\|_{\ell_0^2}\leqslant C\left\|\frac{|\partial\Omega|}{N}E_N\left(z-\langle z\rangle\mathbf{1}\right)\right\|_{\ell_0^2}\\&=C\left\|\begin{pmatrix}v\\v_N\end{pmatrix}-\gamma\frac{|\partial\Omega|}{N}B_N\mathbf{1}-\left\langle\begin{pmatrix}v\\v_N\end{pmatrix}\right\rangle+\gamma\left\langle\frac{|\partial\Omega|}{N}B_N\mathbf{1}\right\rangle\right\|_{\ell_0^2}\\&\leqslant C\left(\left(\frac{1}{N}\sum_{i=1}^N|v_i|^2\right)^{\frac{1}{2}}+|\gamma|\right)\\&=C\left(\left(\frac{1}{N}\sum_{i=1}^{N-1}|v_i|^2+\frac{1}{N}\left|\frac{1}{r_N^N}\sum_{i=1}^{N-1}r_N^iv_i+\frac{r_N^{N+1}}{r_N^N}\gamma\right|^2\right)^{\frac{1}{2}}+|\gamma|\right)\\&\leqslant C\left(\|v\|_{\ell^2}+\frac{1}{\sqrt{N}}\left|\sum_{i=1}^{N-1}r_N^iv_i\right|+|\gamma|\right).\end{aligned}$$

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As $\|z\|_{\ell^2} - |\gamma| \leqslant \|z - \gamma\mathbf{1}\|_{\ell^2_0}$ and

$$\begin{aligned}\frac{1}{\sqrt{N}}\left|\sum_{i=1}^{N-1} r_N^i v_i\right| &\leqslant \frac{1}{\sqrt{N}}\left|\sum_{i=1}^{N-1}\left(r_N^i - \langle r_N'\rangle\right) v_i\right| + \frac{N-1}{\sqrt{N}}\left|\langle r_N'\rangle\langle v\rangle\right| \\ &\leqslant \sqrt{\frac{N-1}{N}}\,\|v\|_{\ell^2}\left(\sum_{i=1}^{N-1}\left(r_N^i - \langle r_N'\rangle\right)^2\right)^{\frac{1}{2}} + \frac{N-1}{\sqrt{N}}\left|\langle r_N'\rangle\langle v\rangle\right| \\ &= \mathcal{O}\left(N^{-\frac{1}{2}}\right)\|v\|_{\ell^2} + \mathcal{O}\left(N^{\frac{1}{2}}\right)|\langle v\rangle|,\end{aligned}$$

where we have used (4.19), we conclude that

$$\|z\|_{\ell^2} \leqslant C\left(\|v\|_{\ell^2} + |\gamma| + \sqrt{N}\,|\langle v\rangle|\right).$$

Finally, concerning the uniqueness of a solution to (4.16), let us consider z and $\tilde{z}$ two solutions of (4.16). Then, $\Phi(z-\tilde{z}) = \begin{pmatrix} 0_{\mathbb{R}^{N-1}} \\ x \\ 0 \end{pmatrix}$ (for some $x \in \mathbb{R}$) belongs to $\operatorname{Im}\Phi$ if only if $x = 0$. By injectivity of Φ, we conclude that necessarily $z = \tilde{z}$, thereby completing the proof of the proposition. ∎

5. Weak convergence of discretized singular integral operators. The results in this section will serve to show that $(u_R - u^N_{\text{app}}) \cdot n|_{\partial\Omega}$ vanishes in a weak sense.

The coming proposition establishes some weak convergence of the discretization of the singular integral operator B defined in (3.1).

PROPOSITION 5.1. *For any $N \geqslant 2$, consider a well distributed mesh $(s_1^N, \ldots, s_N^N) \in [0, |\partial\Omega|)^N$, $(\tilde{s}_1^N, \ldots, \tilde{s}_N^N) \in [0, |\partial\Omega|)^N$ satisfying* (2.9) *and, according to Proposition* 4.7*, consider the solution $\gamma^N = (\gamma_1^N, \ldots, \gamma_N^N) \in \mathbb{R}^N$ to the system* (4.1) *for some periodic function $f \in C^\kappa([0, |\partial\Omega|])$, where $\kappa \geqslant 2$ is introduced in* (2.9)*, with zero mean value $\int_0^{|\partial\Omega|} f(s)ds = 0$ and some $\gamma \in \mathbb{R}$. We define the approximations*

$$\begin{aligned} f^N_{\text{app}}(s) &:= \frac{1}{N}\sum_{j=1}^N \gamma_j^N \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2} \cdot \tau(l(s)), \\ g^N_{\text{app}}(s) &:= \frac{1}{N}\sum_{j=1}^N \gamma_j^N \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2} \cdot n(l(s)). \end{aligned} \tag{5.1}$$

Then, for any periodic test function $\varphi \in C^\infty([0, |\partial\Omega|])$,

$$\left|\int_0^{|\partial\Omega|}(f^N_{\text{app}} - f)\varphi\right| \leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa} + |\gamma|\right)\|\varphi\|_{C^{\kappa+1}},$$

$$\left|\int_0^{|\partial\Omega|}(g^N_{\text{app}} - AB^{-1}f - \pi\gamma H\cdot\tau)\varphi\right| \leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa} + |\gamma|\right)\|\varphi\|_{L^2},$$

where we identify the variable x with the variable s whenever $x = l(s) \in \partial\Omega$, the singular integrals are defined in the sense of Cauchy's principal value, and H is given by the limiting values from Ω of the harmonic vector field defined by (1.6).

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Proof. Let $\varphi \in C^\infty([0, |\partial\Omega|])$ be a periodic test function. Then, we decompose

$$\begin{aligned}\int_0^{|\partial\Omega|} (f_{\rm app}^N - f)\varphi = &\left(\int_0^{|\partial\Omega|} f_{\rm app}^N \varphi - \frac{|\partial\Omega|}{N}\sum_{i=1}^N f_{\rm app}^N(\tilde s_i^N)\varphi(\tilde s_i^N)\right)\\ &- \left(\int_0^{|\partial\Omega|} f\varphi - \frac{|\partial\Omega|}{N}\sum_{i=1}^N f(\tilde s_i^N)\varphi(\tilde s_i^N)\right)\\ &+ \frac{|\partial\Omega|}{N}\sum_{i=1}^{N-1}\left(f_{\rm app}^N(\tilde s_i^N) - f(\tilde s_i^N)\right)\varphi(\tilde s_i^N)\\ &+ \frac{|\partial\Omega|}{N}\left(f_{\rm app}^N(\tilde s_N^N) - f(\tilde s_N^N)\right)\varphi(\tilde s_N^N)\\ =:&\, D_1 + D_2 + D_3 + D_4.\end{aligned}$$

It is readily seen that D_3 is null for $f_{\rm app}^N(\tilde s_i^N) = f(\tilde s_i^N)$, for all $i = 1, \dots, N-1$, by construction (see (4.1)).

Next, note that D_2 is the error of approximation of the integral $\int_0^{|\partial\Omega|} f\varphi$ by its Riemann sum. Therefore, a direct application of Corollary A.2 yields

$$|D_2| \leqslant \frac{C}{N^\kappa}\|f\varphi\|_{C^\kappa} \leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}\|\varphi\|_{C^\kappa}. \tag{5.2}$$

As for the term D_1, it is first rewritten, exploiting (3.5) and (4.14), as

$$\begin{aligned}D_1 &= \int_0^{|\partial\Omega|} f_{\rm app}^N\varphi - \frac{|\partial\Omega|}{N}\sum_{i=1}^N f_{\rm app}^N(\tilde s_i^N)\varphi(\tilde s_i^N)\\ &= \frac{1}{N}\sum_{j=1}^N \gamma_j^N \int_0^{|\partial\Omega|} \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2}\cdot\tau(l(s))\varphi(s)ds\\ &\quad - \frac{|\partial\Omega|}{N^2}\sum_{i,j=1}^N \gamma_j^N \frac{l(\tilde s_i^N) - l(s_j^N)}{\left|l(\tilde s_i^N) - l(s_j^N)\right|^2}\cdot\tau\left(l(\tilde s_i^N)\right)\varphi(\tilde s_i^N)\\ &= \int_0^{|\partial\Omega|} F(s)ds - \frac{|\partial\Omega|}{N}\sum_{i=1}^N F(\tilde s_i^N)\\ &\quad + \frac{1}{N}\sum_{j=1}^N \gamma_j^N \underbrace{\left(\int_0^{|\partial\Omega|} \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2}\cdot\tau(l(s))\,ds\right)}_{=0 \quad \text{by (3.5)}}\varphi(s_j^N)\\ &\quad - \frac{1}{N}\sum_{j=1}^N \gamma_j^N \underbrace{\left(\frac{|\partial\Omega|}{N}\sum_{i=1}^N \frac{l(\tilde s_i^N) - l(s_j^N)}{\left|l(\tilde s_i^N) - l(s_j^N)\right|^2}\cdot\tau\left(l(\tilde s_i^N)\right)\right)}_{=\mathcal{O}(N^{-\kappa}) \quad \text{in } \ell^\infty \text{ by (4.14)}}\varphi(s_j^N)\\ &= \int_0^{|\partial\Omega|} F(s)ds - \frac{|\partial\Omega|}{N}\sum_{i=1}^N F(\tilde s_i^N) + \mathcal{O}\left(\frac{\left\|\gamma^N\right\|_{\ell^1}\|\varphi\|_{L^\infty}}{N^\kappa}\right),\end{aligned}$$

where

$$F(s) = \frac{1}{N}\sum_{j=1}^N \gamma_j^N \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2}\cdot\tau(l(s))\left(\varphi(s) - \varphi(s_j^N)\right).$$

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Note that the integrand

$$s \mapsto \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2} \cdot \tau(l(s)) \left(\varphi(s) - \varphi(s_j^N)\right)$$

above is now regular, thus assuring that the corresponding Riemann sums converge. It therefore follows from Corollary A.2 that

$$\begin{aligned}
|D_1| &\leqslant \frac{C}{N^\kappa} \|F\|_{C^\kappa} + \frac{C}{N^\kappa} \left\|\gamma^N\right\|_{\ell^1} \|\varphi\|_{L^\infty} \\
&\leqslant \frac{C}{N^\kappa} \left\|\gamma^N\right\|_{\ell^1} \\
&\quad \times \sup_{s_* \in [0,|\partial\Omega|]} \left\| \frac{(s - s_*)(l(s) - l(s_*))}{|l(s) - l(s_*)|^2} \cdot \tau(l(s)) \right\|_{C_s^\kappa([0,|\partial\Omega|])} \|\varphi'\|_{C^\kappa} \\
&\quad + \frac{C}{N^\kappa} \left\|\gamma^N\right\|_{\ell^1} \|\varphi\|_{L^\infty} \\
&\leqslant \frac{C}{N^\kappa} \left\|\gamma^N\right\|_{\ell^1} \|\varphi\|_{C^{\kappa+1}} .
\end{aligned}$$

Then, further utilizing estimate (4.17), Corollary A.2, that $\kappa \geqslant \frac{1}{2}$, and the fact that f has zero mean value, we infer

$$\begin{aligned}
|D_1| &\leqslant \frac{C}{N^\kappa} \left(\|f\|_{L^\infty} + |\gamma| + \sqrt{N} \left| \frac{1}{N-1} \sum_{i=1}^{N-1} f(\tilde{s}_i^N) \right| \right) \|\varphi\|_{C^{\kappa+1}} \\
&\leqslant \frac{C}{N^\kappa} \left(\|f\|_{L^\infty} + |\gamma| + \sqrt{N} \left| \int_0^{|\partial\Omega|} f(s) ds - \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} f(\tilde{s}_i^N) \right| \right) \|\varphi\|_{C^{\kappa+1}} \qquad (5.3) \\
&\leqslant \frac{C}{N^\kappa} \left(\|f\|_{C^\kappa} + |\gamma| \right) \|\varphi\|_{C^{\kappa+1}} .
\end{aligned}$$

Finally, regarding D_4, recalling that, by (4.1),

$$\begin{aligned}
\sum_{i=1}^{N-1} f(\tilde{s}_i^N) &= \frac{1}{N} \sum_{i=1}^{N-1} \sum_{j=1}^{N} \gamma_j^N \frac{l(\tilde{s}_i^N) - l(s_j^N)}{\left|l(\tilde{s}_i^N) - l(s_j^N)\right|^2} \cdot \tau\left(l(\tilde{s}_i^N)\right) \\
&= \left\langle B_N \gamma^N \right\rangle - \frac{1}{N} \sum_{j=1}^{N} \gamma_j^N \frac{l(\tilde{s}_N^N) - l(s_j^N)}{\left|l(\tilde{s}_N^N) - l(s_j^N)\right|^2} \cdot \tau\left(l(\tilde{s}_N^N)\right) \\
&= \left\langle B_N \gamma^N \right\rangle - f_{\text{app}}^N\left(\tilde{s}_N^N\right),
\end{aligned}$$

we find

$$\begin{aligned}
D_4 &= \frac{|\partial\Omega|}{N} \left(f_{\text{app}}^N(\tilde{s}_N^N) - f(\tilde{s}_N^N) \right) \varphi(\tilde{s}_N^N) \\
&= \left(\left\langle \frac{|\partial\Omega|}{N} B_N \gamma^N \right\rangle - \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} f(\tilde{s}_i^N) \right) \varphi(\tilde{s}_N^N) \\
&= \left(\left\langle \frac{|\partial\Omega|}{N} B_N \gamma^N \right\rangle + \int_0^{|\partial\Omega|} f(s) ds - \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} f(\tilde{s}_i^N) \right) \varphi(\tilde{s}_N^N).
\end{aligned}$$

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Hence, utilizing (4.8) and Corollary A.2 again,

$$|D_4| \leqslant \left(\frac{C}{N^\kappa}\left\|\gamma^N\right\|_{\ell^1} + \frac{C}{N^\kappa}\|f\|_{C^\kappa}\right)\|\varphi\|_{L^\infty}.$$

Therefore, repeating the control of $\|\gamma^N\|_{\ell^1}$ performed in (5.3) and based on (4.17), we arrive at

$$|D_4| \leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa} + |\gamma|\right)\|\varphi\|_{L^\infty}. \tag{5.4}$$

On the whole, since $D_3 = 0$, combining (5.2), (5.3), and (5.4), we deduce that

$$\left|\int_0^{|\partial\Omega|}(f^N_{\text{app}} - f)\varphi\right| \leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa} + |\gamma|\right)\|\varphi\|_{C^{\kappa+1}}, \tag{5.5}$$

which concludes the convergence estimate for f^N_{app}.

As for g^N_{app}, we first write

$$\int_0^{|\partial\Omega|} g^N_{\text{app}}\varphi = \frac{1}{N}\sum_{j=1}^N \gamma_j^N A^*\varphi\left(l\left(s_j^N\right)\right),$$

where we identify $\varphi(x)$ with $\varphi(s)$ whenever $x = l(s) \in \partial\Omega$.

Now, recall from section 3 that $B \in \mathcal{L}\left(L^2_0\right)$ is invertible. Moreover, it is readily seen, by (3.3) and (3.6), that the adjoint operators of A and B over $L^2_0(\partial\Omega)$ are, respectively, given by

$$\begin{aligned} A^\#\varphi(x) &= A^*\varphi(x) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} A^*\varphi(y)dy, \\ B^\#\varphi(x) &= B^*\varphi(x) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} B^*\varphi(y)dy. \end{aligned}$$

It follows that $B^\# \in \mathcal{L}\left(L^2_0\right)$ is invertible, as well.

We then obtain, by (3.25) and utilizing the adjointness (3.3) and (3.6) of A and B in the $L^2(\partial\Omega)$ structure,

$$\begin{aligned} \int_0^{|\partial\Omega|} g^N_{\text{app}}\varphi &= \frac{1}{N}\sum_{j=1}^N \gamma_j^N A^\#\varphi\left(l\left(s_j^N\right)\right) + \frac{\gamma}{|\partial\Omega|}\int_{\partial\Omega} A^*\varphi \\ &= \frac{1}{N}\sum_{j=1}^N \gamma_j^N \left[B^*\left(B^\#\right)^{-1}A^\#\varphi\right]\left(l\left(s_j^N\right)\right) \\ &\quad - \frac{\gamma}{|\partial\Omega|}\int_{\partial\Omega} B^*\left(B^\#\right)^{-1}A^\#\varphi + \frac{\gamma}{|\partial\Omega|}\int_{\partial\Omega} A^*\varphi \\ &= \int_0^{|\partial\Omega|} f^N_{\text{app}}\left(B^\#\right)^{-1}A^\#\varphi - \frac{\gamma}{|\partial\Omega|}\int_{\partial\Omega} B^{-1}B1A^\#\varphi + \frac{\gamma}{|\partial\Omega|}\int_{\partial\Omega} A^*\varphi \\ &= \int_0^{|\partial\Omega|} f^N_{\text{app}}\left(B^\#\right)^{-1}A^\#\varphi + \frac{\gamma}{|\partial\Omega|}\int_{\partial\Omega} \underbrace{\left[1 - B^{-1}B1\right]}_{=|\partial\Omega|H\cdot\tau \text{ by (3.25)}} A^*\varphi \\ &\quad + \frac{\gamma}{|\partial\Omega|^2}\underbrace{\left[\int_{\partial\Omega} B^{-1}B1\right]}_{=0}\left[\int_{\partial\Omega} A^*\varphi\right] \end{aligned}$$

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$$= \int_0^{|\partial\Omega|} f_{\text{app}}^N \left(B^{\#}\right)^{-1} A^{\#}\varphi + \gamma \int_{\partial\Omega} \underbrace{A[H\cdot\tau]}_{=\pi H\cdot\tau \text{ by } (3.25)} \varphi$$

$$= \int_0^{|\partial\Omega|} \left(f_{\text{app}}^N - f\right) \left(B^{\#}\right)^{-1} A^{\#}\varphi + \int_0^{|\partial\Omega|} AB^{-1} f\varphi + \pi\gamma \int_{\partial\Omega} H\cdot\tau\varphi,$$

where, as already emphasized, the values of H on $\partial\Omega$ are given by its limiting values from inside Ω. Therefore, according to (5.5), we obtain that

$$\left|\int_0^{|\partial\Omega|} (g_{\text{app}}^N - AB^{-1}f - \pi\gamma H\cdot\tau)\varphi\right| \leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa} + |\gamma|\right)\left\|\left(B^{\#}\right)^{-1} A^{\#}\varphi\right\|_{C^{\kappa+1}}.$$

There only remains to estimate the regularity of $\psi = \left(B^{\#}\right)^{-1} A^{\#}\varphi \in L_0^2(\partial\Omega)$. To this end, we rewrite

$$B^*\psi = A^*\varphi + \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\left(B^*\psi - A^*\varphi\right),$$

whence, utilizing (3.5) and (3.11), we infer that

$$\pi^2\psi = A^{*2}\psi + A^*B^*\varphi.$$

Further recalling that A^* is a regularizing operator, for it has a smooth kernel, we deduce that

$$\left\|\left(B^{\#}\right)^{-1} A^{\#}\varphi\right\|_{C^{\kappa+1}} = \|\psi\|_{C^{\kappa+1}} \leqslant C\|\varphi\|_{L^2},$$

which concludes the proof of the proposition. ∎

Remark. Note that the use of (4.14), which is a consequence of Lemma 4.1, to estimate D_1 in the above proof is the reason why it is not possible to relax condition (2.9) for a well distributed mesh if one aims at a convergence rate $\mathcal{O}\left(N^{-\kappa}\right)$ in Theorem 2.1.

6. Proof of Theorem 2.1. We proceed now to the demonstration of our main result—Theorem 2.1—on the approximation of the boundary of an exterior domain by point vortices in system (1.7).

First, for given $\omega \in C_c^{0,\alpha}$ and $\gamma \in \mathbb{R}$, recall that the full plane flow $u_P \in C^1\left(\overline{\Omega}\right)$ is obtained from (2.2) and that the $|\partial\Omega|$-periodic function $f \in C^\infty([0,|\partial\Omega|])$, which has zero mean value, is defined by $f(s) = 2\pi[u_P\cdot n](l(s))$ for all $s\in[0,|\partial\Omega|]$. Therefore, with this given f, according to Proposition 4.7, we find a unique solution $\gamma^N \in \mathbb{R}^N$ of (4.1).

Next, the approximate flow u_{app}^N is introduced by (2.7) and verifies

$$u_{\text{app}}^N(x)\cdot n(x) = -\frac{1}{2\pi} f_{\text{app}}^N(s),$$
$$u_{\text{app}}^N(x)\cdot\tau(x) = \frac{1}{2\pi} g_{\text{app}}^N(s),$$

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where $x = l(s) \in \partial\Omega$ and $f_{\rm app}^N$, $g_{\rm app}^N$ are defined by (5.1). Utilizing identity (3.35) to rewrite the discrete Biot–Savart kernel of $u_{\rm app}^N$, we find that

$$\begin{aligned} u_{\rm app}^N(x) &= \frac{1}{2\pi^2}\sum_{j=1}^N \frac{\gamma_j^N}{N} \\ &\quad\times\left(\int_{\partial\Omega} \frac{x_j^N - z}{\left|x_j^N - z\right|^2}\cdot\tau(z)\frac{x-z}{|x-z|^2}dz - \int_{\partial\Omega}\frac{x_j^N - z}{\left|x_j^N - z\right|^2}\cdot n(z)\frac{(x-z)^\perp}{|x-z|^2}dz\right) \\ &= -\frac{1}{2\pi^2}\int_0^{|\partial\Omega|} f_{\rm app}^N(s_*)\frac{x - l(s_*)}{\left|x - l(s_*)\right|^2}ds_* \\ &\quad+\frac{1}{2\pi^2}\int_0^{|\partial\Omega|} g_{\rm app}^N(s_*)\frac{(x-l(s_*))^\perp}{|x-l(s_*)|^2}ds_*, \quad \text{on } \Omega. \end{aligned}$$

Furthermore, recall that, according to (3.26) and (3.36), the remainder flow u_R can be expressed as

$$\begin{aligned} u_R(x) = &\frac{1}{2\pi^2}\int_{\partial\Omega}\frac{(x-y)^\perp}{|x-y|^2}\left[AB^{-1}f + \gamma\pi H\cdot\tau\right](y)dy \\ &- \frac{1}{2\pi^2}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}f(y)dy, \quad \text{on } \Omega, \end{aligned}$$

whereby

$$\begin{aligned} \left(u_R - u_{\rm app}^N\right)(x) = &\frac{1}{2\pi^2}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}\left(f_{\rm app}^N - f\right)(y)dy \\ &+\frac{1}{2\pi^2}\int_{\partial\Omega}\frac{(x-y)^\perp}{|x-y|^2}\left(AB^{-1}f + \gamma\pi H\cdot\tau - g_{\rm app}^N\right)(y)dy, \quad \text{on } \Omega. \end{aligned}$$

Therefore, in view of Proposition 5.1, we deduce that, for any fixed $x \in \Omega$,

$$\begin{aligned} \left|\left(u_R - u_{\rm app}^N\right)(x)\right| &\leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa} + |\gamma|\right)\left\|\frac{x-y}{|x-y|^2}\right\|_{C_y^{\kappa+1}} \\ &\leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa} + |\gamma|\right)\sup_{y\in\partial\Omega}\left(\frac{1}{|x-y|} + \frac{1}{|x-y|^{\kappa+2}}\right), \end{aligned}$$

where the constant $C > 0$ is independent of x, ω, and γ. Since the support of ω is bounded away from $\partial\Omega$, it holds that

$$\begin{aligned} \|f\|_{C^\kappa} &\leqslant C\sup_{x\in\partial\Omega}\int_{\mathbb{R}^2}\left(\frac{1}{|x-y|} + \frac{1}{|x-y|^{\kappa+1}}\right)|\omega(y)|\,dy \\ &\leqslant \frac{C}{\operatorname{dist}(\operatorname{supp}\omega, \partial\Omega)}\left(1 + \frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)^\kappa}\right)\|\omega\|_{L^1(\mathbb{R}^2)}. \end{aligned}$$

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It follows that, for any closed set $K \subset \Omega$,

$$\|u_R - u^N_{\mathrm{app}}\|_{L^\infty(K)} \leqslant \frac{C}{N^\kappa}\left(\frac{1}{\operatorname{dist}(K,\partial\Omega)} + \frac{1}{\operatorname{dist}(K,\partial\Omega)^{\kappa+2}}\right) \times \left(\left(\frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)} + \frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)^{\kappa+1}}\right)\|\omega\|_{L^1(\mathbb{R}^2)} + |\gamma|\right),$$

which, extending the above estimate to all $\omega \in L^1_c(\Omega)$ by a standard density argument, concludes the proof of the theorem.

7. Proofs of dynamic theorems. We give now complete justifications of Theorems 2.2 and 2.3. Both results heavily rely on the static approximation result Theorem 2.1.

7.1. Wellposedness of vortex approximation. We have already mentioned, earlier in section 2.2, that a classical estimate (see, e.g., [24, Lemma 4.2]) shows that the support in x of the classical solution ω to (2.11) remains uniformly bounded away from the boundary $\partial\Omega$, i.e., $\omega \in C^1_c([0,t_1]\times\Omega)$. We will need to adapt this estimate to the vortex approximation (2.12) and therefore, for later use, we begin by reproducing here this control on the support of ω.

LEMMA 7.1. *Let $\omega \in C^1_c\left([0,t_1]\times\overline{\Omega}\right)$ be the unique classical solution to* (2.11). *Then, it holds that, for all $t\in[0,t_1]$,*

$$\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega) \geqslant C' e^{-C\int_0^t \|u\|_{C^1(\Omega)}} \operatorname{dist}(\operatorname{supp}\omega_0,\partial\Omega) \tag{7.1}$$

for some independent constants $C, C' > 0$. In particular, it follows that $\omega \in C^1_c([0,t_1]\times\Omega)$.

Proof. Recall that a *characteristic curve* (or simply a *characteristic*) for (2.11) is a function

$$X(s,x) \in C^1\left([0,t_2]\times\Omega;\Omega\right),$$

solving the differential equation

$$\frac{dX}{ds} = u(s,X) \tag{7.2}$$

for some given initial data $X(0,x)=x$ and some existence time $0 < t_2 \leqslant t_1$ (t_2 may a priori depend on x). Since the velocity field u is of class C^1, standard results from the theory of ordinary differential equations (see [6], for instance) guarantee the existence, uniqueness, and regularity of such characteristics for any given initial data.

We show below that characteristics actually exist up to time t_1, i.e., that it is always possible to take $t_2 = t_1$. This is a consequence of the fact that the flow u is tangent to the boundary $\partial\Omega$, and it does not otherwise hold for general flows. Indeed, loosely speaking, it is easy to force $t_2 < t_1$ by creating a constant flow that takes particles and crashes them into the boundary arbitrarily fast (take a constant vector field of large magnitude pointing from the initial data x straight into $\partial\Omega$).

It is also easy to see that a tangency condition on u should be enough to guarantee that characteristics never hit the boundary. This follows from the uniqueness of

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solutions to the above differential equation. Indeed, it suffices to note that a tangent flow produces trajectories that remain on $\partial\Omega$ provided they start on the boundary. Therefore, a characteristic can only hit the boundary if it is on the boundary all along.

The estimate we now provide is more precise and yields an accurate control on the distance from the characteristic to the boundary $\operatorname{dist}(X(s,x),\partial\Omega)$ by exploiting the tangency of u.

Employing the conformal map $T:\Omega\to\{|x|>1\}$ used in (1.8) and defining the curves

$$Y(s,T(x)) := T(X(s,x)) \in C^1([0,t_2]\times\Omega;\{|x|>1\}),$$

we obtain solutions of the differential equation

$$\frac{dY}{ds} = DT\left(T^{-1}Y\right) u\left(s,T^{-1}Y\right)$$

for the initial data $Y(0,T(x)) = T(x)$. Now, recalling that $DT^t(x)T(x)$ is normal to $\partial\Omega$ at $x\in\partial\Omega$ (see (1.9); this property is easily obtained by differentiating the relation $|T(x)|=1$ on $\partial\Omega$) and that $u(s,x)$ is tangent to $\partial\Omega$ at the same location, we compute that

$$\begin{aligned}
\frac{d(|Y|-1)}{ds} &= DT\left(T^{-1}Y\right)u\left(s,T^{-1}Y\right)\cdot\frac{Y}{|Y|} = u^t\left(s,T^{-1}Y\right)DT^t\left(T^{-1}Y\right)\frac{Y}{|Y|}\\
&= \left(u^t\left(s,T^{-1}Y\right)DT^t\left(T^{-1}Y\right) - u^t\left(s,T^{-1}\frac{Y}{|Y|}\right)DT^t\left(T^{-1}\frac{Y}{|Y|}\right)\right)\frac{Y}{|Y|}\\
&\geqslant -C_T\|u\|_{C^1(\Omega)}\left|Y-\frac{Y}{|Y|}\right| = -C_T\|u\|_{C^1(\Omega)}(|Y|-1),
\end{aligned}\tag{7.3}$$

where the constant $C_T>0$ only depends on T. It follows that

$$|Y(s,T(x))|-1 \geqslant e^{-C_T\int_0^s\|u\|_{C^1(\Omega)}}(|T(x)|-1),$$

whereby, for any $x_0\in\partial\Omega$,

$$\begin{aligned}
&|X(s,x)-x_0|\\
&\quad\geqslant C_T'|T(X(s,x))-T(x_0)|\\
&\quad\geqslant C_T'e^{-C_T\int_0^s\|u\|_{C^1(\Omega)}}(|T(x)|-1) = C_T'e^{-C_T\int_0^s\|u\|_{C^1(\Omega)}}\left|T(x)-\frac{T(x)}{|T(x)|}\right|\\
&\quad\geqslant C_T''e^{-C_T\int_0^s\|u\|_{C^1(\Omega)}}\left|x-T^{-1}\left(\frac{T(x)}{|T(x)|}\right)\right| \geqslant C_T''e^{-C_T\int_0^s\|u\|_{C^1(\Omega)}}\operatorname{dist}(x,\partial\Omega)
\end{aligned}$$

with some constants $C_T', C_T''>0$ only depending on T. Further minimizing the above left-hand side over $x_0\in\partial\Omega$, we finally conclude that, for any $x\in\Omega$,

$$\operatorname{dist}(X(s,x),\partial\Omega) \geqslant C_T''e^{-C_T\int_0^s\|u\|_{C^1(\Omega)}}\operatorname{dist}(x,\partial\Omega),\tag{7.4}$$

which implies, in particular, that $X(t_2,x)\in\Omega$ for all $x\in\Omega$. Therefore, by continuing the characteristic beyond the existence time t_2 (which is always possible because u is bounded on $[0,t_1]\times\Omega$; see [6, Chapter 1, Theorem 4.1], for instance), we may always assume that $t_2 = t_1$.

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Now, for any fixed $0 \leqslant t \leqslant t_1$, it holds that the mapping $X(t,\cdot) : \Omega \to \Omega$ is a C^1-diffeomorphism preserving the Lebesgue measure, for u is solenoidal (see [6, Chapter 1, Theorem 7.2]). We denote its inverse by $X^{-1}(t,\cdot) : \Omega \to \Omega$. In particular, recasting the transport equation (2.11) in Lagrangian coordinates

$$\begin{cases} \dfrac{d}{dt}\omega\left(t, X(t,x)\right) = 0, \\ \omega(t=0) = \omega_0, \end{cases}$$

we deduce that

$$\omega\left(t, X(t,x)\right) = \omega_0(x) \quad \text{and} \quad \omega(t,x) = \omega_0\left(X^{-1}(t,x)\right)$$

for all $0 \leqslant t \leqslant t_1$ and $x \in \Omega$. We therefore conclude from (7.4) that

$$\begin{aligned} \operatorname{dist}\left(\operatorname{supp}\omega, \partial\Omega\right) &= \inf_{\substack{x\in\Omega \\ \omega(t,x)\neq 0}} \operatorname{dist}\left(x, \partial\Omega\right) \\ &\geqslant C_T'' e^{-C_T \int_0^t \|u\|_{C^1(\Omega)}} \inf_{\substack{x\in\Omega \\ \omega_0\left(X^{-1}(t,x)\right)\neq 0}} \operatorname{dist}\left(X^{-1}(t,x), \partial\Omega\right) \\ &= C_T'' e^{-C_T \int_0^t \|u\|_{C^1(\Omega)}} \operatorname{dist}\left(\operatorname{supp}\omega_0, \partial\Omega\right), \end{aligned}$$

which, as announced, establishes that $\omega \in C_c^1\left([0,t_1]\times\Omega\right)$ and completes the proof of the lemma. ∎

We move on now to the justification of the wellposedness of the vortex approximation (2.12) asserted in Theorem 2.2. To this end, we begin with a few classical lemmas providing precise estimates on velocity flows. The next result is standard (see [26, Lemmas 4.5 and 4.6], for instance).

LEMMA 7.2. *For any vortex density $\omega \in C_c^1(\Omega)$, one has the estimates*

$$\left\|K_{\mathbb{R}^2}\left[\omega\right]\right\|_{L^\infty(\mathbb{R}^2)} \leqslant C \left\|\omega\right\|_{L^1\cap L^\infty(\mathbb{R}^2)} \tag{7.5}$$

and

$$\left\|\nabla K_{\mathbb{R}^2}\left[\omega\right]\right\|_{L^\infty(\mathbb{R}^2)} \leqslant C\left(1 + \left\|\omega\right\|_{L^1\cap L^\infty(\mathbb{R}^2)} + \left\|\omega\right\|_{L^\infty(\mathbb{R}^2)} \log\left(1 + \left\|\nabla\omega\right\|_{L^\infty(\mathbb{R}^2)}\right)\right) \tag{7.6}$$

for some independent constant $C > 0$.

Proof. The first estimate (7.5) is obtained straightforwardly by isolating the integrable singularity of the Biot–Savart kernel by a ball of fixed radius centered at the singularity and, then, by estimating the contributions of the integrand of $K_{\mathbb{R}^2}[\omega]$ within this ball and on its exterior separately.

The second estimate (7.6) is more delicate. To justify it, we first compute that, for any radii $0 < R_0 < R \leqslant 1$,

$$\begin{aligned} &\nabla K_{\mathbb{R}^2}\left[\omega\right] \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{y^\perp}{|y|^2} \otimes \nabla\omega(x-y)dy \\ &= \frac{1}{2\pi}\int_{\{|y|\leqslant R_0\}} \frac{y^\perp}{|y|^2} \otimes \nabla\omega(x-y)dy + \frac{1}{2\pi}\int_{\{|y|=R_0\}} \frac{1}{|y|^3} \begin{pmatrix} -y_1y_2 & -y_2^2 \\ y_1^2 & y_1y_2 \end{pmatrix} \omega(x-y)dy \end{aligned}$$

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$$
\begin{aligned}
&+\ \frac{1}{2\pi}\int_{\{|y|>R_0\}}\frac{1}{|y|^4}\begin{pmatrix}2y_1y_2 & y_2^2-y_1^2\\ y_2^2-y_1^2 & -2y_1y_2\end{pmatrix}\omega(x-y)dy\\
&=\frac{1}{2\pi}\int_{\{|y|\leqslant R_0\}}\frac{y^\perp}{|y|^2}\otimes\nabla\omega(x-y)dy\\
&\quad+\frac{1}{2\pi}\int_{\{|y|=R_0\}}\frac{1}{|y|^3}\begin{pmatrix}-y_1y_2 & -y_2^2\\ y_1^2 & y_1y_2\end{pmatrix}(\omega(x-y)-\omega(x))\,dy+\begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\omega(x)\\
&\quad+\frac{1}{2\pi}\int_{\{R_0<|y|\leqslant R\}}\frac{1}{|y|^4}\begin{pmatrix}2y_1y_2 & y_2^2-y_1^2\\ y_2^2-y_1^2 & -2y_1y_2\end{pmatrix}(\omega(x-y)-\omega(x))\,dy\\
&\quad+\frac{1}{2\pi}\int_{\{|y|>R\}}\frac{1}{|y|^4}\begin{pmatrix}2y_1y_2 & y_2^2-y_1^2\\ y_2^2-y_1^2 & -2y_1y_2\end{pmatrix}\omega(x-y)dy,
\end{aligned}
$$

and then let $R_0\to 0$ to yield

$$
\begin{aligned}
\nabla K_{\mathbb{R}^2}[\omega]=&\begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\omega(x)\\
&+\frac{1}{2\pi}\int_{\{|y|\leqslant R\}}\frac{1}{|y|^4}\begin{pmatrix}2y_1y_2 & y_2^2-y_1^2\\ y_2^2-y_1^2 & -2y_1y_2\end{pmatrix}(\omega(x-y)-\omega(x))\,dy\\
&+\frac{1}{2\pi}\int_{\{|y|>R\}}\frac{1}{|y|^4}\begin{pmatrix}2y_1y_2 & y_2^2-y_1^2\\ y_2^2-y_1^2 & -2y_1y_2\end{pmatrix}\omega(x-y)dy.
\end{aligned}\tag{7.7}
$$

It follows that

$$
\begin{aligned}
\|\nabla K_{\mathbb{R}^2}[\omega]\|_{L^\infty(\mathbb{R}^2)}&\leqslant C\left(\|\omega\|_{L^1\cap L^\infty(\mathbb{R}^2)}+R\|\nabla\omega\|_{L^\infty(\mathbb{R}^2)}+\log\left(R^{-1}\right)\|\omega\|_{L^\infty(\mathbb{R}^2)}\right)\\
&\leqslant C\left(1+\|\omega\|_{L^1\cap L^\infty(\mathbb{R}^2)}+\|\omega\|_{L^\infty(\mathbb{R}^2)}\log\left(1+\|\nabla\omega\|_{L^\infty(\mathbb{R}^2)}\right)\right),
\end{aligned}
$$

where we optimized the last estimate by setting $R=\frac{1}{1+\|\nabla\omega\|_{L^\infty(\mathbb{R}^2)}}$, which completes the proof. ∎

LEMMA 7.3. *For any vortex density $\omega\in C_c^1(\Omega)$ and any circulation $\gamma\in\mathbb{R}$, one has the estimates*

$$
\left\|u_{\mathrm{app}}^N[\omega,\gamma]\right\|_{L^\infty(K)}\leqslant\frac{C}{\operatorname{dist}(K,\partial\Omega)}\left(\|\omega\|_{L^1\cap L^\infty(\mathbb{R}^2)}+|\gamma|\right)\tag{7.8}
$$

and

$$
\left\|\nabla u_{\mathrm{app}}^N[\omega,\gamma]\right\|_{L^\infty(K)}\leqslant\frac{C}{\operatorname{dist}(K,\partial\Omega)^2}\left(\|\omega\|_{L^1\cap L^\infty(\mathbb{R}^2)}+|\gamma|\right)\tag{7.9}
$$

for some independent constant $C>0$.

Proof. Using (4.17) from Proposition 4.7 and the convergence of Riemann sums for $C^{0,\alpha}$ functions (see, e.g., [1, Lemma 4.1]), one has that, for any compact set $K\subset\Omega$,

$$
\begin{aligned}
&\left\|u_{\mathrm{app}}^N[\omega,\gamma]\right\|_{L^\infty(K)}\\
&\leqslant\frac{C}{\operatorname{dist}(K,\partial\Omega)}\left\|\gamma^N\right\|_{\ell^1}\\
&\leqslant\frac{C}{\operatorname{dist}(K,\partial\Omega)}\left(\|K_{\mathbb{R}^2}[\omega]\cdot n\|_{L^\infty(\partial\Omega)}+|\gamma|+\sqrt{N}\left|\frac{1}{N-1}\sum_{i=1}^{N-1}K_{\mathbb{R}^2}[\omega]\cdot n\left(\tilde{x}_i^N\right)\right|\right)
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \frac{C}{\operatorname{dist}(K,\partial\Omega)}\Bigg(\|K_{\mathbb{R}^2}[\omega]\cdot n\|_{L^\infty(\partial\Omega)} + |\gamma| \\
&\qquad + \sqrt{N}\left|\frac{|\partial\Omega|}{N}\sum_{i=1}^{N} K_{\mathbb{R}^2}[\omega]\cdot n\left(\tilde{x}_i^N\right) - \int_{\partial\Omega} K_{\mathbb{R}^2}[\omega]\cdot n(x)dx\right|\Bigg) \\
&\leqslant \frac{C}{\operatorname{dist}(K,\partial\Omega)}\left(\|K_{\mathbb{R}^2}[\omega]\cdot n\|_{C^{0,\frac{1}{2}}(\partial\Omega)} + |\gamma|\right) \\
&\leqslant \frac{C}{\operatorname{dist}(K,\partial\Omega)}\left(\left\|\int_{\mathbb{R}^2}\left(\frac{1}{|x-y|}+\frac{1}{|x-y|^{\frac{3}{2}}}\right)|\omega(y)|\,dy\right\|_{L^\infty(\partial\Omega)} + |\gamma|\right) \\
&\leqslant \frac{C}{\operatorname{dist}(K,\partial\Omega)}\left(\|\omega\|_{L^1\cap L^\infty(\mathbb{R}^2)} + |\gamma|\right)
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
\left\|\nabla u_{\text{app}}^N[\omega,\gamma]\right\|_{L^\infty(K)} &\leqslant \frac{C}{\operatorname{dist}(K,\partial\Omega)^2}\left\|\gamma^N\right\|_{\ell^1} \\
&\leqslant \frac{C}{\operatorname{dist}(K,\partial\Omega)^2}\left(\|\omega\|_{L^1\cap L^\infty(\mathbb{R}^2)} + |\gamma|\right),
\end{aligned}
$$

which completes the proof of the lemma. ∎

LEMMA 7.4. *For any vortex density $\omega \in C_c^1(\Omega)$ and any circulation $\gamma \in \mathbb{R}$, considering the velocity flow u defined by* (1.8)*, one has the estimates*

$$
\begin{aligned}
\|u\|_{L^\infty(\Omega)} &\leqslant C\left(\|\omega\|_{L^1\cap L^\infty(\Omega)} + |\gamma|\right), \\
\|\nabla u\|_{L^\infty(\Omega)} &\leqslant C\left(\|\omega\|_{L^1\cap L^\infty(\Omega)} + \|\nabla\omega\|_{L^1\cap L^\infty(\Omega)} + |\gamma|\right),
\end{aligned}
\tag{7.10}
$$

and, for all $x,h\in\mathbb{R}^2$ such that $|h|\leqslant 1$ and $[x,x+h]\subset\Omega$,

$$
|u(x+h)-u(x)| \leqslant C\left(|\gamma| + \left(1+|x|^2\right)\|\omega\|_{L^1\cap L^\infty(\Omega)}\right)|h|\,(1-\log|h|)
\tag{7.11}
$$

for some independent constant $C>0$.

Proof. The first estimates (7.10) are straightforward and deduced directly from (1.8), employing (1.10).

Let us therefore focus on the more refined control (7.11). Using that

$$
\frac{|b-b^*|}{|a-b^*|} \leqslant \frac{|b-b^*|}{\left|\frac{b}{|b|}-b^*\right|} = 1+|b|
$$

for any $a,b\in\mathbb{R}^2$ such that $|a|,|b|>1$, we first compute that

$$
\begin{aligned}
|\nabla_x G_\Omega(x,y)| &\leqslant \frac{|DT(x)|\,|T(y)-T(y)^*|}{|T(x)-T(y)|\,|T(x)-T(y)^*|} \leqslant \frac{|DT(x)|\,(1+|T(y)|)}{|T(x)-T(y)|} \\
&\leqslant C_T\frac{1+|y|}{|x-y|} \leqslant C_T\left(1+\frac{1+|x|}{|x-y|}\right), \\
\left|D_x^2 G_\Omega(x,y)\right| &\leqslant \frac{\left|D^2T(x)\right|\,|T(y)-T(y)^*|}{|T(x)-T(y)|\,|T(x)-T(y)^*|}
\end{aligned}
$$

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$$
\begin{aligned}
&+7\frac{|DT(x)|^2\,|T(y)-T(y)^*|\,(|T(x)-T(y)|+|T(x)-T(y)^*|)}{|T(x)-T(y)|^2\,|T(x)-T(y)^*|^2}\\
&\leqslant \frac{\left|D^2T(x)\right|(1+|T(y)|)}{|T(x)-T(y)|}+14\frac{|DT(x)|^2\,(1+|T(y)|)^2}{|T(x)-T(y)|^2}\\
&\leqslant C_T\left(\frac{1+|y|}{|x-y|}+\frac{1+|y|^2}{|x-y|^2}\right)\leqslant C_T'\left(1+\frac{1+|x|^2}{|x-y|^2}\right)
\end{aligned}
$$

for some constants $C_T, C_T' > 0$ only depending on T.

Then, utilizing these estimates on the Green function, we deduce the quasi-Lipschitz estimate, for all $x, h \in \mathbb{R}^2$ such that $|h| \leqslant 1$ and $[x, x+h] \subset \Omega$:

$$
\begin{aligned}
&|u(x+h)-u(x)|\\
&\lesssim |\alpha||h| + \int_\Omega |\nabla_x G_\Omega(x+h,y)-\nabla_x G_\Omega(x,y)|\,|\omega(y)|\,dy\\
&\lesssim \left(|\gamma|+\|\omega\|_{L^1(\Omega)}\right)|h| + \int_{\{|x-y|\leqslant 2|h|\}} \left(|\nabla_x G_\Omega(x+h,y)|+|\nabla_x G_\Omega(x,y)|\right)|\omega(y)|\,dy\\
&\quad + \int_{\{|x-y|>2|h|\}} |\nabla_x G_\Omega(x+h,y)-\nabla_x G_\Omega(x,y)|\,|\omega(y)|\,dy\\
&\lesssim \left(|\gamma|+\|\omega\|_{L^1\cap L^\infty(\Omega)}\right)|h| + (1+|x|)\,\|\omega\|_{L^\infty(\Omega)}\int_{\{|x-y|\leqslant 3|h|\}}\frac{1}{|x-y|}dy\\
&\quad + |h|\sup_{z\in[x,x+h]}\int_{\{|x-y|>2|h|\}}\left|D_x^2 G_\Omega(z,y)\right|\,|\omega(y)|\,dy\\
&\lesssim \left(|\gamma|+(1+|x|)\,\|\omega\|_{L^1\cap L^\infty(\Omega)}\right)|h| + |h|\left(1+|x|^2\right)\int_{\{|x-y|>2|h|\}}\frac{1}{|x-y|^2}\,|\omega(y)|\,dy\\
&\lesssim \left(|\gamma|+\left(1+|x|^2\right)\|\omega\|_{L^1\cap L^\infty(\Omega)}\right)|h|\,(1-\log|h|)\,,
\end{aligned}
$$

where $\lesssim$ denotes the possible presence of various multiplicative independent constants, which completes the proof of the lemma. ∎

For any given initial data $\omega_0 \in C_c^1(\Omega)$ and some possibly small parameter $0 < \varepsilon \leqslant 1$ (how small is decided later on), we introduce now the subspace $C_{\omega_0,\varepsilon} \subset C_c^1([0,t_1]\times\Omega)$ as follows: a vortex density $\xi \in C_c^1([0,t_1]\times\Omega)$ belongs to $C_{\omega_0,\varepsilon}$ if and only if

- $\|\xi\|_{L^1(\Omega)} = \|\omega_0\|_{L^1(\Omega)}$ and $\|\xi\|_{L^\infty(\Omega)} = \|\omega_0\|_{L^\infty(\Omega)}$ for every $t \in [0,t_1]$,
- $\xi(0,x) = \omega_0(x)$ for every $x \in \Omega$,
- $\operatorname{supp}\xi(t,\cdot) \subset \Omega \cap \left\{\varepsilon < \operatorname{dist}(x,\partial\Omega) < \varepsilon^{-1}\right\}$ for every $t \in [0,t_1]$.

Observe finally that the subspace $C_{\omega_0,\varepsilon}$ inherits its topology from the metric topology of $C_c^1([0,t_1]\times\Omega)$.

The following proposition is a suitable wellposedness result for a linearized version of (2.12) in the metric subspace $C_{\omega_0,\varepsilon}$.

PROPOSITION 7.5. *Let $\omega_0 \in C_c^1(\Omega)$, $\gamma \in \mathbb{R}$, $0 < \varepsilon \leqslant 1$ and consider any fixed time $t_1 > 0$. Then, for a well distributed mesh on $\partial\Omega$, provided ε is sufficiently small, there exists $N_1 \geqslant N_0$ (N_0 is determined in Theorem 2.1) such that, for any $N \geqslant N_1$ and for any vortex density $\omega \in C_{\omega_0,\varepsilon}$, there is a unique classical solution $\xi \in C_{\omega_0,\varepsilon}$ to the linear equation*

$$
\begin{cases}\partial_t\xi + u\cdot\nabla\xi = 0,\\ \xi(t=0)=\omega_0\end{cases}
\tag{7.12}
$$

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with a velocity flow

$$u = K_{\mathbb{R}^2}[\omega] + u_{\text{app}}^N[\omega, \gamma].$$

Proof. Let

$$Z(s,x) \in C^1\left([0,t_2] \times \Omega; \Omega\right)$$

be the characteristic curve corresponding to u, i.e., the curve solving the differential equation

$$\frac{dZ}{ds} = u\left(s, Z\right)$$

for some given initial data $Z(0,x) = x \in \Omega$ and some existence time $0 < t_2 \leqslant t_1$. As before (see the proof of Lemma 7.1), since the velocity field u is of class C^1 on Ω (but not on $\overline{\Omega}$, though), standard results from the theory of ordinary differential equations guarantee the existence, uniqueness, and regularity of such curves for any given initial data. It seems a priori that the time t_2 may depend on x. However, since the support of ω_0 is bounded and does not intersect $\partial\Omega$, and u is uniformly bounded on compact subsets $K \subset \Omega$, by (7.5) and (7.8), independently of $\omega \in C_{\omega_0,\varepsilon}$, it is possible to set the time t_2 so small that $Z(t,x)$ for all $x \in \operatorname{supp}\omega_0$ is defined on $[0,t_2]$ and remains uniformly bounded away from $\partial\Omega$.

In fact, we show now that $Z(t,x)$ for each $x \in \operatorname{supp}\omega_0$ remains uniformly bounded away from $\partial\Omega$ no matter how large $0 < t_2 \leqslant t_1$ is. To this end, consider the modified Lagrangian flow Y defined by

$$Y\left(s, T(x)\right) := T\left(Z(s,x)\right) \in C^1\left([0,t_2] \times \operatorname{supp}\omega_0; \{|x| > 1\}\right),$$

solving the differential equation

$$\frac{dY}{ds} = DT\left(T^{-1}Y\right) u\left(s, T^{-1}Y\right)$$

for the initial data $Y(0,T(x)) = T(x)$, with $x \in \operatorname{supp}\omega_0$, and the velocity flow

$$v = K_\Omega[\omega] + \left(\gamma + \int_\Omega \omega(y)dy\right) H(x).$$

Then, in view of Theorem 2.1 with $\kappa = 2$ and employing (7.10)–(7.11), a variation of estimate (7.3) yields that

$$\begin{aligned}
&\left|\frac{d\left(|Y|-1\right)}{ds}\right| \\
&\quad = \left|v^t\left(s, T^{-1}Y\right) DT^t\left(T^{-1}Y\right)\frac{Y}{|Y|} + (u-v)^t\left(s, T^{-1}Y\right) DT^t\left(T^{-1}Y\right)\frac{Y}{|Y|}\right| \\
&\quad = \left|\left(v^t\left(s, T^{-1}Y\right) DT^t\left(T^{-1}Y\right) - v^t\left(s, T^{-1}\frac{Y}{|Y|}\right) DT^t\left(T^{-1}\frac{Y}{|Y|}\right)\right)\frac{Y}{|Y|}\right. \\
&\qquad \left. + (u-v)^t\left(s, T^{-1}Y\right) DT^t\left(T^{-1}Y\right)\frac{Y}{|Y|}\right| \\
&\quad \leqslant C\left\|v\right\|_{L^\infty(\Omega)}\left|Y - \frac{Y}{|Y|}\right| + C\left|v\left(s, T^{-1}Y\right) - v\left(s, T^{-1}\frac{Y}{|Y|}\right)\right|
\end{aligned}$$

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$$+ C |u - v| \left(s, T^{-1}Y\right)$$
$$\leqslant C (|Y| - 1) (1 + |\log (|Y| - 1)|) + \frac{C}{N^2\varepsilon^3} \left(\frac{1}{|Y| - 1} + \frac{1}{(|Y| - 1)^4} \right),$$

where $C > 0$ denotes various constants possibly depending on T, ω_0, and γ, but independent of ω, ε, and N. Further denoting $y = (|Y| - 1)^5$ for convenience and rearranging the preceding estimate yields

$$\left| \frac{dy}{ds} \right| \leqslant C \left(y (1 + |\log y|) + \frac{1}{N^2\varepsilon^3} \right)$$
$$\leqslant C \left(y + \frac{1}{N^2\varepsilon^3} \right) \left(1 + \left| \log \left(y + \frac{1}{N^2\varepsilon^3} \right) \right| \right),$$

and, therefore,

$$\left| \frac{d \log \left(1 + \left|\log \left(y + \frac{1}{N^2\varepsilon^3}\right)\right|\right)}{ds} \right| \leqslant C.$$

It follows that, for all $x \in \operatorname{supp} \omega_0$,

$$y + \frac{1}{N^2\varepsilon^3} \geqslant e^{1 - Ce^{Cs}},$$

where $C \geqslant 1$ only depends on T, ω_0, and γ, whence

$$C' |Z(s,x) - x_0|^5 \geqslant |Y(s,T(x)) - T(x_0)|^5$$
$$\geqslant (|Y(s,T(x))| - 1)^5 \geqslant e^{1 - Ce^{Cs}} - \frac{1}{N^2\varepsilon^3}$$

for any $x_0 \in \partial\Omega$, where $C' > 0$ only depends on T. Further taking the infimum of the above left-hand side over all $x_0 \in \partial\Omega$ yields that

$$C' \operatorname{dist}(Z(s,x), \partial\Omega)^5 \geqslant e^{1 - Ce^{Cs}} - \frac{1}{N^2\varepsilon^3} \geqslant e^{1 - Ce^{Ct_1}} - \frac{1}{N^2\varepsilon^3}$$

for all $x \in \operatorname{supp} \omega_0$ and $s \in [0, t_2]$.

Now comes the time to set the value of ε so small that the above right-hand side is larger than $C'\varepsilon^5$. More precisely, the parameter ε is first chosen so that $e^{1 - Ce^{Ct_1}} \geqslant 2C'\varepsilon^5$. Once ε is set, it is readily seen that there exists $N_1 \geqslant N_0$ such that $e^{1 - Ce^{Ct_1}} - \frac{1}{N^2\varepsilon^3} \geqslant 2C'\varepsilon^5 - \frac{1}{N^2\varepsilon^3} > C'\varepsilon^5$ for all $N \geqslant N_1$. In particular, since this implies that $\operatorname{dist}(Z(s,x), \partial\Omega) > \varepsilon$ as long as $Z(s,x)$ exists within $[0, t_1]$, according to classical results on the continuation of solutions to differential equations (see [6, Chapter 1, Theorem 4.1], for instance), since u is continuous and uniformly bounded pointwise on compact sets independently of $\omega \in C_{\omega_0,\varepsilon}$, we conclude that the solution $Z(s,x)$ exists over $[0, t_1]$. If necessary, it is possible to further reduce the value of ε so that

$$\varepsilon < \operatorname{dist}(Z(s,x), \partial\Omega) < \varepsilon^{-1} \quad \text{for all } x \in \operatorname{supp} \omega_0 \text{ and } s \in [0, t_1]. \tag{7.13}$$

For any fixed $t \in [0, t_1]$, it holds that the mapping $x \mapsto Z(t,x)$ is a C^1-diffeomorphism, preserving the Lebesgue measure, from $\operatorname{supp} \omega_0$ onto its own image. We denote its inverse by $Z^{-1}(t, \cdot)$ and we define the new vortex density $\xi \in$

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$C_c^1([0,t_1]\times\Omega)$ by

$$\begin{cases} \xi(t,x) = \omega_0\left(Z^{-1}(t,x)\right) & \text{if } x \in Z\left(t, \operatorname{supp}\omega_0\right), \\ \xi(t,x) = 0 & \text{otherwise.} \end{cases} \tag{7.14}$$

Then, by virtue of (7.13), one easily verifies that ξ belongs to $C_{\omega_0,\varepsilon}$ and solves the transport equation (7.12).

Finally, the fact that any classical solution of (7.12) necessarily satisfies (7.14) easily yields the uniqueness of ξ, which concludes the proof of the proposition. ∎

We are now ready to proceed to the actual proof of Theorem 2.2.

Proof of Theorem 2.2. This demonstration is somewhat lengthy and, so, we split it into several steps:

1. First, we build an approximating sequence using a standard iteration procedure based on the wellposedness of the linear transport equation established in Proposition 7.5.
2. Second, we establish uniform C^1-bounds on this approximating sequence.
3. Next, we show that it is actually a Cauchy sequence in C^0, which allows us to pass to the limit in the iteration scheme and obtain a solution of (2.12) in the sense of distributions.
4. Finally, we explain why this solution is in fact of class C^1 and provide some concluding remarks.

Construction of approximating sequence. Now, in order to establish the existence of the classical solution to (2.12), we first build an approximating sequence $\{\xi_n\}_{n\geqslant 0}$ of vortex densities within the complete metric subspace $C_{\omega_0,\varepsilon} \subset C_c^1([0,t_1]\times\Omega)$ defined above.

The first term $\xi_0 \in C_{\omega_0,\varepsilon}$ of the approximating sequence is simply given by $\xi_0(t,x) = \omega_0(x)$ for all $(t,x) \in [0,t_1]\times\Omega$. Then, for each $\xi_n \in C_{\omega_0,\varepsilon}$, the following term $\xi_{n+1} \in C_{\omega_0,\varepsilon}$ is defined, by virtue of Proposition 7.5, assuming $\varepsilon > 0$ is sufficiently small while N is large enough, as the unique solution to

$$\begin{cases} \partial_t \xi_{n+1} + \mu_n \cdot \nabla \xi_{n+1} = 0, \\ \xi_{n+1}(t=0) = \omega_0, \end{cases}$$

where the velocity flow μ_n is given by

$$\mu_n = K_{\mathbb{R}^2}[\xi_n] + u^N_{\mathrm{app}}[\xi_n, \gamma].$$

Uniform boundedness in C^1. We show now that $\{\xi_n\}_{n\geqslant 0}$ is uniformly bounded in $C^1([0,t_1]\times\Omega)$.

To this end, observe that the ξ_n's also solve (in the sense of distributions) the equation for $i = 1,2$:

$$\partial_t \partial_{x_i} \xi_{n+1} + \mu_n \cdot \nabla \partial_{x_i} \xi_{n+1} = -\partial_{x_i} \mu_n \cdot \nabla \xi_{n+1}.$$

It follows that, for any $[a,b] \subset [0,t_1]$,

$$\partial_{x_i} \xi_{n+1}\left(b, Z_n(b,x)\right) = \partial_{x_i} \xi_{n+1}\left(a, Z_n(a,x)\right) - \int_a^b \partial_{x_i} \mu_n \cdot \nabla \xi_{n+1}\left(s, Z_n(s,x)\right) ds,$$

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where

$$Z_n(s,x) \in C^1\left([0,t_2]\times\Omega;\Omega\right)$$

is the characteristic curve corresponding to μ_n, i.e., the curve solving the differential equation

$$\frac{dZ_n}{ds} = \mu_n\left(s, Z_n\right)$$

for some given initial data $Z(0,x) = x \in \Omega$ and some existence time $0 < t_2 \leqslant t_1$. Recall that, as shown in the proof of Proposition 7.5, it is possible to set $t_2 = t_1$ for all $x \in \operatorname{supp}\omega_0$.

Then, by Grönwall's lemma, we obtain

$$\begin{aligned}
&\left|\nabla\xi_{n+1}\left(b, Z_n(b,x)\right)\right| \leqslant \left|\nabla\xi_{n+1}\left(a, Z_n(a,x)\right)\right| e^{\int_a^b |\nabla\mu_n(s,Z_n(s,x))|ds},\\
&\left|\partial_t\xi_{n+1}\left(b, Z_n(b,x)\right)\right| \leqslant \left|\nabla\xi_{n+1}\left(a, Z_n(a,x)\right)\right| \left|\mu_n\left(b, Z_n(b,x)\right)\right| e^{\int_a^b |\nabla\mu_n(s,Z_n(s,x))|ds}
\end{aligned}$$

for all $x \in \operatorname{supp}\omega_0$. Further combining the preceding estimates with (7.5)–(7.6) and (7.8)–(7.9), we conclude that

$$\begin{aligned}
&\left\|\xi_{n+1}\right\|_{C^1([a,b]\times\Omega)}\\
&\quad\leqslant \left\|\omega_0\right\|_{L^\infty(\Omega)} + \frac{C}{\varepsilon}\left\|\nabla\xi_{n+1}(a,\cdot)\right\|_{L^\infty(\Omega)}\left(1 + \left\|\omega_0\right\|_{L^1\cap L^\infty(\Omega)} + |\gamma|\right)\\
&\qquad\times e^{(b-a)\frac{C}{\varepsilon^2}\left(1+\|\omega_0\|_{L^1\cap L^\infty(\Omega)}+|\gamma|\right)+(b-a)C\|\omega_0\|_{L^\infty(\mathbb{R}^2)}\log\left(1+\|\nabla\xi_n\|_{L^\infty([a,b]\times\mathbb{R}^2)}\right)}.
\end{aligned}$$

Therefore, we deduce

$$\left\|\xi_{n+1}\right\|_{C^1([a,b]\times\Omega)} \leqslant C_0 + C_0\left\|\nabla\xi_{n+1}(a,\cdot)\right\|_{L^\infty(\Omega)}\left(1 + \left\|\xi_n\right\|_{C^1([a,b]\times\Omega)}\right)^{C_0(b-a)},$$

where $C_0 > 0$ may only depend on $\|\omega_0\|_{L^1\cap L^\infty(\Omega)}$, γ, ε, and t_1 but remains independent of ξ_n, ξ_{n+1}, and $[a,b]$. It follows that setting $(b-a)$ sufficiently small so that, say,

$$C_0(b-a) \leqslant \frac{1}{2}, \tag{7.15}$$

yields

$$\left\|\xi_{n+1}\right\|_{C^1([a,b]\times\Omega)} \leqslant \frac{1}{2} + C_0 + \frac{C_0^2}{2}\left\|\nabla\xi_{n+1}(a,\cdot)\right\|^2_{L^\infty(\Omega)} + \frac{1}{2}\left\|\xi_n\right\|_{C^1([a,b]\times\Omega)},$$

whence, for each $k = 0,\dots,n$,

$$\begin{aligned}
\left\|\xi_{n+1}\right\|_{C^1([a,b]\times\Omega)} &\leqslant \left(\frac{1}{2} + C_0 + \frac{C_0^2}{2}\left\|\nabla\xi_{n+1}(a,\cdot)\right\|^2_{L^\infty(\Omega)}\right) + \frac{1}{2}\left\|\xi_n\right\|_{C^1([a,b]\times\Omega)}\\
&\leqslant \left(\frac{1}{2} + C_0 + \frac{C_0^2}{2}\sup_{p\geqslant 0}\left\|\nabla\xi_p(a,\cdot)\right\|^2_{L^\infty(\Omega)}\right)\left(\sum_{j=0}^k 2^{-j}\right)\\
&\quad + \frac{1}{2^{k+1}}\left\|\xi_{n-k}\right\|_{C^1([a,b]\times\Omega)}\\
&\leqslant 1 + 2C_0 + C_0^2\sup_{p\geqslant 0}\left\|\nabla\xi_p(a,\cdot)\right\|^2_{L^\infty(\Omega)} + \frac{1}{2^{n+1}}\left\|\omega_0\right\|_{C^1(\Omega)}.
\end{aligned}$$

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Since the initial data ω_0 belongs to $C^1(\Omega)$, the constant C_0 only depends on fixed parameters, and the bound (7.15) on the maximal length of $[a,b]$ only involves C_0, we deduce that we may propagate the preceding C^1-bound on $[a,b]$ to the whole interval $[0,t_1]$. This yields a uniform bound

$$\sup_{n\geqslant 0}\|\xi_n\|_{C^1([0,t_1]\times\Omega)}<\infty. \tag{7.16}$$

Convergence properties. We have thus produced an approximating sequence $\{\xi_n\}_{n\geqslant 0}\subset C_{\omega_0,\varepsilon}$ bounded, by virtue of (7.16), in the metric topology induced by $C^1([0,t_1]\times\Omega)$. We analyze now its convergence properties.

To this end, note that

$$\begin{aligned}\partial_t(\xi_{n+1}-\xi_n)+\mu_n\cdot\nabla(\xi_{n+1}-\xi_n)&=(\mu_{n-1}-\mu_n)\cdot\nabla\xi_n,\\ \partial_t(\xi_{n+1}-\xi_n)+\mu_{n-1}\cdot\nabla(\xi_{n+1}-\xi_n)&=(\mu_{n-1}-\mu_n)\cdot\nabla\xi_{n+1},\end{aligned}$$

whence, for any $[a,b]\subset[0,t_1]$,

$$\begin{aligned}(\xi_{n+1}-\xi_n)(b,Z_n(b,x))&=(\xi_{n+1}-\xi_n)(a,Z_n(a,x))\\&\quad+\int_a^b(\mu_{n-1}-\mu_n)\cdot\nabla\xi_n(s,Z_n(s,x))\,ds,\\ (\xi_{n+1}-\xi_n)(b,Z_{n-1}(b,x))&=(\xi_{n+1}-\xi_n)(a,Z_{n-1}(a,x))\\&\quad+\int_a^b(\mu_{n-1}-\mu_n)\cdot\nabla\xi_{n+1}(s,Z_{n-1}(s,x))\,ds,\end{aligned}$$

which implies, utilizing estimates (7.5), (7.8), and (7.16), for each $k=0,\dots,n-1$,

$$\begin{aligned}&\|\xi_{n+1}-\xi_n\|_{L^\infty([a,b]\times\Omega)}\\&\qquad\leqslant\|(\xi_{n+1}-\xi_n)(a,\cdot)\|_{L^\infty(\Omega)}+C_1(b-a)\|\xi_n-\xi_{n-1}\|_{L^\infty([a,b]\times\Omega)}\\&\qquad\leqslant\sum_{j=0}^{k}(C_1(b-a))^j\|(\xi_{n+1-j}-\xi_{n-j})(a,\cdot)\|_{L^\infty(\Omega)}\\&\qquad\quad+(C_1(b-a))^{k+1}\|\xi_{n-k}-\xi_{n-1-k}\|_{L^\infty([a,b]\times\Omega)}\\&\qquad\leqslant\sum_{j=0}^{n-1}(C_1(b-a))^j\|(\xi_{n+1-j}-\xi_{n-j})(a,\cdot)\|_{L^\infty(\Omega)}\\&\qquad\quad+(C_1(b-a))^{n}\|\xi_1-\xi_0\|_{L^\infty([a,b]\times\Omega)}\end{aligned} \tag{7.17}$$

for some independent constant $C_1>0$. As before, we set $(b-a)$ sufficiently small so that, say,

$$C_1(b-a)\leqslant\frac{1}{2}.$$

In particular, since the ξ_n's all have the same initial data ω_0, we find that

$$\begin{aligned}\|\xi_{n+1}-\xi_n\|_{L^\infty([0,b-a]\times\Omega)}&\leqslant(C_1(b-a))^n\|\xi_1-\xi_0\|_{L^\infty([0,b-a]\times\Omega)}\\&\leqslant\frac{1}{2^{n-1}}\|\omega_0\|_{L^\infty(\Omega)}.\end{aligned}$$

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Therefore, utilizing the elementary identity

$$\sum_{j=0}^{n} \binom{j+k}{k} = \binom{n+k+1}{k+1}$$

for each $n, k \in \mathbb{N}$, we obtain

$$\begin{aligned} \|\xi_{n+1} - \xi_n\|_{L^\infty([k(b-a),(k+1)(b-a)]\times\Omega)} &\leqslant 2\binom{n+k}{k} (C_1(b-a))^n \|\omega_0\|_{L^\infty(\Omega)} \\ &\leqslant \frac{1}{2^{n-1}} \binom{n+k}{k} \|\omega_0\|_{L^\infty(\Omega)}, \end{aligned}$$

whence

$$\begin{aligned} \|\xi_{n+1} - \xi_n\|_{L^\infty([0,t_1]\times\Omega)} &\leqslant \frac{C}{2^{\frac{n}{2}}} && \text{for all } n \geqslant 0, \\ \|\xi_m - \xi_n\|_{L^\infty([0,t_1]\times\Omega)} &\leqslant \frac{C'}{2^{\frac{n}{2}}} && \text{for all } m > n \geqslant 0, \end{aligned}$$

for some independent constants $C, C' > 0$.

It follows that $\{\xi_n\}_{n\geqslant 0}$ is a Cauchy sequence in $C^0([0,t_1]\times\Omega)$ and, therefore, there exists $\omega^N \in C([0,t_1]\times\Omega)$ such that

$$\begin{aligned} &\xi_n \longrightarrow \omega^N \quad \text{in } L^\infty([0,t_1]\times\Omega), \\ &\mu_n \longrightarrow u^N \quad \text{in } L^\infty([0,t_1]\times K) \text{ for any compact set } K \subset \Omega, \end{aligned} \tag{7.18}$$

where u^N is defined by (2.13) and we have used (7.5) and (7.8) to derive the convergence of μ_n from that of ξ_n in both $L^\infty([0,t_1]\times\Omega)$ and $L^1([0,t_1]\times\Omega)$ (recall that the vorticities ξ_n have uniformly bounded supports since they belong to $C_{\omega_0,\varepsilon}$). It is then readily seen that ω^N solves (2.12) in the sense of distributions.

Regularity of solution and conclusion of proof. In order to complete the proof of (global for large N) wellposedness of (2.12) in C^1, there only remains to show that ω^N is actually of class C^1. Indeed, the uniqueness of solutions will then easily ensue from an estimate similar to (7.17).

For the moment, the uniform boundedness of $\{\xi_n\}_{n\geqslant 0}$ in $C^1([0,t_1]\times\Omega)$ only allows us to deduce that ω^N is Lipschitz continuous (in t and x). However, standard estimates (see [9, p. 249], for instance, or use the respresentation formula (7.7)) show that this Lipschitz continuity implies that ∇u^N exists and is continuous in $[0,t_1]\times\Omega$. It follows that the associated characteristic curve $Z^N(t,x)$ solving

$$\frac{dZ^N}{ds} = u^N\left(s, Z^N\right) \tag{7.19}$$

for some given initial data $Z^N(0,x) = x \in \Omega$ belongs to $C^1([0,t_3]\times\Omega;\Omega)$ for some possibly small existence time $0 < t_3 \leqslant t_1$. Moreover, one easily estimates, using (7.13) and (7.18), that, for all $t \in [0,t_3]$ and $x \in \operatorname{supp}\omega_0$,

$$\begin{aligned} \left|Z_n(t,x) - Z^N(t,x)\right| &\leqslant \int_0^t \left|\mu_n(s, Z_n(s,x)) - u^N\left(s, Z^N(s,x)\right)\right| ds \\ &\leqslant \int_0^t \left|\mu_n(s, Z_n(s,x)) - u^N(s, Z_n(s,x))\right| ds \end{aligned}$$

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$$+\int_0^t \left|u^N\left(s, Z_n(s,x)\right) - u^N\left(s, Z^N(s,x)\right)\right| ds$$
$$\leqslant o(1) + C\int_0^t \left|Z_n(s,x) - Z^N(s,x)\right| ds,$$

which implies, through a straighforward application of Grönwall's lemma, that Z_n converges uniformly in $(t,x) \in [0,t_3] \times \operatorname{supp}\omega_0$ toward Z^N. One can therefore assume that $t_3 = t_1$ for $Z^N(t_3,x)$ remains uniformly bounded away from $\partial\Omega$ (at a distance at least ε, to be precise) for each $x \in \operatorname{supp}\omega_0$.

Next, as before, since the mapping $x \mapsto Z^N(t,x)$ is a C^1-diffeomorphism from $\operatorname{supp}\omega_0$ onto its own image, we consider its inverse $(Z^N)^{-1}(t,x)$. It is then readily seen that

$$\begin{cases} \omega^N(t,x) = \omega_0\left(\left(Z^N\right)^{-1}(t,x)\right) & \text{if } x \in Z^N\left(t, \operatorname{supp}\omega_0\right), \\ \omega^N(t,x) = 0 & \text{otherwise,} \end{cases}$$

which establishes that $\omega^N \in C^1_c\left([0,t_1]\times\Omega\right)$ for $\omega_0 \in C^1_c\left(\Omega\right)$ and thereby concludes the proof of Theorem 2.2 on the wellposedness of (2.12) in $C^1_c\left([0,t_1]\times\Omega\right)$. Further observe that, repeating the estimates leading up to (7.16) (replacing ξ_n and ξ_{n+1} by ω^N and μ_n by u^N), it is possible to show that the C^1-bound on ω^N is uniform in N. ◻

7.2. Proof of Theorem 2.3. Considering the difference of (2.11) and (2.12), note that

$$\partial_t\left(\omega - \omega^N\right) + u\cdot\nabla\left(\omega - \omega^N\right) = -\left(u - u^N\right)\cdot\nabla\omega^N,$$
$$\partial_t\left(\omega - \omega^N\right) + u^N\cdot\nabla\left(\omega - \omega^N\right) = -\left(u - u^N\right)\cdot\nabla\omega,$$

whence, for every $(t,x) \in [0,t_1] \times \operatorname{supp}\omega_0$,

$$\begin{aligned} \left(\omega - \omega^N\right)(t, X(t,x)) &= -\int_0^t \left(u - u^N\right)\cdot\nabla\omega^N\left(s, X(s,x)\right) ds, \\ \left(\omega - \omega^N\right)\left(t, Z^N(t,x)\right) &= -\int_0^t \left(u - u^N\right)\cdot\nabla\omega\left(s, Z^N(s,x)\right) ds, \end{aligned} \tag{7.20}$$

where the characteristics $X(t,x)$ and $Z^N(t,x)$ are, respectively, defined by (7.2) and (7.19) and, as previously explained, exist for all $(t,x) \in [0,t_1] \times \operatorname{supp}\omega_0$ provided N is sufficiently large. We also introduce the velocity flow

$$\breve{u}^N = K_{\mathbb{R}^2}[\omega] + u^N_{\mathrm{app}}[\omega,\gamma],$$

where $u^N_{\mathrm{app}}[\omega,\gamma]$ is given by (2.7)–(2.8) for the same prescribed $\gamma \in \mathbb{R}$ and where u_P in the right-hand side of (2.8) is $K_{\mathbb{R}^2}[\omega]$.

By Theorem 2.1, it holds that, for any compact set $K \subset \Omega$,

$$\begin{aligned} &\|u - \breve{u}^N\|_{L^\infty(K)} \\ &\quad\leqslant \frac{C}{N^\kappa}\left(\left(\frac{1}{\operatorname{dist}\left(\operatorname{supp}\omega, \partial\Omega\right)} + \frac{1}{\operatorname{dist}\left(\operatorname{supp}\omega, \partial\Omega\right)^{\kappa+1}}\right)\|\omega\|_{L^1(\mathbb{R}^2)} + |\gamma|\right), \end{aligned} \tag{7.21}$$

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where the constant $C > 0$ may now depend on K. We also estimate, using (7.5) and (7.8), that

$$
\begin{aligned}
\|\breve{u}^N - u^N\|_{L^\infty(K)} &\leqslant \left\|K_{\mathbb{R}^2}\left[\omega-\omega^N\right]\right\|_{L^\infty(K)} + \left\|u^N_{\rm app}\left[\omega-\omega^N,0\right]\right\|_{L^\infty(K)} \\
&\leqslant C\left\|\omega-\omega^N\right\|_{L^1\cap L^\infty(\Omega)} \\
&\leqslant C\left(1+|\operatorname{supp}\omega|+\left|\operatorname{supp}\omega^N\right|\right)\left\|\omega-\omega^N\right\|_{L^\infty(\Omega)}.
\end{aligned}
\tag{7.22}
$$

Therefore, recalling that ω and ω^N are all uniformly bounded in $C^1([0,t_1]\times\Omega)$, combining (7.21) and (7.22) with (7.20), and choosing the compact set K so that $\operatorname{supp}\omega\cup\operatorname{supp}\omega^N\subset K$ for all N, we find that

$$
\begin{aligned}
\left\|(\omega-\omega^N)(t)\right\|_{L^\infty(\Omega)} &= \left\|(\omega-\omega^N)(t)\right\|_{L^\infty(K)} \\
&\leqslant C\int_0^t \left\|(u-u^N)(s)\right\|_{L^\infty(K)}ds \\
&\leqslant \frac{C}{N^\kappa} + C\int_0^t \left\|(\omega-\omega^N)(s)\right\|_{L^\infty(\Omega)}ds.
\end{aligned}
$$

Finally, by Grönwall's lemma, we deduce that

$$
\left\|(\omega-\omega^N)(t)\right\|_{L^\infty(\Omega)} \leqslant \frac{C}{N^\kappa}e^{Ct},
$$

which concludes the proof of the theorem.

8. An alternative approach: The fluid charge method. We wish now to propose an alternative method for approximating u_R by discretizing the boundary of the domain in (2.4). It consists in constructing an approximate flow

$$
\breve{u}^N_{\rm app}(x) := \frac{1}{2\pi}\sum_{j=1}^N \frac{\breve{\gamma}^N_j}{N}\frac{x-x^N_j}{|x-x^N_j|^2} + \gamma H_*(x),
\tag{8.1}
$$

where the positions $(x^N_1, x^N_2, \dots, x^N_N)$ on the boundary $\partial\Omega$ are still determined by (2.5) and we assume that $H_*(x)\in C^\infty(\Omega)$ is a given vector field solving (3.32), such that all its derivatives are continuous up to the boundary $\partial\Omega$, $H_*\cdot n$ has mean zero over $\partial\Omega$, and $H_*\cdot n\in C^\infty(\partial\Omega)$.

For practical purposes, the field $H_*(x)$ should be either known explicitely or previously computed by other means. For instance, one may consider the harmonic vector field $H_* = H(x)$ defined by (1.6) or a single point vortex velocity field $H_*(x) = \frac{(x-x_*)^\perp}{2\pi|x-x_*|^2} = K_{\mathbb{R}^2}[\delta_{x_*}]$ for any given $x_*\in\overline{\Omega}^c$.

Observe that (8.1) is essentially a discretization of (3.33). The clear advantage of discretizing (3.33) over (3.24) resides in that (3.33) only involves the inversion of the regular perturbation of the identity $A+\pi$ whereas (3.24) requires the inversion of the singular integral operator B. We argue below, in section 8.3, that this provides a more efficient discretization method because it often yields better conditioned matrices.

By analogy with the electric field produced by single electric charges, we refer to the building blocks $\frac{x-y}{2\pi|x-y|^2}$ with $y\in\mathbb{R}^2$ of the above flow as *fluid charges*. Recall that a fluid charge satisfies

$$
\begin{cases}
\operatorname{div}\frac{x}{2\pi|x|^2} = \delta(x),\\
\operatorname{curl}\frac{x}{2\pi|x|^2} = 0.
\end{cases}
$$

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8.1. Static convergence of the fluid charge approximation. As before, concerning u^N_{app}, it is a priori not obvious that such a flow $\check{u}^N_{\text{app}}$ can be made a good approximation of u_R. Nonetheless, note that $\check{u}^N_{\text{app}}$ already naturally satisfies, for any smooth simple closed curve c_0 enclosing the obstacle $\mathcal{C}$,

$$\begin{cases} \operatorname{div} \check{u}^N_{\text{app}} = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} \check{u}^N_{\text{app}} = 0 & \text{in} \quad \Omega, \\ \check{u}^N_{\text{app}} \to 0 & \text{as} \quad x \to \infty, \\ \oint_{c_0} \check{u}^N_{\text{app}} \cdot \tau ds = \gamma, \end{cases}$$

where τ denotes the unit tangent vector on c_0. Following the developments leading to (2.8), it would now be tempting to enforce that the boundary condition be satisfied as $N \to \infty$ by setting the density $\check{\gamma}^N = (\check{\gamma}^N_1, \ldots, \check{\gamma}^N_N) \in \mathbb{R}^N$ to be the solution of the following system of N linear equations:

$$\text{(8.2)} \quad \frac{1}{2\pi} \sum_{j=1}^{N} \frac{\check{\gamma}^N_j}{N} \frac{\tilde{x}^N_i - x^N_j}{|\tilde{x}^N_i - x^N_j|^2} \cdot n(\tilde{x}^N_i) = -[(u_P + \gamma H_*) \cdot n](\tilde{x}^N_i) \text{ for all } i = 1, \ldots, N$$

for some appropriate intermediate mesh points $(\tilde{x}^N_1, \tilde{x}^N_2, \ldots, \tilde{x}^N_N)$ on the boundary $\partial\Omega$. This is, however, not feasible in general because the resulting matrix may not be invertible (consider the case of the unit disk where $\frac{x-y}{|x-y|^2} \cdot n(x) = \frac{1}{2}$ for all $x, y \in \partial B(0,1)$). In fact, it turns out that, instead of (8.2), the correct condition on the density $\check{\gamma}^N$ is given by the system

$$\text{(8.3)} \quad \left(\frac{1}{2\pi} \sum_{j=1}^{N} \frac{\check{\gamma}^N_j}{N} \frac{\tilde{x}^N_i - x^N_j}{|\tilde{x}^N_i - x^N_j|^2} \cdot n(\tilde{x}^N_i) \right) + \frac{\check{\gamma}^N_i}{2\,|\partial\Omega|} = -[(u_P + \gamma H_*) \cdot n](\tilde{x}^N_i) \text{ for all } i = 1, \ldots, N,$$

which is inspired by the integral representation (3.33) and the inversion of the operator $A + \pi$. In order to emphasize the dependence of $\check{u}^N_{\text{app}}$ on ω (through u_P) and γ, we may denote $\check{u}^N_{\text{app}} = \check{u}^N_{\text{app}}[\omega, \gamma]$. Note that $\check{u}^N_{\text{app}}$ is linear in (ω, γ).

In the notation introduced in (2.5)–(2.6), this system can be recast as

$$\frac{1}{N} \sum_{j=1}^{N} \check{\gamma}^N_j \frac{l\left(\tilde{s}^N_i\right) - l\left(s^N_j\right)}{\left|l\left(\tilde{s}^N_i\right) - l\left(s^N_j\right)\right|^2} \cdot n\left(l\left(\tilde{s}^N_i\right)\right) + \frac{\pi \check{\gamma}^N_i}{|\partial\Omega|} = f\left(\tilde{s}^N_i\right) \quad \text{for all } i = 1, \ldots, N,$$

where $f(s) = -2\pi[(u_P + \gamma H_*) \cdot n]\,(l(s))$ for all $s \in [0, |\partial\Omega|]$. Equivalently, in the matrix notation of section 4, this system becomes

$$\text{(8.4)} \quad \left(\frac{1}{N} A_N + \frac{\pi}{|\partial\Omega|} \right) \check{\gamma}^N = \left(f\left(\tilde{s}^N_i\right) \right)_{1 \leqslant i \leqslant N}.$$

We use here the convention that, whenever $\tilde{s}^N_i = s^N_j$, the entry of A_N corresponding to

$$\frac{l\left(\tilde{s}^N_i\right) - l\left(s^N_j\right)}{\left|l\left(\tilde{s}^N_i\right) - l\left(s^N_j\right)\right|^2} \cdot n\left(l\left(\tilde{s}^N_i\right)\right)$$

is naturally determined by the limiting value of the smooth kernel $\frac{l(s)-l(t)}{|l(s)-l(t)|^2} \cdot n\left(l\left(s\right)\right)$

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as t and s converge to the same value t_0, say, that is,

$$\lim_{s,t\to t_0} \frac{l(s)-l(t)}{|l(s)-l(t)|^2}\cdot n(l(s)) = \lim_{s,t\to t_0} \frac{-\frac{1}{2}l''(s)(t-s)^2+o(|t-s|^2)}{|l(s)-l(t)|^2}\cdot n(l(s))$$
$$= -\frac{1}{2}l''(t_0)\cdot n(l(t_0)).$$

In particular, by the uniform boundedness of each component of A_N, there exists a constant $C>0$ independent of N such that, for each $1\leqslant p\leqslant\infty$,

$$\frac{1}{N}\|A_N z\|_{\ell^p}\leqslant C\|z\|_{\ell^1} \tag{8.5}$$

for all $z\in\mathbb{R}^N$.

In this section, we are going to adopt a new notion of well distributedness, introduced in the coming definition, which differs from the one defined by (2.9)–(2.10). In order to avoid any possible confusion, we will refer to these newly introduced meshes as being well-$*$ distributed, which will distinguish them from the well distributed meshes determined by (2.9)–(2.10). It is to be emphasized that well-$*$ distributed meshes are used in the present section, section 8 only.

DEFINITION. *We say that the points* $\{x_i^N\}_{1\leqslant i\leqslant N}$ *and* $\{\tilde{x}_i^N\}_{1\leqslant i\leqslant N}$ *given by* $x_i^N := l(s_i^N)$ *and* $\tilde{x}_i^N := l(\tilde{s}_i^N)$, *where* $(s_1^N,\dots,s_N^N)\in\mathbb{R}^N$ *and* $(\tilde{s}_1^N,\dots,\tilde{s}_N^N)\in\mathbb{R}^N$, *are* well-$*$ distributed *on* $\partial\Omega$ *if there exists an integer* $\kappa\geqslant 2$ *such that, as* $N\to\infty$,

$$\max_{i=1,\dots,N}\left|s_i^N-\theta_i^N\right| = \mathcal{O}\left(N^{-\kappa}\right) \quad \textit{and} \quad \max_{i=1,\dots,N}\left|\tilde{s}_i^N-\theta_i^N\right| = \mathcal{O}\left(N^{-\kappa}\right), \tag{8.6}$$

where

$$\theta_i^N = \frac{(i-1)|\partial\Omega|}{N} \quad \textit{for all } i=1,\dots,N. \tag{8.7}$$

Our main result concerning the alternative approximation method discussed in the present section is analog to Theorem 2.1 and states that the approximate flow $\check{u}^N_{\rm app}$, constructed through the procedure (8.3), is a good approximation of u_R provided the vortices are well-$*$ distributed on $\partial\Omega$.

THEOREM 8.1. *Let* $\omega\in L^1_c(\Omega)$ *and* $\gamma\in\mathbb{R}$ *be given. For any* $N\geqslant 2$, *we consider a well-$*$ distributed mesh satisfying* (8.6) *and* u_P *defined in* (2.2).

Then, there exists N_0 *(independent of* ω *and* γ*) such that, for all* $N\geqslant N_0$, *the system* (8.3) *admits a unique solution* $\check{\gamma}^N\in\mathbb{R}^N$. *Moreover, for any closed set* $K\subset\Omega$ *there exists a constant* $C>0$ *independent of* N, K, ω, *and* γ *such that*

$$\begin{aligned}
&\|u_R-\check{u}^N_{\rm app}\|_{L^\infty(K)}\\
&\quad\leqslant \frac{C}{N^\kappa}\left(\frac{1}{\operatorname{dist}(K,\partial\Omega)}+\frac{1}{\operatorname{dist}(K,\partial\Omega)^{\kappa+2}}\right)\\
&\qquad\times\left(\left(\frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)}+\frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)^{\kappa+1}}\right)\|\omega\|_{L^1(\mathbb{R}^2)}+|\gamma|\right),
\end{aligned}$$

where $\check{u}^N_{\rm app}$ *is given by* (8.1) *in terms of* $\check{\gamma}^N$ *and* u_R *is the continuous flow* (2.3).

The proof of Theorem 8.1 relies on Propositions 8.2 and 8.3. It is given per se after the proof of Proposition 8.3 is completed, below.

Adapting the proof of Lemma 4.6 to the present situation and using estimate (3.30) on the spectral radius of $A-\pi$ rather than (3.27) yields the following result.

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PROPOSITION 8.2. *For any integer $N \geqslant 2$, consider a well-$*$ distributed mesh $(s_1^N, \ldots, s_N^N) \in \mathbb{R}^N$, $(\tilde{s}_1^N, \ldots, \tilde{s}_N^N) \in \mathbb{R}^N$. Then, there exist $N_*, k_* \geqslant 1$ and $\delta > 0$ such that*

$$\left\| \left(\frac{|\partial\Omega|}{N} A_N - \pi \right)^k \right\|_{\mathcal{L}(\ell^2)}^{\frac{1}{k}} \leqslant 2\pi - \delta$$

for all $k \geqslant k_*$ and $N \geqslant N_*$.

In particular, provided N is sufficiently large, the Neumann series

$$\left(\frac{|\partial\Omega|}{N} A_N + \pi \right)^{-1} = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\frac{\pi - \frac{|\partial\Omega|}{N} A_N}{2\pi} \right)^k$$

is absolutely convergent in $\mathcal{L}(\ell^2)$ and the inverse operator it defines is bounded in $\mathcal{L}(\ell^2)$ uniformly in N. It follows that, provided N is sufficiently large, the problem

$$z \in \mathbb{R}^N, \qquad \left(\frac{|\partial\Omega|}{N} A_N + \pi \right) z = v,$$

has a unique solution for any given $v \in \mathbb{R}^N$. Moreover, this solution satisfies

$$\|z\|_{\ell^1} \leqslant \|z\|_{\ell^2} \leqslant C\|v\|_{\ell^2} \leqslant C\|v\|_{\ell^\infty}$$

for some independent constant $C > 0$.

Proof. Following the proof of Lemma 4.6, for each $k \geqslant 1$, we denote by $K_k(x, y)$ the kernel of A^k, which is smooth and satisfies, for all $x, y \in \partial\Omega$,

$$\begin{aligned} K_k(x,y) = \int_{\partial\Omega\times\cdots\times\partial\Omega} & \frac{x - y_1}{|x - y_1|^2} \cdot n(x) \\ & \times \left(\prod_{j=1}^{k-2} \frac{y_j - y_{j+1}}{|y_j - y_{j+1}|^2} \cdot n(y_j) \right) \frac{y_{k-1} - y}{|y_{k-1} - y|^2} \cdot n(y_{k-1}) dy_1 \ldots dy_{k-1}. \end{aligned}$$

Therefore, by smoothness, approximating in $L^\infty_{x,y}(\partial\Omega \times \partial\Omega)$ the kernel $\frac{x-y}{|x-y|^2} \cdot n(x)$ by

$$\sum_{i,j=1}^{N} \mathbb{1}_{[\theta_i^N, \theta_{i+1}^N)}(s) \frac{l(\tilde{s}_i^N) - l(s_j^N)}{\left| l(\tilde{s}_i^N) - l(s_j^N) \right|^2} \cdot n\left(l(\tilde{s}_i^N)\right) \mathbb{1}_{[\theta_j^N, \theta_{j+1}^N)}(s_*),$$

where the θ_i^N's are defined in (8.7) and we identify $x = l(s)$ and $y = l(s_*)$, we find, as $N \to \infty$, that $K_k(x, y)$ is arbitrarily close in $L^\infty_{x,y}(\partial\Omega \times \partial\Omega)$ to

$$\begin{aligned} &\sum_{i,j=1}^{N} \mathbb{1}_{[\theta_i^N, \theta_{i+1}^N)}(s) \mathbb{1}_{[\theta_j^N, \theta_{j+1}^N)}(s_*) \\ &\times \left(\frac{|\partial\Omega|}{N} \right)^{k-1} \sum_{j_1,\ldots,j_{k-1}=1}^{N} \frac{l(\tilde{s}_i^N) - l(s_{j_1}^N)}{\left| l(\tilde{s}_i^N) - l(s_{j_1}^N) \right|^2} \cdot n\left(l(\tilde{s}_i^N)\right) \end{aligned}$$

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$$\times \left(\prod_{n=1}^{k-2} \frac{l\left(\tilde{s}_{j_n}^N\right) - l\left(s_{j_{n+1}}^N\right)}{\left| l\left(\tilde{s}_{j_n}^N\right) - l\left(s_{j_{n+1}}^N\right)\right|^2} \cdot n\left(l\left(\tilde{s}_{j_n}^N\right)\right) \right) \frac{l\left(\tilde{s}_{j_{k-1}}^N\right) - l\left(s_j^N\right)}{\left| l\left(\tilde{s}_{j_{k-1}}^N\right) - l\left(s_j^N\right)\right|^2} \cdot n\left(l\left(\tilde{s}_{j_{k-1}}^N\right)\right)$$
$$= \left(\frac{|\partial\Omega|}{N}\right)^{k-1} \sum_{i,j=1}^N \mathbb{1}_{[\theta_i^N, \theta_{i+1}^N)}(s) \left(A_N^k\right)_{ij} \mathbb{1}_{[\theta_j^N, \theta_{j+1}^N)}(s_*).$$

It follows that, for any fixed $k \geqslant 1$ and $\varepsilon > 0$, provided N is sufficiently large, with the convention that $K_0(l(s), l(s_*))$ denotes a Dirac mass at $s = s_*$,

$$\begin{aligned}
&\left\| (A-\pi)^k \right\|_{\mathcal{L}(L^2)} + \varepsilon \\
&\quad = \left\| \sum_{n=0}^k \binom{k}{n} (-\pi)^{k-n} A^n \right\|_{\mathcal{L}(L^2)} + \varepsilon \\
&\quad = \sup_{\varphi \in L^2(\partial\Omega)} \frac{\left\| \sum_{n=0}^k \binom{k}{n} (-\pi)^{k-n} \int_0^{|\partial\Omega|} K_n\left(l(s), l(s_*)\right) \varphi(l(s_*)) ds_* \right\|_{L_s^2}}{\|\varphi(l(s))\|_{L_s^2}} + \varepsilon \\
&\quad \geqslant \sup_{z \in \mathbb{R}^N} \frac{\left\| \sum_{n=0}^k \binom{k}{n} (-\pi)^{k-n} \left(\frac{|\partial\Omega|}{N}\right)^n \sum_{i,j=1}^N \mathbb{1}_{[\theta_i^N, \theta_{i+1}^N)}(s) \left(A_N^n\right)_{ij} z_j \right\|_{L_s^2}}{\left\| \sum_{i=1}^N z_i \mathbb{1}_{[\theta_i^N, \theta_{i+1}^N)}(s) \right\|_{L_s^2}} \\
&\quad = \sup_{z \in \mathbb{R}^N} \frac{\left\| \sum_{n=0}^k \binom{k}{n} (-\pi)^{k-n} \left(\frac{|\partial\Omega|}{N}\right)^n A_N^n z \right\|_{\ell^2}}{\|z\|_{\ell^2}} \\
&\quad = \sup_{z \in \mathbb{R}^N} \frac{\left\| \left(\frac{|\partial\Omega|}{N} A_N - \pi\right)^k z \right\|_{\ell^2}}{\|z\|_{\ell^2}} = \left\| \left(\frac{|\partial\Omega|}{N} A_N - \pi\right)^k \right\|_{\mathcal{L}(\ell^2)}.
\end{aligned}$$

Further deducing from estimate (3.30) that there exist $k_0 \geqslant 1$ and $\delta > 0$ such that $\|(A-\pi)^{k_0}\|_{\mathcal{L}(L^2)}^{\frac{1}{k_0}} \leqslant 2\pi - 3\delta$, we infer that, setting $\varepsilon > 0$ sufficiently small,

$$\left\| \left(\frac{|\partial\Omega|}{N} A_N - \pi\right)^{k_0} \right\|_{\mathcal{L}(\ell^2)} \leqslant \left\| (A-\pi)^{k_0} \right\|_{\mathcal{L}(L^2)} + \varepsilon \leqslant (2\pi - 3\delta)^{k_0} + \varepsilon \leqslant (2\pi - 2\delta)^{k_0}$$

for N sufficiently large.

Now, for any $k \geqslant k_0$, we write $k = pk_0 + q$ with positive integral numbers and $0 \leqslant q \leqslant k_0 - 1$. Then, we obtain

$$\begin{aligned}
\left\| \left(\frac{|\partial\Omega|}{N} A_N - \pi\right)^k \right\|_{\mathcal{L}(\ell^2)} &\leqslant \left\| \left(\frac{|\partial\Omega|}{N} A_N - \pi\right)^{k_0} \right\|_{\mathcal{L}(\ell^2)}^p \left\| \frac{|\partial\Omega|}{N} A_N - \pi \right\|_{\mathcal{L}(\ell^2)}^q \\
&\leqslant (2\pi - 2\delta)^{k-q} \left\| \frac{|\partial\Omega|}{N} A_N - \pi \right\|_{\mathcal{L}(\ell^2)}^q.
\end{aligned}$$

Further using that $N^{-1} A_N$ is a bounded operator over ℓ^2 uniformly in N (see (8.5)), we arrive at, for some fixed constant $C_* > 0$ independent of N and k, and for suffi-

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ciently large k,

$$\left\| \left(\frac{|\partial\Omega|}{N} A_N - \pi \right)^k \right\|_{\mathcal{L}(\ell^2)} \leqslant C_* \left(2\pi - 2\delta\right)^k \leqslant \left(2\pi - \delta\right)^k ,$$

which concludes the proof of the proposition. ∎

The coming result is an adaptation of Proposition 5.1 and establishes the weak convergence of the discretization of the operator $A + \pi$.

Proposition 8.3. *For any integer $N \geqslant 2$, consider a well-$*$ distributed mesh $(s_1^N, \ldots, s_N^N) \in \mathbb{R}^N$, $(\tilde{s}_1^N, \ldots, \tilde{s}_N^N) \in \mathbb{R}^N$ satisfying (8.6) and, according to Proposition 8.2, consider the solution $\check{\gamma}^N = (\check{\gamma}_1^N, \ldots, \check{\gamma}_N^N) \in \mathbb{R}^N$ to the system (8.4) for some periodic function $f \in C^\kappa([0, |\partial\Omega|])$, where $\kappa \geqslant 2$. We define the approximations*

(8.8)
$$\begin{aligned} \check{f}_{\text{app}}^N(s) &:= \frac{1}{N} \sum_{j=1}^N \check{\gamma}_j^N \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2} \cdot \tau(l(s)), \\ \check{g}_{\text{app}}^N(s) &:= \frac{1}{N} \sum_{j=1}^N \check{\gamma}_j^N \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2} \cdot n(l(s)). \end{aligned}$$

Then, for any periodic test function $\varphi \in C^\infty([0, |\partial\Omega|])$,

$$\begin{aligned} \left| \int_0^{|\partial\Omega|} \left(\check{f}_{\text{app}}^N - B(A+\pi)^{-1} f \right) \varphi \right| &\leqslant \frac{C}{N^\kappa} \|f\|_{C^\kappa} \|\varphi\|_{C^{\kappa+1}}, \\ \left| \int_0^{|\partial\Omega|} \left(\check{g}_{\text{app}}^N - A(A+\pi)^{-1} f \right) \varphi \right| &\leqslant \frac{C}{N^\kappa} \|f\|_{C^\kappa} \|\varphi\|_{L^2}, \end{aligned}$$

where we identify the variable x with the variable s whenever $x = l(s) \in \partial\Omega$ and the singular integrals are defined in the sense of Cauchy's principal value.

Proof. For any $h \in C^\kappa([0, |\partial\Omega|])$, an estimate based on Corollary A.2 yields that

$$\begin{aligned} &\left\| \left(Ah\left(\tilde{s}_i^N\right)\right)_{1 \leqslant i \leqslant N} - \frac{|\partial\Omega|}{N} A_N \left(h\left(s_i^N\right)\right)_{1 \leqslant i \leqslant N} \right\|_{\ell^\infty} \\ &\leqslant \sup_{s \in [0, |\partial\Omega|]} \left| \int_0^{|\partial\Omega|} \frac{l(s) - l(s_*)}{\left|l(s) - l(s_*)\right|^2} \cdot n(l(s))\, h(l(s_*))\, ds_* \right. \\ &\quad \left. - \frac{|\partial\Omega|}{N} \sum_{i=1}^N \frac{l(s) - l(s_i^N)}{\left|l(s) - l(s_i^N)\right|^2} \cdot n(l(s))\, h\left(l(s_i^N)\right) \right| \\ &\leqslant \frac{C}{N^\kappa} \sup_{s \in [0, |\partial\Omega|]} \left\| \frac{l(s) - l(s_*)}{\left|l(s) - l(s_*)\right|^2} \cdot n(l(s)) \right\|_{C_{s_*}^\kappa([0, |\partial\Omega|])} \|h\|_{C^\kappa} \leqslant \frac{C}{N^\kappa} \|h\|_{C^\kappa}. \end{aligned}$$

Exploiting (8.6) and that $N^{-1} A_N : \ell^1 \to \ell^\infty$ is a bounded operator (see (8.5)),

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uniformly in N, we also find that

$$\begin{aligned}\left\| \left(Ah\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N} - \frac{|\partial\Omega|}{N}A_N\left(h\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^\infty} &\leqslant \frac{C}{N^\kappa}\|h\|_{C^\kappa} + C\left\|\left(h\left(s_i^N\right)-h\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^\infty}\\ &\leqslant \frac{C}{N^\kappa}\left(\|h\|_{C^\kappa}+\|h\|_{C^1}\right) \leqslant \frac{C}{N^\kappa}\|h\|_{C^\kappa}.\end{aligned}$$

In particular, setting $h=(A+\pi)^{-1}f$ in the preceding estimate, exploiting the relation (8.4), and using the uniform boundedness of the inverse operator

$$\left(\frac{|\partial\Omega|}{N}A_N+\pi\right)^{-1}$$

examined in Proposition 8.2, we deduce that

$$\begin{aligned}&\left\|\frac{\check{\gamma}^N}{|\partial\Omega|} - \left((A+\pi)^{-1}f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^2}\\ &= \left\|\left(\frac{|\partial\Omega|}{N}A_N+\pi\right)^{-1}\left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N} - \left((A+\pi)^{-1}f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^2}\\ &\leqslant C\left\|\left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N} - \left(\frac{|\partial\Omega|}{N}A_N+\pi\right)\left((A+\pi)^{-1}f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^\infty}\\ &= C\left\|\left(A(A+\pi)^{-1}f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N} - \frac{|\partial\Omega|}{N}A_N\left((A+\pi)^{-1}f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^\infty}\\ &\leqslant \frac{C}{N^\kappa}\left\|(A+\pi)^{-1}f\right\|_{C^\kappa} \leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa}+\left\|A(A+\pi)^{-1}f\right\|_{C^\kappa}\right)\\ &\leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa},\end{aligned}$$

where we have used in the last step above that A has a smooth kernel and $(A+\pi)^{-1}$ is bounded over L^2. Further using (8.6) finally yields the similar estimate

$$\begin{aligned}&\left\|\frac{\check{\gamma}^N}{|\partial\Omega|} - \left((A+\pi)^{-1}f\left(s_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^2}\\ &\qquad\leqslant \left\|\frac{\check{\gamma}^N}{|\partial\Omega|} - \left((A+\pi)^{-1}f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^2}\\ &\qquad\quad + \left\|\left((A+\pi)^{-1}f\left(\tilde{s}_i^N\right)-(A+\pi)^{-1}f\left(s_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^2}\\ &\qquad\leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa}+\left\|(A+\pi)^{-1}f\right\|_{C^1}\right) \leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa},\end{aligned} \tag{8.9}$$

where we have used that A is a regularizing operator.

Now, note that

$$\int_0^{|\partial\Omega|}\check{g}^N_{\mathrm{app}}\varphi = \frac{1}{N}\sum_{j=1}^N \check{\gamma}_j^N A^*\varphi\left(l\left(s_j^N\right)\right),$$

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where we identify $\varphi(x)$ with $\varphi(s)$ whenever $x = l(s) \in \partial\Omega$. Then, employing (8.9) and Corollary A.2, we deduce that

$$\begin{aligned}
&\left|\int_0^{|\partial\Omega|} \left(\check{g}^N_{\mathrm{app}} - A\left(A+\pi\right)^{-1} f\right)\varphi\right| \\
&\quad = \left|\frac{1}{N}\sum_{j=1}^N \check{\gamma}^N_j A^*\varphi\left(l\left(s^N_j\right)\right) - \int_0^{|\partial\Omega|} \left(A+\pi\right)^{-1} f A^*\varphi\right| \\
&\quad \leqslant \left|\frac{|\partial\Omega|}{N}\sum_{j=1}^N \left(\frac{\check{\gamma}^N}{|\partial\Omega|} - \left(A+\pi\right)^{-1} f\left(s^N_j\right)\right) A^*\varphi\left(l\left(s^N_j\right)\right)\right| \\
&\qquad + \left|\frac{|\partial\Omega|}{N}\sum_{j=1}^N \left(A+\pi\right)^{-1} f\left(s^N_j\right) A^*\varphi\left(l\left(s^N_j\right)\right) - \int_0^{|\partial\Omega|} \left(A+\pi\right)^{-1} f A^*\varphi\right| \\
&\quad \leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa}\|A^*\varphi\|_{L^\infty} + \left\|\left(A+\pi\right)^{-1} f\right\|_{C^\kappa}\|A^*\varphi\|_{C^\kappa}\right) \\
&\quad \leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}\|\varphi\|_{L^2},
\end{aligned}$$

where we have used, again, in the last step above, that A and A^* are regularizing, which concludes the convergence estimate on $\check{g}^N_{\mathrm{app}}$.

As for $\check{f}^N_{\mathrm{app}}$, observe that

$$\int_0^{|\partial\Omega|} \check{f}^N_{\mathrm{app}}\varphi = \frac{1}{N}\sum_{j=1}^N \check{\gamma}^N_j B^*\varphi\left(l\left(s^N_j\right)\right).$$

Therefore, a similar estimate based on (8.9) and Corollary A.2 yields that

$$\begin{aligned}
&\left|\int_0^{|\partial\Omega|} \left(\check{f}^N_{\mathrm{app}} - B\left(A+\pi\right)^{-1} f\right)\varphi\right| \\
&\quad = \left|\frac{1}{N}\sum_{j=1}^N \check{\gamma}^N_j B^*\varphi\left(l\left(s^N_j\right)\right) - \int_0^{|\partial\Omega|} \left(A+\pi\right)^{-1} f B^*\varphi\right| \\
&\quad \leqslant \left|\frac{|\partial\Omega|}{N}\sum_{j=1}^N \left(\frac{\check{\gamma}^N}{|\partial\Omega|} - \left(A+\pi\right)^{-1} f\left(s^N_j\right)\right) B^*\varphi\left(l\left(s^N_j\right)\right)\right| \\
&\qquad + \left|\frac{|\partial\Omega|}{N}\sum_{j=1}^N \left(A+\pi\right)^{-1} f\left(s^N_j\right) B^*\varphi\left(l\left(s^N_j\right)\right) - \int_0^{|\partial\Omega|} \left(A+\pi\right)^{-1} f B^*\varphi\right| \\
&\quad \leqslant \frac{C}{N^\kappa}\left(\|f\|_{C^\kappa}\|B^*\varphi\|_{L^\infty} + \left\|\left(A+\pi\right)^{-1} f\right\|_{C^\kappa}\|B^*\varphi\|_{C^\kappa}\right) \\
&\quad \leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}\|B^*\varphi\|_{C^\kappa}.
\end{aligned}$$

This will complete the demonstration provided we show that $B^*\varphi$ is sufficiently regular, which is not obvious since B^* has a singular kernel. We have already discussed, in section 3, the Plemelj–Privalov theorem, which guarantees that $B^*\varphi \in C^{0,\alpha}$ provided

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$\varphi \in C^{0,\alpha}$ for any given $0 < \alpha < 1$. This, however, is not sufficient for our purpose and we need now to establish some higher regularity estimate on $B^*\varphi$. To this end, employing (3.5), we first write

$$
\begin{aligned}
B^*\varphi(s) &= \int_0^{|\partial\Omega|} \frac{l(s) - l(s_*)}{|l(s) - l(s_*)|^2} \cdot \tau(l(s_*)) \left(\varphi(s) - \varphi(s_*)\right) ds_* \\
&= \int_0^{|\partial\Omega|} \int_0^1 \left| \frac{l(s) - l(s_*)}{s - s_*} \right|^{-2} \frac{l(s) - l(s_*)}{s - s_*} \cdot \tau(l(s_*)) \varphi'(s_* + t(s - s_*)) dt ds_*.
\end{aligned}
$$

Then, upon noticing that $\frac{l(s)-l(s_*)}{s-s_*}$ and $\tau(l(s_*))$ are smooth and that $\frac{l(s)-l(s_*)}{s-s_*}$ remains bounded away from zero, this representation of $B^*\varphi$ easily yields that

$$
\|B^*\varphi\|_{C^\kappa} \leqslant C \|\varphi'\|_{C^\kappa} \leqslant C \|\varphi\|_{C^{\kappa+1}},
$$

which ends the proof. ◻

We are now in a position to give a complete justification of Theorem 8.1.

Proof of Theorem 8.1. We follow here the method of proof of Theorem 2.1 presented in section 6.

First, for given $\omega \in C_c^{0,\alpha}$ and $\gamma \in \mathbb{R}$, recall that the full plane flow $u_P \in C^1\left(\overline{\Omega}\right)$ is obtained from (2.2) and that the $|\partial\Omega|$-periodic function $f \in C^\infty\left([0, |\partial\Omega|]\right)$ is defined by $f(s) = -2\pi[(u_P + \gamma H_*) \cdot n]\left(l(s)\right)$ for all $s \in [0, |\partial\Omega|]$. Therefore, with this given f, according to Proposition 8.2, we find a unique solution $\check{\gamma}^N \in \mathbb{R}^N$ of (8.4).

Next, the approximate flow $\check{u}^N_{\mathrm{app}}$ is introduced by (8.1) and verifies

$$
\begin{aligned}
\check{u}^N_{\mathrm{app}}(x) \cdot \tau(x) &= \frac{1}{2\pi} \check{f}^N_{\mathrm{app}}(s) + \gamma H_* \cdot \tau, \\
\check{u}^N_{\mathrm{app}}(x) \cdot n(x) &= \frac{1}{2\pi} \check{g}^N_{\mathrm{app}}(s) + \gamma H_* \cdot n,
\end{aligned}
$$

where $x = l(s) \in \partial\Omega$ and $\check{f}^N_{\mathrm{app}}$, $\check{g}^N_{\mathrm{app}}$ are defined by (8.8). Recall that the values of H_* on $\partial\Omega$ are determined by its limiting values from within Ω. Utilizing identity (3.35) to rewrite the discrete kernel of $\check{u}^N_{\mathrm{app}}$, we find that

$$
\begin{aligned}
\check{u}^N_{\mathrm{app}}(x) = \frac{-1}{2\pi^2} &\sum_{j=1}^N \frac{\check{\gamma}^N_j}{N} \\
&\times \left(\int_{\partial\Omega} \frac{x^N_j - z}{\left|x^N_j - z\right|^2} \cdot \tau(z) \frac{(x-z)^\perp}{|x-z|^2} dz + \int_{\partial\Omega} \frac{x^N_j - z}{\left|x^N_j - z\right|^2} \cdot n(z) \frac{x-z}{|x-z|^2} dz \right) \\
&+ \gamma H_*(x) \\
= \frac{1}{2\pi^2} &\int_0^{|\partial\Omega|} \check{f}^N_{\mathrm{app}}(s_*) \frac{\left(x - l(s_*)\right)^\perp}{\left|x - l(s_*)\right|^2} ds_* \\
&+ \frac{1}{2\pi^2} \int_0^{|\partial\Omega|} \check{g}^N_{\mathrm{app}}(s_*) \frac{x - l(s_*)}{|x - l(s_*)|^2} ds_* + \gamma H_*(x), \quad \text{on } \Omega.
\end{aligned}
$$

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Furthermore, recall that, according to (3.37), the remainder flow u_R can be expressed as

$$u_R(x) = \frac{1}{2\pi^2}\int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} B\left(A+\pi\right)^{-1} f(y)dy + \frac{1}{2\pi^2}\int_{\partial\Omega} \frac{x-y}{|x-y|^2} A\left(A+\pi\right)^{-1} f(y)dy + \gamma H_*(x), \quad \text{on } \Omega,$$

whereby

$$\left(\check{u}^N_{\text{app}} - u_R\right)(x) = \frac{1}{2\pi^2}\int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2}\left(\check{f}^N_{\text{app}} - B\left(A+\pi\right)^{-1} f\right)(y)dy + \frac{1}{2\pi^2}\int_{\partial\Omega} \frac{x-y}{|x-y|^2}\left(\check{g}^N_{\text{app}} - A\left(A+\pi\right)^{-1} f\right)(y)dy, \quad \text{on } \Omega.$$

Therefore, in view of Proposition 8.3, we deduce that, for any fixed $x \in \Omega$,

$$\begin{aligned}\left|\left(\check{u}^N_{\text{app}} - u_R\right)(x)\right| &\leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}\left\|\frac{x-y}{|x-y|^2}\right\|_{C^{\kappa+1}_y} \\ &\leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}\sup_{y\in\partial\Omega}\left(\frac{1}{|x-y|} + \frac{1}{|x-y|^{\kappa+2}}\right),\end{aligned}$$

where the constant $C>0$ is independent of x, ω, and γ. Since the support of ω is bounded away from $\partial\Omega$, it holds that

$$\begin{aligned}\|f\|_{C^\kappa} &\leqslant C\sup_{x\in\partial\Omega}\int_{\mathbb{R}^2}\left(\frac{1}{|x-y|} + \frac{1}{|x-y|^{\kappa+1}}\right)|\omega(y)|\,dy + C|\gamma|\,\|H_*\cdot n\|_{C^\kappa} \\ &\leqslant \frac{C}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)}\left(1 + \frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)^\kappa}\right)\|\omega\|_{L^1(\mathbb{R}^2)} + C|\gamma|\,\|H_*\cdot n\|_{C^\kappa}.\end{aligned}$$

It follows that, for any closed set $K\subset\Omega$,

$$\begin{aligned}&\|u_R - \check{u}^N_{\text{app}}\|_{L^\infty(K)} \\ &\quad\leqslant \frac{C}{N^\kappa}\left(\frac{1}{\operatorname{dist}(K,\partial\Omega)} + \frac{1}{\operatorname{dist}(K,\partial\Omega)^{\kappa+2}}\right) \\ &\qquad\times\left(\left(\frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)} + \frac{1}{\operatorname{dist}(\operatorname{supp}\omega,\partial\Omega)^{\kappa+1}}\right)\|\omega\|_{L^1(\mathbb{R}^2)} + |\gamma|\right),\end{aligned}$$

which, extending the above estimate to all $\omega \in L^1_c(\Omega)$ by a standard density argument, concludes the proof of the theorem. ∎

8.2. A further refinement of the fluid charge method. With the purpose of improving the accuracy and efficiency of potential numerical methods based on the preceding theorems, we develop now a refinement of Theorem 8.1 which is simply based upon noticing that, by (3.8), the subspace $L^2_0(\partial\Omega)\subset L^2(\partial\Omega)$ is invariant under the action of $A+\pi$. In particular, defining the averaging operator over $\partial\Omega$ by

$$\langle h\rangle = \int_{\partial\Omega} h(y)dy$$

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for any $h \in L^1(\partial\Omega)$, since the operators $A+\pi$ and $A-\lambda\langle\cdot\rangle+\pi$ for any given smooth function $\lambda(x)$ on $\partial\Omega$ coincide on $L_0^2(\partial\Omega)$, it clearly holds that $A-\lambda\langle\cdot\rangle+\pi : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega)$ has a bounded inverse given by

$$(A-\lambda\langle\cdot\rangle+\pi)^{-1} = (A+\pi)^{-1} : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega).$$

It follows that (3.33) may be recast, for any smooth $\lambda(x)$, as

$$u_R(x) = -\int_{\partial\Omega} \frac{x-y}{|x-y|^2} (A-\lambda\langle\cdot\rangle+\pi)^{-1} [(u_P+\gamma H_*)\cdot n](y)dy + \gamma H_*(x) \tag{8.10}$$

for $(u_P+\gamma H_*)\cdot n$ in (3.33) has mean value zero (recall that u_P is divergence free in $\mathbb{R}^2$).

In fact, if $\langle\lambda\rangle \neq 2\pi$, then it is straightforward to verify that $A-\lambda\langle\cdot\rangle+\pi : L^2(\partial\Omega) \to L^2(\partial\Omega)$ also has a bounded inverse given by

$$(A-\lambda\langle\cdot\rangle+\pi)^{-1} h = (A+\pi)^{-1}\left[h + \frac{\langle h\rangle}{2\pi-\langle\lambda\rangle}\lambda\right] \tag{8.11}$$

for any $h \in L^2(\partial\Omega)$.

Aiming at discretizing (8.10) rather than (3.33), we propose now to build an approximate flow

$$\check{\breve{u}}_{\mathrm{app}}^N(x) := \frac{1}{2\pi}\sum_{j=1}^N \frac{\check{\breve{\gamma}}_j^N}{N}\frac{x-x_j^N}{|x-x_j^N|^2} + \gamma H_*(x), \tag{8.12}$$

where the density $\check{\breve{\gamma}}^N = (\check{\breve{\gamma}}_1^N,\dots,\check{\breve{\gamma}}_N^N) \in \mathbb{R}^N$ solves the following system of N linear equations, for any prescribed smooth function $\lambda(x)$ such that $\langle\lambda\rangle \neq 2\pi$:

$$\begin{aligned}\left(\frac{1}{2\pi}\sum_{j=1}^N \frac{\check{\breve{\gamma}}_j^N}{N}\left(\frac{\tilde{x}_i^N-x_j^N}{|\tilde{x}_i^N-x_j^N|^2}\cdot n(\tilde{x}_i^N) - \lambda\left(\tilde{x}_i^N\right)\right)\right) + \frac{\check{\breve{\gamma}}_i^N}{2|\partial\Omega|}&\\ = -[(u_P+\gamma H_*)\cdot n](\tilde{x}_i^N) \text{ for all } i=1,\dots,N.&\end{aligned} \tag{8.13}$$

As usual, we may use the notation $\check{\breve{u}}_{\mathrm{app}}^N = \check{\breve{u}}_{\mathrm{app}}^N[\omega,\gamma]$ to emphasize the linear dependence of $\check{\breve{u}}_{\mathrm{app}}^N$ in (ω,γ).

Equivalently, this system may be recast as

$$\left(\frac{1}{N}A_N - \lambda^N\langle\cdot\rangle + \frac{\pi}{|\partial\Omega|}\right)\check{\breve{\gamma}}^N = \left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}, \tag{8.14}$$

where $f(s) = -2\pi[(u_P+\gamma H_*)\cdot n](l(s))$ for all $s \in [0,|\partial\Omega|]$, and the vector $\lambda^N = \left(\lambda_1^N,\dots,\lambda_N^N\right) \in \mathbb{R}^N$ is defined by $\lambda_i^N = \lambda\left(\tilde{x}_i^N\right)$ for all $i=1,\dots,N$.

We arrive now at the following main theorem concerning the convergence of the approximate flow $\check{\breve{u}}_{\mathrm{app}}^N$ defined in (8.12).

THEOREM 8.4. *Let $\omega \in L_c^1(\Omega)$, $\gamma \in \mathbb{R}$, and a smooth function λ such that $\langle\lambda\rangle \neq 2\pi$ be given. For any $N \geqslant 2$, we consider a well-$*$ distributed mesh satisfying* (8.6) *and u_P defined in* (2.2).

Then, there exists N_0 (independent of ω and γ) such that, for all $N \geqslant N_0$, the system (8.13) *admits a unique solution $\check{\breve{\gamma}}^N \in \mathbb{R}^N$. Moreover, for any closed set $K \subset \Omega$*

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there exists a constant $C > 0$ independent of N, K, ω, and γ such that

$$\|u_R - \check{\tilde{u}}^N_{\mathrm{app}}\|_{L^\infty(K)} \leqslant \frac{C}{N^\kappa}\left(\frac{1}{\mathrm{dist}\,(K,\partial\Omega)} + \frac{1}{\mathrm{dist}\,(K,\partial\Omega)^{\kappa+2}}\right) \times \left(\left(\frac{1}{\mathrm{dist}\,(\mathrm{supp}\,\omega,\partial\Omega)} + \frac{1}{\mathrm{dist}\,(\mathrm{supp}\,\omega,\partial\Omega)^{\kappa+1}}\right)\|\omega\|_{L^1(\mathbb{R}^2)} + |\gamma|\right),$$

where $\check{\tilde{u}}^N_{\mathrm{app}}$ is given by (8.12) *in terms of $\check{\tilde{\gamma}}^N$ and u_R is the continuous flow* (2.3).

The justification of the above theorem is based on Propositions 8.5 and 8.6, which are established below. Using these results, the proof of the above theorem follows the same steps mutatis mutandis as the proof of Theorem 8.1 and, so, we leave it to the reader.

The next result is a simple extension of Proposition 8.2 and allows us to solve (8.14).

PROPOSITION 8.5. *For any integer $N \geqslant 2$, consider a well-$*$ distributed mesh $(s_1^N,\dots,s_N^N) \in \mathbb{R}^N$, $(\tilde{s}_1^N,\dots,\tilde{s}_N^N) \in \mathbb{R}^N$ and let λ be a smooth function such that $\langle\lambda\rangle \neq 2\pi$.*

Then, provided N is sufficiently large, the operator $\frac{1}{N}A_N - \lambda^N\langle\cdot\rangle + \frac{\pi}{|\partial\Omega|} \in \mathcal{L}\left(\ell^2\right)$ is invertible. Furthermore, its inverse operator is bounded in $\mathcal{L}\left(\ell^2\right)$ uniformly in N. It follows that, provided N is sufficiently large, the problem

$$z \in \mathbb{R}^N, \quad \left(\frac{1}{N}A_N - \lambda^N\langle\cdot\rangle + \frac{\pi}{|\partial\Omega|}\right) z = v$$

has a unique solution for any given $v \in \mathbb{R}^N$. Moreover, this solution satisfies

$$\|z\|_{\ell^1} \leqslant \|z\|_{\ell^2} \leqslant C\|v\|_{\ell^2} \leqslant C\|v\|_{\ell^\infty}$$

for some independent constant $C > 0$.

Proof. In accordance with Proposition 8.2, the operator $(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|})$ is invertible provided N is large. Therefore, any solution $z \in \mathbb{R}^N$ of the above system has to satisfy

$$z - \langle z\rangle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N = \left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} v,$$

whence

$$\langle z\rangle\left(1 - \left\langle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N\right\rangle\right) = \left\langle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} v\right\rangle. \tag{8.15}$$

It follows that, provided

$$\left\langle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N\right\rangle \neq 1,$$

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any solution has to be given by the formula

$$z = \left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\left(v + \frac{\left\langle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} v\right\rangle}{1 - \left\langle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N\right\rangle}\lambda^N\right),$$

which is a discrete analog to (8.11). It is straightforward to check that, for any given $v \in \mathbb{R}^N$, the above identity also provides a viable solution $z \in \mathbb{R}^N$.

Therefore, it only remains to verify that

$$\left\langle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N\right\rangle \neq 1.$$

To this end, observe first that

$$\left(\frac{|\partial\Omega|}{N}A_N^*\mathbf{1}\right)_j$$

with $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$, for each $j = 1, \dots, N$, is a discretization of the integral $(A^*1)(l(s_j^N))$. Since $A^*1 \equiv \pi$, by (3.7), we conclude, by the uniform convergence of Riemann sums for smooth functions (see Corollary A.2), that

$$\left\|\frac{|\partial\Omega|}{N}A_N^*\mathbf{1} - \pi\mathbf{1}\right\|_{\ell^\infty} = \mathcal{O}\left(N^{-\kappa}\right).$$

This elementary estimate is similar to (4.14), the difference being that, here, the mesh is not well distributed and the control does not concern B_N^*. By duality, it follows that, for all $z \in \mathbb{R}^N$,

$$\left|\left\langle\left(\frac{1}{N}A_N - \frac{\pi}{|\partial\Omega|}\right)z\right\rangle\right| \leqslant \frac{C}{N^\kappa}\|z\|_{\ell^1} \tag{8.16}$$

for some uniform constant $C > 0$. Then, we deduce

$$\begin{aligned}
&\left|\left\langle\lambda^N\right\rangle - \frac{2\pi}{|\partial\Omega|}\left\langle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N\right\rangle\right| \\
&\qquad = \left|\left\langle\left(\frac{1}{N}A_N - \frac{\pi}{|\partial\Omega|}\right)\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N\right\rangle\right| \\
&\qquad \leqslant \frac{C}{N^\kappa}\left\|\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N\right\|_{\ell^1},
\end{aligned}$$

which, since $|\partial\Omega|\left\langle\lambda^N\right\rangle = \langle\lambda\rangle + o(1)$ by the convergence of Riemann sums for continuous functions, and by the uniform boundedness of $(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|})^{-1} \in \mathcal{L}(\ell^2)$ asserted in Proposition 8.2, implies that

$$\left|\left\langle\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1}\lambda^N\right\rangle - \frac{\langle\lambda\rangle}{2\pi}\right| = o(1). \tag{8.17}$$

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Recalling that $\langle\lambda\rangle \neq 2\pi$, we finally deduce that

$$\left\langle \left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} \lambda^N \right\rangle \neq 1$$

for large values of N.

Since

$$\left\langle \left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} \lambda^N \right\rangle$$

is, in fact, uniformly bounded away from 1, as $N \to \infty$, the uniform boundedness in N of the inverse operator easily ensues, which concludes the proof of the proposition. ∎

It is also possible to obtain an extension of Proposition 8.3 concerning the weak convergence of the discretization of the operator $A - \lambda\langle\cdot\rangle + \pi$.

PROPOSITION 8.6. *For any integer $N \geqslant 2$, consider a well-$*$ distributed mesh $(s_1^N, \ldots, s_N^N) \in \mathbb{R}^N$, $(\tilde{s}_1^N, \ldots, \tilde{s}_N^N) \in \mathbb{R}^N$ satisfying* (8.6) *and, according to Proposition* 8.5, *consider the solution $\check{\tilde{\gamma}}^N = (\check{\tilde{\gamma}}_1^N, \ldots, \check{\tilde{\gamma}}_N^N) \in \mathbb{R}^N$ to the system* (8.14) *for some periodic function $f \in C^\kappa([0, |\partial\Omega|])$, where $\kappa \geqslant 2$ with zero mean value $\int_0^{|\partial\Omega|} f(s)ds = 0$ and some smooth function λ such that $\langle\lambda\rangle \neq 2\pi$. We define the approximations*

$$\check{\tilde{f}}_{\mathrm{app}}^N(s) := \frac{1}{N}\sum_{j=1}^N \check{\tilde{\gamma}}_j^N \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2} \cdot \tau(l(s)),$$

$$\check{\tilde{g}}_{\mathrm{app}}^N(s) := \frac{1}{N}\sum_{j=1}^N \check{\tilde{\gamma}}_j^N \frac{l(s) - l(s_j^N)}{\left|l(s) - l(s_j^N)\right|^2} \cdot n(l(s)).$$

Then, for any periodic test function $\varphi \in C^\infty([0, |\partial\Omega|])$,

$$\left|\int_0^{|\partial\Omega|} \left(\check{\tilde{f}}_{\mathrm{app}}^N - B(A+\pi)^{-1} f\right)\varphi\right| \leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}\|\varphi\|_{C^{\kappa+1}},$$

$$\left|\int_0^{|\partial\Omega|} \left(\check{\tilde{g}}_{\mathrm{app}}^N - A(A+\pi)^{-1} f\right)\varphi\right| \leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}\|\varphi\|_{L^2},$$

where we identify the variable x with the variable s whenever $x = l(s) \in \partial\Omega$ and the singular integrals are defined in the sense of Cauchy's principal value.

Proof. First, we estimate, by (8.16) and by the uniform boundedness of the operator $(\frac{|\partial\Omega|}{N}A_N + \pi)^{-1}$ in $\mathcal{L}(\ell^2)$ established in Proposition 8.2,

$$\begin{aligned}
&\left|\left\langle \left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} \left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N} \right\rangle - \frac{|\partial\Omega|}{2\pi}\left\langle \left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\rangle\right| \\
&\quad= \left|\left\langle \frac{|\partial\Omega|}{2\pi}\left(\frac{1}{N}A_N - \frac{\pi}{|\partial\Omega|}\right)\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} \left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\rangle\right| \\
&\quad\leqslant \frac{C}{N^\kappa}\left\|\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} \left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^1} \\
&\quad\leqslant \frac{C}{N^\kappa}\left\|\left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\|_{\ell^2} \leqslant \frac{C}{N^\kappa}\|f\|_{L^\infty},
\end{aligned}$$

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whence, since f has zero mean value over $\partial\Omega$, by the convergence of Riemann sums for smooth functions (see Corollary A.2),

$$\left|\left\langle \left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} \left(f\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N}\right\rangle\right| \leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}.$$

Further using (8.15) and recalling from (8.17) that

$$\left\langle \left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)^{-1} \lambda^N \right\rangle$$

remains bounded away from 1, as $N\to\infty$, we conclude that

$$\left|\langle\check{\tilde{\gamma}}^N\rangle\right| \leqslant \frac{C}{N^\kappa}\|f\|_{C^\kappa}. \tag{8.18}$$

Next, according to (8.14), it holds that

$$\left(\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}\right)\check{\tilde{\gamma}}^N = \left(h^N\left(\tilde{s}_i^N\right)\right)_{1\leqslant i\leqslant N},$$

where the periodic function $h^N \in C^\kappa([0,|\partial\Omega|])$ for each N is defined by $h^N(s) = f(s) + \lambda(l(s))\langle\check{\tilde{\gamma}}^N\rangle$ for all $s\in[0,|\partial\Omega|]$. Therefore, by Proposition 8.3, we infer that

$$\left|\int_0^{|\partial\Omega|}\left(\check{\tilde{f}}^N_{\mathrm{app}} - B(A+\pi)^{-1}h^N\right)\varphi\right| \leqslant \frac{C}{N^\kappa}\left\|h^N\right\|_{C^\kappa}\|\varphi\|_{C^{\kappa+1}},$$

$$\left|\int_0^{|\partial\Omega|}\left(\check{\tilde{g}}^N_{\mathrm{app}} - A(A+\pi)^{-1}h^N\right)\varphi\right| \leqslant \frac{C}{N^\kappa}\left\|h^N\right\|_{C^\kappa}\|\varphi\|_{L^2},$$

which, by the boundedness of A, B, and $(A+\pi)^{-1}$ over $L^2(\partial\Omega)$, implies that

$$\left|\int_0^{|\partial\Omega|}\left(\check{\tilde{f}}^N_{\mathrm{app}} - B(A+\pi)^{-1}f\right)\varphi\right| \leqslant C\left(\frac{1}{N^\kappa}\|f\|_{C^\kappa} + \left|\langle\check{\tilde{\gamma}}^N\rangle\right|\right)\|\varphi\|_{C^{\kappa+1}},$$

$$\left|\int_0^{|\partial\Omega|}\left(\check{\tilde{g}}^N_{\mathrm{app}} - A(A+\pi)^{-1}f\right)\varphi\right| \leqslant C\left(\frac{1}{N^\kappa}\|f\|_{C^\kappa} + \left|\langle\check{\tilde{\gamma}}^N\rangle\right|\right)\|\varphi\|_{L^2}.$$

Finally, incorporating (8.18) into the preceding estimate completes the proof of the proposition. ∎

8.3. Good conditioning of discretized systems. Theorem 8.1 provides an alternative method for building approximate flows which may, in many cases, yield efficient numerical methods outperforming the corresponding methods based on Theorem 2.1. Indeed, Theorem 8.1 requires the resolution of systems given by the matrices $\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}$ (for a well-$*$ distributed mesh). The fact that the coefficients of $\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}$ on its diagonal are of order $\mathcal{O}(1)$ and, thus, dominate those off the diagonal, which are of order $\mathcal{O}(N^{-1})$, guarantees good conditioning properties, which allow one to solve the corresponding systems with good numerical accuracy.

More precisely, supposing, for instance, that $\partial\Omega$ is strictly convex so that the kernel of the operator A, defined in (3.1), satisfies $\frac{x-y}{|x-y|^2}\cdot n(x) \geqslant C_0 > 0$ for all

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$x, y \in \partial\Omega$ and for some $C_0 > 0$, we see that, for each given $j = 1, \ldots, N$, according to (3.7) and Corollary A.2, for sufficiently large N,

$$\begin{aligned}\sum_{\substack{1\leqslant i\leqslant N\\ i\neq j}} \left|\left(\frac{|\partial\Omega|}{N}A_N + \pi\right)_{ij}\right| &= \frac{|\partial\Omega|}{N}\sum_{\substack{1\leqslant i\leqslant N\\ i\neq j}} (A_N)_{ij} \\ &= \int_{\partial\Omega} \frac{x - x_j^N}{\left|x - x_j^N\right|^2}\cdot n(x)dx + \mathcal{O}\left(N^{-2}\right) - \frac{|\partial\Omega|}{N}(A_N)_{jj} \\ &\leqslant \int_{\partial\Omega} \frac{x - x_j^N}{\left|x - x_j^N\right|^2}\cdot n(x)dx + \underbrace{\mathcal{O}\left(N^{-2}\right) - \frac{C_0\,|\partial\Omega|}{N}}_{<0 \text{ for large } N} \\ &< \int_{\partial\Omega} \frac{x - x_j^N}{\left|x - x_j^N\right|^2}\cdot n(x)dx = \pi < \left|\left(\frac{|\partial\Omega|}{N}A_N + \pi\right)_{jj}\right|.\end{aligned} \tag{8.19}$$

In other words, whenever $\partial\Omega$ is striclty convex, the matrix $\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}$ is strictly diagonally dominant with respect to columns, which opens the door to efficient and accurate numerical resolution methods for the corresponding systems. In particular, the LU decomposition exists and no pivoting is necessary in Gaussian elimination (see [16, section 4.1]).

Such diagonal dominance properties never hold for the methods based on Theorem 2.1, which require the resolution of large systems whose coefficients stem from the nonintegrable kernel of the singular operator B defined in (3.1) and are therefore prone to large numerical errors.

Like Theorem 8.1, Theorem 8.4 provides an alternative method for building approximate flows. In fact, Theorem 8.4 is more general and reduces to Theorem 8.1 by setting $\lambda \equiv 0$ therein. Numerically, the extra degree of freedom provided by the parameter λ is significant, for it may lead, in numerous case (depending on the geometry of $\partial\Omega$), to large linear systems whose coefficient matrices can be better conditioned with an appropriate choice of λ. As previously mentioned, the case $\lambda \equiv 0$ is sufficient to produce well conditioned systems in strictly convex geometries (recall that $\frac{1}{N}A_N + \frac{\pi}{|\partial\Omega|}$ is strictly diagonally dominant with respect to columns whenever $\partial\Omega$ is strictly convex; see (8.19)). Now, we are also able to handle some nonconvex geometries by appropriately setting $\lambda \not\equiv 0$.

Indeed, let us suppose, for instance, that the geometry of $\partial\Omega$ is such that the following analytical condition is satisfied:

$$\sup_{y\in\partial\Omega}\int_{\partial\Omega}\left|\frac{x-y}{|x-y|^2}\cdot n(x) - \lambda(x)\right| dx < \pi \tag{8.20}$$

for some smooth λ. Observe that, by (3.7), it necessarily holds that $0 < \langle\lambda\rangle < 2\pi$. Then, we see, for each given $j = 1, \ldots, N$, according to (3.7) and Corollary A.2, for sufficiently large N, that

$$\begin{aligned}&\sum_{\substack{1\leqslant i\leqslant N\\ i\neq j}} \left|\left(\frac{|\partial\Omega|}{N}A_N - |\partial\Omega|\,\lambda^N\langle\cdot\rangle + \pi\right)_{ij}\right| \\ &\qquad = \frac{|\partial\Omega|}{N}\sum_{\substack{1\leqslant i\leqslant N\\ i\neq j}}\left|(A_N)_{ij} - \lambda_i^N\right|\end{aligned}$$

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$$= \int_{\partial\Omega} \left| \frac{x - x_j^N}{\left|x - x_j^N\right|^2} \cdot n(x) - \lambda(x) \right| dx + \mathcal{O}\left(N^{-1}\right) - \frac{|\partial\Omega|}{N} \left| (A_N)_{jj} - \lambda_j^N \right|$$
$$< \pi - \frac{|\partial\Omega|}{N} \left| (A_N)_{jj} - \lambda_j^N \right| \leqslant \left| \frac{|\partial\Omega|}{N} \left((A_N)_{jj} - \lambda_j^N \right) + \pi \right|$$
$$= \left| \left(\frac{|\partial\Omega|}{N} A_N - |\partial\Omega| \, \lambda^N \langle \cdot \rangle + \pi \right)_{jj} \right|.$$

In other words, the matrix $\frac{1}{N} A_N - \lambda^N \langle \cdot \rangle + \frac{\pi}{|\partial\Omega|}$ is strictly diagonally dominant with respect to columns, for large N, as soon as (8.20) is satisfied. Again, we insist on the fact that this property leads to well conditioned systems and thus a significant potential improvement of the corresponding numerical resolution.

8.4. Geometric interpretation of (8.20). For simplicity, we first consider the case where $\lambda(x)$ in (8.20) is identically equal to a constant which we also denote by $0 < \lambda < \frac{2\pi}{|\partial\Omega|}$.

Then, it is readily seen that the L^1-condition (8.20) is implied by the stricter L^2-condition

$$\sup_{y \in \partial\Omega} \int_{\partial\Omega} \left(\frac{x - y}{|x - y|^2} \cdot n(x) - \lambda \right)^2 dx < \frac{\pi^2}{|\partial\Omega|},$$

whose left-hand side is quadratic in λ and therefore minimized, in view of (3.7), by the value $\lambda = \frac{\pi}{|\partial\Omega|}$. It follows that (8.20) holds with $\lambda(x) \equiv \frac{\pi}{|\partial\Omega|}$ provided

$$\sup_{y \in \partial\Omega} \int_{\partial\Omega} \left(\frac{x - y}{|x - y|^2} \cdot n(x) \right)^2 dx < \frac{2\pi^2}{|\partial\Omega|},$$

or, even more stringently,

$$\sup_{x, y \in \partial\Omega} \left| \frac{x - y}{|x - y|^2} \cdot n(x) \right| < \sqrt{2} \frac{\pi}{|\partial\Omega|}. \tag{8.21}$$

Notice that $\frac{x-y}{|x-y|^2} \cdot n(x) = \frac{\pi}{|\partial\Omega|}$ for all $x, y \in \partial\Omega$ if $\partial\Omega$ is a circle. Therefore, we may interpret the preceding conditions with $\lambda(x) \equiv \frac{\pi}{|\partial\Omega|}$ as a requirement that $\partial\Omega$ does not deviate too much from a circle of equal circumference.

More precisely, for any $R \in \mathbb{R} \setminus \{0\}$, one easily verifies that the constraint

$$\frac{x - y}{|x - y|^2} \cdot n(x) = \frac{1}{2R} \tag{8.22}$$

is equivalent to the relation

$$|y - (x - Rn(x))| = |R|. \tag{8.23}$$

Recalling that $\frac{x-y}{|x-y|^2} \cdot n(x)$ for each fixed $y \in \partial\Omega$ has an average value of $\frac{\pi}{|\partial\Omega|}$ over

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$x \in \partial\Omega$, we further introduce $R_{\sup} \in (0, \frac{|\partial\Omega|}{2\pi}]$ and $R_{\inf} \in (-\infty, 0) \cup [\frac{|\partial\Omega|}{2\pi}, \infty]$ defined by

$$\sup_{x,y\in\partial\Omega}\left(\frac{x-y}{|x-y|^2}\cdot n(x)\right) = \frac{1}{2R_{\sup}},$$
$$\inf_{x,y\in\partial\Omega}\left(\frac{x-y}{|x-y|^2}\cdot n(x)\right) = \frac{1}{2R_{\inf}}.$$

Observe that $R_{\inf} \in [\frac{|\partial\Omega|}{2\pi}, \infty]$ if $\overline{\Omega}^c$ is convex, whereas $R_{\inf} \in (-\infty, 0)$ if $\overline{\Omega}^c$ is nonconvex. In view of the equivalence between between (8.22) and (8.23), we have the following properties:

- $R_{\sup}$ is the largest radius $R > 0$ such that, for each $x \in \partial\Omega$, the domain $\overline{\Omega}^c$ contains an open ball of radius R tangent to $\partial\Omega$ at x,
- if $\overline{\Omega}^c$ is convex, $R_{\inf}$ is the smallest radius $R > 0$ such that, for each $x \in \partial\Omega$, the domain $\overline{\Omega}^c$ is contained in an open ball of radius R tangent to $\partial\Omega$ at x,
- if $\overline{\Omega}^c$ is nonconvex, $R_{\inf}$ is negative, and $|R_{\inf}|$ is the largest radius $R > 0$ such that, for each $x \in \partial\Omega$, the exterior domain Ω contains an open ball of radius R tangent to $\partial\Omega$ at x.

Thus, we arrive at the following geometric interpretation: condition (8.21) holds if and only if there exists $\frac{|\partial\Omega|}{2\pi\sqrt{2}} < R \leqslant \frac{|\partial\Omega|}{2\pi}$ such that, for each $x \in \partial\Omega$, there are two open balls of radius R, one contained in Ω and the other in $\overline{\Omega}^c$, both tangent to $\partial\Omega$ at x (note that $R > \frac{|\partial\Omega|}{2\pi}$ is not possible by the isoperimetric inequality). This criterion includes a large variety of nonconvex geometries.

The preceding analysis can also offer a more general geometric interpretation of (8.20) with nonconstant parameters $\lambda(x)$. For instance, considering the following reasonable choice of parameter,

$$\lambda(x) = (1-\sigma)\sup_{y\in\partial\Omega}\left(\frac{x-y}{|x-y|^2}\cdot n(x)\right) + \sigma\inf_{y\in\partial\Omega}\left(\frac{x-y}{|x-y|^2}\cdot n(x)\right),$$

for some given $0 \leqslant \sigma \leqslant 1$, it is readily seen that (8.20) holds provided that

$$(1-\sigma)\left(\int_{\partial\Omega}\sup_{y\in\partial\Omega}\left(\frac{x-y}{|x-y|^2}\cdot n(x)\right)dx - \pi\right) + \sigma\left(\pi - \int_{\partial\Omega}\inf_{y\in\partial\Omega}\left(\frac{x-y}{|x-y|^2}\cdot n(x)\right)dx\right) < \pi,$$

or, even more stringently,

$$(1-\sigma)\left(\frac{1}{2R_{\sup}} - \frac{\pi}{|\partial\Omega|}\right) + \sigma\left(\frac{\pi}{|\partial\Omega|} - \frac{1}{2R_{\inf}}\right) < \frac{\pi}{|\partial\Omega|}. \tag{8.24}$$

Note, then, that setting $\sigma = 1$ reduces the above condition to the simple requirement that $\partial\Omega$ be strictly convex, i.e., $R_{\inf} > 0$, whereas the value $\sigma = 0$ yields the new criterion

$$R_{\sup} > \frac{|\partial\Omega|}{4\pi}, \tag{8.25}$$

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which is much less restrictive than (8.21). The geometric interpretation of (8.25) is as follows: condition (8.25) holds if and only if there exists $\frac{|\partial\Omega|}{4\pi} < R \leqslant \frac{|\partial\Omega|}{2\pi}$ such that, for each $x \in \partial\Omega$, there is an open ball of radius R contained in $\overline{\Omega}^c$ and tangent to $\partial\Omega$ at x.

Finally, other values $0 < \sigma < 1$ in (8.24) can be loosely interpreted as an interpolation of the geometric conditions for $\sigma = 0$ and $\sigma = 1$.

8.5. Dynamic convergence of the fluid charge approximation. We end this section on the fluid charge method by providing dynamic theorems which are analog to Theorems 2.2 and 2.3 on the vortex method.

To this end, for any prescribed smooth λ such that $\langle\lambda\rangle \neq 2\pi$ and for sufficiently large integers N (at least as large as N_0 determined by Theorem 8.4 so that (8.13) is invertible; see also Proposition 8.5), we consider the approximate system

$$\begin{cases} \partial_t \breve{\breve{\omega}}^N + \breve{\breve{u}}^N \cdot \nabla \breve{\breve{\omega}}^N = 0, \\ \breve{\breve{\omega}}^N(t=0) = \omega_0 \end{cases} \tag{8.26}$$

for some initial data $\omega_0 \in C_c^1(\Omega)$ extended by zero outside Ω and with a velocity flow

$$\breve{\breve{u}}^N = K_{\mathbb{R}^2}\left[\breve{\breve{\omega}}^N\right] + \breve{\breve{u}}^N_{\text{app}}\left[\breve{\breve{\omega}}^N, \gamma\right],$$

where $\breve{\breve{u}}^N_{\text{app}}\left[\breve{\breve{\omega}}^N, \gamma\right]$ is given by (8.12)–(8.13), for some prescribed $\gamma \in \mathbb{R}$ and where u_P in the right-hand side of (8.13) is now $K_{\mathbb{R}^2}\left[\breve{\breve{\omega}}^N\right]$.

By repeating the arguments leading up to Theorem 2.2, we deduce the following corresponding result. Its proof only requires slight adaptations from the proof of Theorem 2.2 and so we omit it.

THEOREM 8.7. *Let $\omega_0 \in C_c^1(\Omega)$, $\gamma \in \mathbb{R}$, a smooth function λ be such that $\langle\lambda\rangle \neq 2\pi$ and consider any fixed time $t_1 > 0$. Then, for a well-$*$ distributed mesh on $\partial\Omega$, there exists $N_1 \geqslant N_0$ (N_0 is determined in Theorem 8.4) such that, for any $N \geqslant N_1$, there is a unique classical solution $\breve{\breve{\omega}}^N \in C_c^1([0,t_1]\times\Omega)$ to (8.26). Moreover, the sequence of solutions $\left\{\breve{\breve{\omega}}^N\right\}_{N\geqslant N_1}$ is uniformly bounded in $C_c^1([0,t_1]\times\Omega)$.*

The following theorem establishes the convergence of system (8.26) toward system (2.11) as $N \to \infty$. Its proof is similar to the justification of Theorem 2.3 and so we leave it to the reader.

THEOREM 8.8. *Let $\omega_0 \in C_c^1(\Omega)$, $\gamma \in \mathbb{R}$, a smooth function λ be such that $\langle\lambda\rangle \neq 2\pi$ and consider any fixed time $t_1 > 0$. Then, for a well-$*$ distributed mesh on $\partial\Omega$, as $N \to \infty$, the unique classical solution $\breve{\breve{\omega}}^N \in C_c^1([0,t_1]\times\Omega)$ to (8.26) converges uniformly toward the unique classical solution $\omega \in C_c^1([0,t_1]\times\Omega)$ to (2.11). More precisely, it holds that*

$$\left\|\omega - \breve{\breve{\omega}}^N\right\|_{L^\infty([0,t_1]\times\Omega)} = \mathcal{O}\left(N^{-\kappa}\right).$$

Appendix A. Convergence rates of Riemann sums. The following elementary lemma is a reminder about standard estimates on the rate of convergence of Riemann sums.

LEMMA A.1. *Consider the uniformly distributed mesh $(\theta_1^N,\dots,\theta_N^N) \in [0,|\partial\Omega|)^N$, $(\tilde\theta_1^N,\dots,\tilde\theta_N^N) \in [0,|\partial\Omega|)^N$ defined by (2.10) and let g be a smooth periodic function on $[0,|\partial\Omega|]$.*

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Then, for any $\kappa \geqslant 2$ *and* $N \geqslant 2$

$$\left| \int_0^{|\partial\Omega|} g(\theta) d\theta - \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} g(\tilde{\theta}_i^N) \right| \leqslant \frac{\pi^2 |\partial\Omega|^{\kappa+1}}{3(2\pi N)^\kappa} \|g^{(\kappa)}\|_{L^\infty}.$$

Proof. As g is of class C^1, its Fourier series converges uniformly to g:

$$g(x) = \sum_{n=-\infty}^{+\infty} c_n(g) e^{i2\pi nx/|\partial\Omega|}, \quad \text{where } c_n(g) = \frac{1}{|\partial\Omega|} \int_0^{|\partial\Omega|} g(t) e^{-i2\pi nt/|\partial\Omega|}\, dt.$$

Hence we compute for any N

$$\begin{aligned} \frac{|\partial\Omega|}{N} \sum_{j=1}^{N} g(\tilde{\theta}_j^N) &= \frac{|\partial\Omega|}{N} \sum_{j=1}^{N} \sum_{n=-\infty}^{+\infty} c_n(g) e^{i2\pi n \tilde{\theta}_j^N/|\partial\Omega|} \\ &= |\partial\Omega| c_0(g) + \frac{|\partial\Omega|}{N} \sum_{n\neq 0} c_n(g) e^{i\pi n/N} \sum_{j=0}^{N-1} (e^{i2\pi n/N})^j \\ &= |\partial\Omega| c_0(g) + |\partial\Omega| \sum_{n\neq 0} c_{nN}(g)(-1)^n. \end{aligned}$$

By κ integrations by parts in the definition of $c_{nN}(g)$, we get for any $\kappa \geqslant 2$

$$\begin{aligned} \left| \int_0^{|\partial\Omega|} g(\theta) d\theta - \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} g(\tilde{\theta}_i^N) \right| &\leqslant \frac{|\partial\Omega|^{\kappa+1}}{(2\pi N)^\kappa} \|g^{(\kappa)}\|_{L^\infty} \sum_{n\neq 0} \frac{1}{|n|^\kappa} \\ &\leqslant \frac{|\partial\Omega|^{\kappa+1}}{(2\pi N)^\kappa} \|g^{(\kappa)}\|_{L^\infty} \frac{\pi^2}{3}, \end{aligned}$$

which ends the proof of the lemma. ∎

The preceding lemma can also be easily adapted to more general meshes, which is the content of the following result.

Corollary A.2. *For any* $N \geqslant 2$*, consider a mesh* $(\tilde{s}_1^N, \ldots, \tilde{s}_N^N) \in [0, |\partial\Omega|)^N$ *satisfying*

$$\max_{i=1,\ldots,N} \left| \tilde{s}_i^N - \tilde{\theta}_i^N \right| = \mathcal{O}\left(N^{-\kappa}\right)$$

for some $\kappa \geqslant 2$ *and let* g *be a smooth periodic function on* $[0, |\partial\Omega|]$.

Then, there exists a constant $C > 0$ *depending only on* κ *and*

$$\sup_{N\geqslant 2} \max_{i=1,\ldots,N} N^\kappa \left| \tilde{s}_i^N - \tilde{\theta}_i^N \right|$$

such that

$$\left| \int_0^{|\partial\Omega|} g(s) ds - \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} g(\tilde{s}_i^N) \right| \leqslant \frac{C}{N^\kappa} \|g\|_{C^\kappa}$$

for any $N \geqslant 2$.

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Proof. By Lemma A.1, it is readily seen that

$$\begin{aligned}
&\left| \int_0^{|\partial\Omega|} g(s)ds - \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} g(\tilde{s}_i^N) \right| \\
&\qquad \leqslant \left| \int_0^{|\partial\Omega|} g(\theta)d\theta - \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} g(\tilde{\theta}_i^N) \right| + \frac{|\partial\Omega|}{N} \sum_{i=1}^{N} \left| g(\tilde{\theta}_i^N) - g(\tilde{s}_i^N) \right| \\
&\qquad \leqslant \frac{C}{N^\kappa} \|g\|_{C^\kappa} + |\partial\Omega| \, \|g\|_{C^1} \left(\max_{i=1,\dots,N} \left| \tilde{s}_i^N - \tilde{\theta}_i^N \right| \right) \\
&\qquad \leqslant \frac{C}{N^\kappa} \|g\|_{C^\kappa},
\end{aligned}$$

which concludes the proof. ∎

REFERENCES

[1] D. Arsénio, E. Dormy, and C. Lacave, *The vortex method for 2D ideal flows in the exterior of a disk*, Journées équations aux dérivées partielles, 2014.

[2] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren Math. Wiss. 343, Springer-Verlag, Berlin, 2011.

[3] S. M. Belotserkovsky and I. K. Lifanov, *Method of Discrete Vortices*, CRC Press, Boca Raton, FL, 1993.

[4] T. Chang and K. Lee, *Spectral properties of the layer potentials on Lipschitz domains*, Illinois J. Math., 52 (2008), pp. 463–472.

[5] J.-P. Choquin, G.-H. Cottet, and S. Mas-Gallic, *On the validity of vortex methods for nonsmooth flows*, in Vortex Methods, Lecture Notes in Math. 1360, Springer-Verlag, Berlin, 1988, pp. 56–67.

[6] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[7] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. of Math. (2), 116 (1982), pp. 361–387.

[8] G.-H. Cottet and P. D. Koumoutsakos, *Vortex Methods: Theory and Practice*, Cambridge University Press, Cambridge, 2000.

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics: Partial Differential Equations* Vol II, Wiley Classics Libr., Wiley, New York, 1989.

[10] E. Fabes, M. Jodeit, and N. Riviere, *Potential techniques for boundary value problems on C^1-domains*, Acta Math., 141 (1978), pp. 165–186.

[11] E. Fabes, M. jun. Jodeit, and J. E. Lewis, *Double layer potentials for domains with corners and edges*, Indiana Univ. Math. J., 26 (1977), pp. 95–114.

[12] E. Fabes, M. Sand, and J. K. Seo, *The spectral radius of the classical layer potentials on convex domains*, in Partial Differential Equations with Minimal Smoothness and Applications, Springer-Verlag, Berlin, 1992, pp. 129–137.

[13] D. Gérard-Varet and C. Lacave, *The two dimensional Euler equations on singular exterior domains*, Arch. Ration. Mech. Anal., 218 (2015), pp. 1609–1631.

[14] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics Math., Springer-Verlag, Berlin, 2001.

[15] A. Ginevsky and A. Zhelannikov, *Vortex Wakes of Aircrafts*, Springer-Verlag, Berlin, 2009.

[16] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th ed., Johns Hopkins Stud. Math. Sci., Johns Hopkins University Press, Baltimore, MD, 2013.

[17] J. Goodman, T. Y. Hou, and J. Lowengrub, *Convergence of the point vortex method for the 2-D Euler equations*, Comm. Pure Appl. Math., 43 (1990), pp. 415–430.

[18] L. Grafakos, *Classical Fourier analysis.* 2nd ed., Springer-Verlag, Berlin, 2008.

[19] J. D. Gray and S. A. Morris, *When is a function that satisfies the Cauchy-Riemann equations analytic?*, Amer. Math. Monthly, 85 (1978), pp. 246–256.

[20] C. Hirsch, *Numerical Computation of Internal and External Flows*, Wiley Ser. Numer. Methods Eng., Wiley, New York, 1988.

[21] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes, *Two dimensional incompressible ideal flow around a small obstacle*, Comm. Partial Differential Equations, 28 (2003), pp. 349–379.

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[22] O. Kellogg, *Foundations of Potential Theory*, Springer-Verlag, Berlin, 1967.

[23] K. Kikuchi, *Exterior problem for the two-dimensional Euler equation*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 30 (1983), pp. 63–92.

[24] C. Lacave, E. Miot, and C. Wang, *Uniqueness for the two-dimensional Euler equations on domains with corners*, Indiana Univ. Math. J., 63 (2014), pp. 1725–1756.

[25] J.-G. Liu and Z. Xin, *Convergence of the point vortex method for 2-D vortex sheet*, Math. Comp., 70 (2001), pp. 595–606.

[26] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts Appl. Math. 27, Cambridge University Press, Cambridge, 2002.

[27] C. Marchioro and M. Pulvirenti, *On the vortex-wave system*, in Mechanics, Analysis and Geometry: 200 Years After Lagrange, North-Holland Delta Ser., North-Holland, Amsterdam, 1991, pp. 79–95.

[28] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Appl. Math. Sci. 96, Springer-Verlag, New York, 1994.

[29] N. I. Muskhelishvili, *Singular Integral Equations: Boundary Problems of Functions Theory and Their Applications to Mathematical Physics*, Wolters-Noordhoff Publishing, Groningen, 1972.

[30] M. Reed and B. Simon, *Functional Analysis*, Methods of Modern Mathematical Physics, Vol. 1, 2nd ed., Academic Press, New York, 1980.

[31] S. Schochet, *The point-vortex method for periodic weak solutions of the 2-D Euler equations*, Comm. Pure Appl. Math., 49 (1996), pp. 911–965.

[32] E. F. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction*, 2nd ed., Springer-Verlag, Berlin, 1999.

[33] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), pp. 572–611.

[34] H. Whitney, *Functions differentiable on the boundaries of regions*, Ann. of Math. (2), 35 (1934), pp. 482–485.

[35] K. Yosida, *Functional Analysis*, 6th ed., Grundlehren Math. Wiss. 123, Springer-Verlag, Berlin, 1980.

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