ON THE INTEGRAL HODGE CONJECTURE FOR 3-FOLDS

by

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Date: 12/10/04

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics
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ABSTRACT

(Mathematics)

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Abstract

We consider the following adaptation of Hodge's original conjecture : given a smooth, complex, connected, projective variety X, does every cohomology class in $H^{2k}(X, \mathbb{Z})$ / $H^{2k}(X, \mathbb{Z})_{tors}$ of Hodge type (k, k) have an algebraic representative? We call this adaptation the non-torsion version of the integral Hodge conjecture.

We focus on the conjecture in the case of 3-folds. The only example where this was shown not to hold is due to János Kollár, with the example being a very general hypersurfaces in $\mathbb{P}^4$. Using an adaptation of Kollár's argument, we show that very general hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ fail the non-torsion version of the integral Hodge conjecture. We also show that this condition holds for abelian 3-folds and for Fano 3-folds.

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Chapter 1

Introduction

One of the most important problems in algebraic geometry is to know what subvarieties exist in a variety X. More generally, we want to understand the k-cycles of X, which are integral linear combinations of subvarieties of dimension k. In general, obtaining results for k-cycles is a difficult task. When X is a complex, projective variety, we try to make the problem more tractable by moving from k-cycles on X to k-cycles on X modulo homological equivalence, so that, associated to each such cycle is a homology class in $H_{2k}(X, \mathbb{Z})$. If, in addition, X is smooth and connected, we may associate to k-cycles a cohomology class in $H^{2n-2k}(X, \mathbb{Z})$, where $n = \dim X$, using Poincaré Duality.

When passing to cohomology classes for a smooth, complex, connected, projective variety X, we may tensor $H^{2n-2k}(X, \mathbb{Z})$ with the complex numbers to obtain classes in $H^{2n-2k}(X, \mathbb{C})$, thus allowing us to employ Hodge theory. Not all classes in $H^{2n-2k}(X, \mathbb{C})$ are associated to k-cycles. In fact, a necessary condition is that every such class must have Hodge type $(n-k, n-k)$. If $k = n-1$, then this is also a sufficient condition by the Lefschetz-$(1,1)$ Theorem. In [19], Hodge asked if this was a sufficient condition for all k. In other words, Hodge made the following conjecture : given a smooth, complex, connected, projective variety X, every class in $H^{2k}(X,\mathbb{Z})$ of Hodge type (k,k) is algebraic, or the class associated to an $n-k$-cycle. This conjecture is what I will call Hodge's original conjecture or the integral Hodge conjecture.

Hodge's original conjecture was shown to be false by Atiyah and Hirzebruch in 1962 in [3]. Specifically, for $k \geq 2$, they construct smooth, complex, connected, projective varieties X with torsion classes $\gamma \in H^{2k}(X,\mathbb{Z})$ that are not the cohomology classes of codimension k cycles. The counterexample makes essential use of torsion classes, so that the method used to construct the counterexample does not generalize to non-torsion classes. This led to the following reformulation of the Hodge Conjecture in terms of rational cohomology classes : given a smooth, complex, connected, projective variety X, every class $\gamma \in H^{2k}(X,\mathbb{Q})$ of Hodge type (k,k) is algebraic. Note that torsion classes become the zero class when tensored with $\mathbb{Q}$, and thus the torsion classes constructed by Atiyah and Hirzebruch do not provide a counterexample to this reformulation of the Hodge Conjecture.

The above reformulation of the Hodge conjecture is not the only possible reformulation. A natural question to ask is if the only counterexamples to Hodge's original conjecture are torsion classes. This requires restating Hodge's original conjecture in terms of classes $\gamma \in H^{2k}(X,\mathbb{Z})/H^{2k}(X,\mathbb{Z})_{tors}$. We say that a smooth, complex, connected, projective variety X satisfies the non-torsion version of the integral Hodge conjecture, or X satisfies NHC, if all classes $\gamma \in H^{2k}(X,\mathbb{Z})/H^{2k}(X,\mathbb{Z})_{tors}$ of Hodge type (k, k) are represented by a cycle of codimension k.

The non-torsion version of the conjecture is also false, as was shown by Kollár in [4]. As described in Section 2.2, Kollár's counterexample is a smooth hypersurface in $\mathbb{P}^4$. Since X is a hypersurface, $H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$ by the Lefschetz Hyperplane Theorem. Hence, all classes in $H^4(X, \mathbb{Z})$ are non-torsion classes of Hodge type $(2, 2)$. The counterexample in this case will be the generator of $H^4(X, \mathbb{Z})$. In fact, Kollár proves a stronger result, namely that, "very general" hypersurfaces X of degree d in $\mathbb{P}^4$, with additional conditions on d, fail Hodge's original conjecture.

Beyond the previously mentioned work of Atiyah and Hirzebruch and of Kollár, very little is known about when Hodge's original conjecture or when the non-torsion integral Hodge conjecture holds, even in the case of 3-folds. Our goal is to prove or disprove the non-torsion version of the integral Hodge conjecture for various types of 3-folds. To accomplish this goal, we will need background material about how the (non-torsion) integral Hodge conjecture behaves under surjective maps and in smooth families of smooth, complex, connected, projective varieties. We also consider the behavior of the (non-torsion) integral Hodge conjecture under birational equivalence, and we show that failure of the (non-torsion) integral Hodge conjecture is a birational invariant for smooth, complex, connected, projective varieties X of dimension n when considering classes of Hodge type $(n-1, n-1)$. For classes of Hodge type $(n-k, n-k)$ and $k \geq 2$, the (non-torsion) integral Hodge conjecture can fail to be a birational invariant, though this will not be an obstacle for our purposes, since we focus on the case of 3-folds, and failure of the (non-torsion) integral Hodge conjecture can only occur in 3-folds for $k = 1$ by the Lefschetz-$(1, 1)$ Theorem. This material, along with background definitions, is covered in Chapter 2.

Chapter 3 is where we prove the non-torsion version of the integral Hodge conjecture for abelian 3-folds and for Fano 3-folds. In the case of abelian 3-folds, Hodge's original conjecture and the non-torsion version of the conjecture are equivalent, since the cohomology of an abelian variety has no torsion. The main tool used, Proposition 3.1.8, states that failure of the integral Hodge conjecture for abelian varieties reduces to failure of the integral Hodge conjecture for principally polarized abelian varieties. The proof of Proposition 3.1.8 uses the Fourier transformation on the cohomology of an abelian variety.

We also prove the non-torsion version of the integral Hodge conjecture for Fano 3-folds. As an auxiliary step, we prove the non-torsion version of the integral Hodge conjecture for certain types of conic bundles, a conic bundle being a surjective morphism $f : X \to S$, where X is smooth and of dimension 3, and where the fibers of f are isomorphic to degree 2 curves in $\mathbb{P}^2$. We prove the non-torsion integral Hodge conjecture for conic bundles having singular fibers and for conic bundles with a base B that is simply connected. We then use the work of Mori and Mukai and of Iskovskih and Sokurov which classifies Fano 3-folds to prove the non-torsion integral Hodge conjecture for Fano 3-folds.

In Chapter 4, we consider Hodge's original conjecture for smooth hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$. We first establish that the conjecture is true for hypersurfaces of bidegree (a, b) when a and b are relatively prime. Then, using a method adapted from Kollár's original argument, we prove that very general hypersurfaces $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (da, dr), with some additional conditions on d, a, and r, fail Hodge's original conjecture.

Chapter 2

Background

2.1 Preliminary Definitions and Notation

Most of the material in this section is background material needed to state the integral Hodge conjecture. In this, as well as in later chapters, we will use two versions of the conjecture. The first, abbreviated by IHC_k, is the version of the conjecture as originally stated by Hodge in [19]. The second, abbreviated by NHC_k, asks if all integral classes of Hodge type are algebraic modulo torsion. In practice, we will work with the second version of the conjecture, since it is easier to develop techniques to prove or disprove it. In fact, proving the original Hodge conjecture in the cases that follow in later chapters will entail proving the non-torsion version, and then showing that the two versions of the conjecture are equivalent.

Notation 2.1.1. *Let X be a smooth, complex, projective variety of dimension n. Let Z_kX be the group of k-cycles on X. Let $[\cdot] : Z_kX \to H^{2n-2k}(X, \mathbb{Z})$ be the map of cycles to cycles modulo homological equivalence. Given $D \in Z_kX$, denote also by*

$[D]$ *the image of the map*

$$Z_kX \to H^{2n-2k}(X,\mathbb{Z}) \to H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}.$$

Definition 2.1.2. *Let X be a smooth, complex, connected, projective variety of dimension n. There exists a map*

$$H^{2n-2k}(X,\mathbb{Z}) \to H^{2n-2k}(X,\mathbb{C}),$$

and the map descends to an injective map

$$H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors} \hookrightarrow H^{2n-2k}(X,\mathbb{C}).$$

If the image of a class $\alpha \in H^{2n-2k}(X,\mathbb{Z})$ has pure Hodge type (a,a), where $a = n-k$, we say that α has Hodge type (a,a). Similarly, given $\beta \in H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$, if β injects into $H^{2n-2k}(X,\mathbb{C})$ to give a class of pure Hodge type (a,a), we say that β has Hodge type (a,a).

Definition 2.1.3. *A class $\alpha \in H^{2n-2k}(X,\mathbb{Z})$ is algebraic if $\alpha = [D]$ for some $D \in Z_kX$. Similarly, a class $\beta \in H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$ is algebraic if $\beta = [D]$ for some $D \in Z_kX$.*

Definition 2.1.4. *A smooth, complex, connected, projective variety X of dimension n satisfies the non-torsion version of the integral Hodge conjecture for k-cycles, or X satisfies NHC_k, if every class $\beta \in H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$ of Hodge type $(n-k,n-k)$ is algebraic.*

Definition 2.1.5. *A smooth, complex, connected, projective variety X of dimension n satisfies the integral Hodge conjecture for k-cycles, or X satisfies IHC_k, if every class $\alpha \in H^{2n-2k}(X,\mathbb{Z})$ of Hodge type $(n-k,n-k)$ is algebraic.*

Notation 2.1.6. *Given* $\alpha \in H^k(X,\mathbb{Z})$ *and* $\beta \in H^l(X,\mathbb{Z})$*, let* $\alpha \smile \beta \in H^{k+l}(X,\mathbb{Z})$ *denote the cup product. If* X *is a smooth, connected, projective variety of dimension* n*, then* $H^{2n}(X,\mathbb{Z}) \simeq \mathbb{Z}$ *and is generated by* $[x]$*, where* $x \in X$*. In this case, if* $\alpha \in H^{2n}(X,\mathbb{Z})$ *and* $\alpha = m \cdot [x]$*, we suppress the* $[x]$ *and write* $\alpha = m$*.*

Remark 2.1.7. *An irreducible variety over the complex numbers cannot be the union of countably many proper subvarieties. Note that this follows from the Baire Category Theorem (Theorem 5.6. of [28]). This fact is needed for the following definition :*

Definition 2.1.8. *We say a complex, projective variety* X *is* <u>*very general*</u> *if* X *is a member of an irreducible family of varieties which lies in the complement of a countable union of proper subvarieties of the family. When referring to very general varieties, the proper subvarieties will usually not be given explicitly.*

2.2 Kollár's Example

The first example of a complex, projective 3-fold X having a non-torsion, non-algebraic class $\alpha \in H^4(X,\mathbb{Z})$ of Hodge type $(2,2)$ was given by Janos Kollár in [4]. In this case, $X \subset \mathbb{P}^4$ is a very general hypersurface of degree d. The precise statement proved by Kollár in [4] is the following:

Proposition 2.2.1. *Suppose there exists a smooth, complex, connected, projective variety* X *of dimension 3 and a very ample divisor* L *on* X *such that :*

1. $L^3 = d$.

2. *For every curve* $C \subset X$, $k \mid [L] \smile [C]$.

Then, very general hypersurfaces $Y \subset \mathbb{P}^4$ of degree d have the property that, for all curves $C \subset Y$, $k | 6 \cdot \deg C$.

Remark 2.2.2. *Kollár's original paper gives a sketch of the proof. In [30], Soulé and Voisin present Kollár's proof with more of the details fleshed out.*

The proof entails using the very ample divisor L to embed X in a large projective space, and then projection from a linear subspace gives a (highly singular) hypersurface $Z \subset \mathbb{P}^4$ of degree L^3. If $\phi : X \to Z$ is the projection, then generically ϕ will be 1:1, 2:1 on a divisor of X, $3:1$ on a curve in X, and $4:1$ on points of X. This follows from Corollary 2 of [27]. Condition 2 carries over to the degree of curves on Z, up to possibly a factor of 2 or 3 coming from ϕ, which is why the 6 term appears. Finally, this multiplicity condition then holds for very general hypersurfaces of degree L^3.

If a hypersurface Y satisfies the conditions of Proposition 2.2.1, then Y fails IHC_1. This follows from the weak Lefschetz Theorem, and the argument can be found at the end of Chapter 5 in [24]. Note that, by the Lefschetz Hyperplane Theorem, $H^4(Y, \mathbb{Z}) \simeq \mathbb{Z}$, and thus IHC_1 and NHC_1 are equivalent in this case.

2.3 Integral Hodge Conjecture and Surjective Morphisms

Lemma 2.3.1. *Let X be a smooth, complex, connected, projective variety with $\dim X = n$. Then the following are equivalent:*

1. *X fails NHC_k.*
2. *There exists a class $L \in H^{2k}(X, \mathbb{Z})/H^{2k}(X, \mathbb{Z})_{tors}$, a class $\alpha \in H^{2n-2k}(X, \mathbb{Z})/H^{2n-2k}(X, \mathbb{Z})_{tors}$ of Hodge type $(n-k, n-k)$, and a non-negative integer $d \neq 1$*

such that :

(a) $L \smile \alpha = 1$.

(b) If $a \in H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$ *is algebraic, then* $d \mid L \smile a$.

Proof. The case (2) $\Rightarrow$ (1) is trivial, since, under the hypotheses in (2), $\alpha \in H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$ can't be an algebraic class.

Now suppose that X fails NHC_k. Then there exists a class $\alpha \in H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$ of Hodge type $(n-k, n-k)$ that is not algebraic. Without loss of generality, we may assume that α is not a non-trivial multiple of another integral class.

There are now two cases to consider. First, suppose that there exists a strictly positive integer such that $d \cdot \alpha$ is algebraic. Let d be the smallest strictly positive integer such that $d \cdot \alpha$ is algebraic. Let $A \subset H^{2n-2k}(X,\mathbb{Z})/ H^{2n-2k}(X,\mathbb{Z})_{tors}$ be the $\mathbb{Z}$-submodule of algebraic classes. Since $d \cdot \alpha$ is not the non-trivial multiple of another algebraic class, we may choose a basis of A extending from $d \cdot \alpha$ given by $\{d \cdot \alpha, a_1, \ldots, a_s\}$. Now let $F \in Hom(H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}, \mathbb{Z})$ be a homomorphism satisfying $F(\alpha) = 1$ and $F(a_i) = 0$ for all $i = 1, \ldots, s$. Then by Prop. 3.37 of [17], there exists a class $L \in H^{2k}(X,\mathbb{Z})/H^{2k}(X,\mathbb{Z})_{tors}$ such that the homomorphism F is given by $F(c) = L \smile c$. In particular, $L \smile \alpha = 1$ and $L \smile a_i = 0$ for $i = 1, \ldots, s$. Since $\{d \cdot \alpha, a_1, \ldots, a_s\}$ generates A, every algebraic class in $H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$ is an integer linear combination of the classes $\{d \cdot \alpha, a_1, \ldots, a_s\}$. Let

$$w = w_0 \cdot (d \cdot \alpha) + \sum_{i=1}^{s} w_i \cdot a_i$$

be an algebraic class with $w_i \in \mathbb{Z}$ for $i = 0, \ldots, s$. Then $L \smile w = d \cdot w_0$ is a multiple of d, and since $L \smile \alpha = 1$, both conditions of 2. are satisfied in the case that a non-zero, integer multiple of α is algebraic.

Now suppose that the only integer multiple of α that is not algebraic is $0 \cdot \alpha$. Then let $d = 0$. As in the previous case, let $\{a_1, \ldots, a_s\}$ be a basis for the $\mathbb{Z}$-submodule of algebraic classes in $H^{2n-2k}(X,\mathbb{Z})/\ H^{2n-2k}(X,\mathbb{Z})_{tors}$. Now let $F \in Hom(H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}, \mathbb{Z})$ be a homomorphism satisfying $F(\alpha) = 1$ and $F(a_i) = 0$ for all $i = 1, \ldots, s$. Then by Prop. 3.37 of [17], there exists a class $L \in H^{2k}(X,\mathbb{Z})/H^{2k}(X,\mathbb{Z})_{tors}$ such that the homomorphism F is given by $F(c) = L \smile c$. In this case, $L \smile \alpha = 1$ and, since $L \smile a_i = 0$ for all $i = 1, \ldots, s$, $L \smile w = 0$ for all algebraic classes $w \in H^{2n-2k}(X,\mathbb{Z})/\ H^{2n-2k}(X,\mathbb{Z})_{tors}$. Hence, L satisfies both conditions of 2., with d being equal to zero in this case. Thus the lemma is proved in all cases. □

Lemma 2.3.2. *Let $f : X \to Y$ be a finite, surjective map of degree r between smooth, complex, connected, projective varieties of dimension n. Suppose there exists a class $\alpha \in H^{2n-2k}(X,\mathbb{Z})/\ H^{2n-2k}(X,\mathbb{Z})_{tors}$ which is a non-algebraic class of Hodge type $(n-k, n-k)$. If a non-zero integer multiple of α is algebraic, let d be the smallest strictly positive integer such that $d \cdot \alpha$ is algebraic. If no non-zero multiple of α is algebraic, let $d = 0$. Then, if r is not a multiple of d, $f^*\alpha \in H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$ is not algebraic, and X fails NHC_k.*

Proof. Let f, r, d, and α be as above. By Lemma 2.3.1, there exists a class $L \in H^{2k}(Y,\mathbb{Z})/H^{2k}(Y,\mathbb{Z})_{tors}$ such that $L \smile \alpha = 1$ and $d \mid L \smile a$ for all algebraic classes $a \in H^{2n-2k}(Y,\mathbb{Z})/H^{2n-2k}(Y,\mathbb{Z})_{tors}$. Let $L' := f^*L$, and let $\alpha' := f^*\alpha$. Then

$L' \smile \alpha' = r$. Also, if $a \in H^{2n-2k}(X,\mathbb{Z})/H^{2n-2k}(X,\mathbb{Z})_{tors}$ is algebraic, then, by the projection formula, $L' \smile a = f_*(L' \smile a) = L \smile f_*a$ is a multiple of d. Since r is not a multiple of d, the class α' is not algebraic. The class α is of Hodge type $(n-k, n-k)$, and hence $f^*\alpha$ is also of Hodge type $(n-k, n-k)$. Therefore, X fails NHC_k. □

Example 2.3.3. *In Lemma 2.3.2, we made the assumption that the degree of f is not a multiple of d. This assumption is necessary, since, without it, it is possible to have a finite surjective map $f : X \to Y$ and a non-algebraic class $\alpha \in H^{2n-2k}(Y,\mathbb{Z})/H^{2n-2k}(Y,\mathbb{Z})_{tors}$ of Hodge type $(n-k, n-k)$ such that $f^*\alpha$ is algebraic.*

As an example, let $Y \subset \mathbb{P}^4$ be a very general hypersurface of degree d failing the integral Hodge conjecture, obtained through Kollár's construction in Section 2.2. Let $p : \tilde{Y} \to Y$ be a blowup of Y along D, the transverse intersection of two hyperplane sections, so that we obtain a Lefschetz pencil $\pi : \tilde{Y} \to \mathbb{P}^1$. Now let C be a smooth, connected curve with a surjective, degree d morphism $h : C \to \mathbb{P}^1$ that is totally ramified at d distinct points. In this case, we may take $C \subset \mathbb{P}^2$ to be the zero locus of the degree d homogeneous form

$$x_2^d - f_1(x_0, x_1) \cdot \ldots \cdot f_d(x_0, x_1),$$

where the $f_j(x_0, x_1)$ are linear forms vanishing for distinct values of $[x_0 : x_1]$. The morphism h is then the restriction to C of projection from the point $[1 : 0 : 0] \in \mathbb{P}^2$.

Let $X := C \times_{\mathbb{P}^1} \tilde{Y}$:

$$\begin{array}{ccccc} X & \xrightarrow{g} & \tilde{Y} & \xrightarrow{p} & Y \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow & & \\ C & \xrightarrow{h} & \mathbb{P}^1 & & \end{array}$$

Let $f := p \circ g : X \to Y$. *Note that* f *is a finite, surjective morphism of degree* d. *Also, if* $b_1, \ldots, b_d \in \mathbb{P}^1$ *are the branch points of the map* $h : C \to \mathbb{P}^1$, *then* f *is totally ramified at the hyperplane sections* $\tilde{H}_j := \pi^{-1}(b_j)$ *for* $j = 1, \ldots, d$. *We choose the map* h *and the points* $b_1, \ldots, b_d$ *so that each* $\tilde{H}_j$ *is smooth. Let* $H_j := p(\tilde{H}_j)$. *Note that* $\tilde{H}_j$ *is then the strict transform of* H_j, *which is smooth. Moreover,* D *is then the transverse intersection of* H_j *with another hyperplane section. Let* $R_j := (\pi \circ g)^{-1}(b_j)$, *and let* E *be the exceptional locus of* $p : \tilde{Y} \to Y$. *Finally, let* $H \subset Y$ *be a general hyperplane section.*

Since $D \subset H_j$ *is a tranverse intersection, and since each* R_j *is totally ramified over* $\tilde{H}_j$, *we have the following relations on the level of cohomology :*

$$[\tilde{H}_j] = p^*[H] - [E] \in H^2(\tilde{Y}, \mathbb{Z})/H^2(\tilde{Y}, \mathbb{Z})_{tors}$$

$$g^*[\tilde{H}_j] = d \cdot [R_j] \in H^2(X, \mathbb{Z})/H^2(X, \mathbb{Z})_{tors}$$

Since Y *was chosen to fail the integral Hodge conjecture, and since* Y *is a hypersurface in* $\mathbb{P}^4$, $H^4(Y, \mathbb{Z})$ *is generated by the non-algebraic class* $\alpha := \frac{1}{d}[H]^2$. *We claim that* $f^*\alpha$ *is an algebraic class. Indeed, note that*

$$\begin{aligned} f^*[H] \smile d \cdot [R_1] &= f^*[H] \smile g^*[\tilde{H}_1] \\ d \cdot (f^*[H] \smile [R_1]) &= f^*[H] \smile g^*(p^*[H] - [E]) \\ &= f^*[H] \smile (f^*[H] - g^*[E]) \\ &= f^*[H]^2 - f^*[H] \smile g^*[E] \end{aligned}$$

$$= f^*[H]^2 - g^*(p^*[H] \smile [E])$$
$$= f^*[H]^2 - g^*(d \cdot F)$$
$$= d \cdot f^*\alpha - d \cdot g^*F,$$

where F is the class associated to a fiber in the bundle $E \longrightarrow D$. Taking the above equality and dividing by d, we obtain $f^*\alpha = g^*F + f^*[H] \smile [R_1]$. The cup product of algebraic classes is algebraic, and thus $f^*\alpha \in H^4(X,\mathbb{Z})/H^4(X,\mathbb{Z})_{tors}$ is algebraic, even though $\alpha \in H^4(Y,\mathbb{Z})/H^4(Y,\mathbb{Z})_{tors}$ is not algebraic.

2.4 Birational Equivalence

In this section, we will let X be a smooth, complex, connected, projective variety, let $D \subset X$ be a smooth subvariety of codimension $r \geq 2$, and let $\tilde{X}$ be the blowup of X along D with associated blowdown map $f : \tilde{X} \longrightarrow X$. Let E be the exceptional locus of the blowup, let $g : E \longrightarrow D$ be the restriction of f to E, and let $j : E \hookrightarrow \tilde{X}$ and $i : D \hookrightarrow X$ be inclusion maps. Finally, let $\mu \in H^2(E,\mathbb{Z})$ be the first Chern class of the tautological line bundle associated to the projective bundle $g : E \longrightarrow D$. The blowups and maps are given in the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{j} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{i} & X \end{array}$$

The main results of this section are Corollaries 2.4.3 and 2.4.4, which state that IHC_1 and NHC_1 are birational invariants, respectively. The main tool in the proof is computing the integral cohomology of $\tilde{X}$ in terms of the integral cohomologies of

X and D. This is the content of Proposition 2.4.1. Its proof is taken from Chapter 4, Section 6 of [13], with some of the details of their proof filled in.

At the end of the section, we present a counterexample, due to [30], that neither NHC_k nor IHC_k is a birational invariant for $k \geq 2$.

Proposition 2.4.1. *Let $X, D, \tilde{X}$ be as above. Then*

$$H^i(\tilde{X}, \mathbb{Z}) \simeq f^*H^i(X, \mathbb{Z}) \oplus j_*(g^*H^{i-2}(D, \mathbb{Z}) \oplus \cdots \oplus g^*H^{i-2r+2}(D, \mathbb{Z}) \smile \mu^{r-2}).$$

Proof. Let $\alpha : X \to \mathbb{R}$ be a rug function for D (see Appendix A for the definition of rug functions and algebraic neighborhoods). Then $\alpha \circ f : \tilde{X} \to \mathbb{R}$ is a rug function for E. There exists a $\delta \in \mathbb{R}_{\geq 0}$ such that neither α nor $\alpha \circ f$ have a critical value in $[0, \delta]$. Then $U := \alpha^{-1}([0, \delta])$ and $U' := f^{-1}(U) = (\alpha \circ f)^{-1}([0, \delta])$ are algebraic neighborhoods of D and E, respectively.

Let $V := X \setminus D$ and $V' := \tilde{X} \setminus E$, and consider the Mayer-Vietoris sequences associated to $X = U \cup V$ and $\tilde{X} = U' \cup V'$ (all cohomology is in $\mathbb{Z}$ coefficients) :

$$\begin{array}{ccccccc} \to & H^i(\tilde{X}) & \to & H^i(U') \oplus H^i(V') & \xrightarrow{\phi_i'} & H^i(U' \cap V') & \to \\ & \uparrow_{f^*} & & \uparrow_{f^*} & & \uparrow_{f^*} & \\ \to & H^i(X) & \to & H^i(U) \oplus H^i(V) & \xrightarrow{\phi_i} & H^i(U \cap V) & \to \end{array} \tag{2.1}$$

By Proposition A.0.7, $D \hookrightarrow U$ and $E \hookrightarrow U'$ are homotopy equivalences, and hence $H^*(U) \simeq H^*(D)$ and $H^*(U') \simeq H^*(E)$. Note also that $f : V' \to V$ is an isomorphism, so f^* induces an isomorphism $H^*(V, \mathbb{Z}) \to H^*(V', \mathbb{Z})$. Similarly, f^* induces an isomorphism $H^*(U \cap V, \mathbb{Z}) \to H^*(U' \cap V', \mathbb{Z})$. We may then rewrite (2.1) as :

$$\begin{array}{ccccccc} \to H^i(\tilde{X}) \to & H^i(E) \oplus H^i(V) & \xrightarrow{\phi_i'} & H^i(U \cap V) & \xrightarrow{\theta_i'} & H^{i+1}(\tilde{X}) & \to \\ \uparrow_{f^*} & \uparrow_{f^*} & & \uparrow_{f^*} & & \uparrow_{f^*} & \\ \to H^i(X) \to & H^i(D) \oplus H^i(V) & \xrightarrow{\phi_i} & H^i(U \cap V) & \xrightarrow{\theta_i} & H^{i+1}(X) & \to \end{array} \tag{2.2}$$

We now want to show that there is a short exact sequence :

$$0 \to H^i(X) \xrightarrow{f^*} H^i(\tilde{X}) \xrightarrow{j^*} H^i(E)/f^*H^i(D) \to 0 \tag{2.3}$$

Note first that $f^* : H^i(X,\mathbb{Z}) \to H^i(\tilde{X},\mathbb{Z})$ is injective, since $f_* \circ f^*$ is the identity on $H^i(X,\mathbb{Z})$. In addition, the composition $j^* \circ f^*$ in (2.3) is the zero map, since we obtain j^* from the map $H^i(\tilde{X}) \to H^i(E) \oplus H^i(V)$ in the Mayer-Vietoris sequence comprising the top row of (2.2).

It remains to show that the map j^* in (2.3) is surjective. Suppose that $b = \phi_i'(a)$ for some $a \in H^i(E) \oplus H^i(V)$. Let $c \in H^i(U \cap V)$ be given by $f^*(c) = b$. Then $\theta_i'(b) = 0$ by the exactness of the top row of (2.2). The map $f^* : H^{i+1}(X) \hookrightarrow H^{i+1}(\tilde{X})$ is injective, and thus $\theta_i(c) = 0$ as well. Since the bottom row of (2.2) is exact, there exists an element $d \in H^i(D) \oplus H^i(V)$ such that $\phi_i(d) = c$. The diagram (2.2) is commutative, and hence $\phi_i'(f^*d) = b$. The element b was arbitrary, and thus the map

$$(H^i(E) \oplus H^i(V))/f^*(H^i(D) \oplus H^i(V)) \xrightarrow{\phi_i'} H^i(U \cap V)$$

is the zero map. Therefore, we have the following short exact sequence :

$$0 \to H^i(X) \xrightarrow{f^*} H^i(\tilde{X}) \xrightarrow{j^*} H^i(E)/f^*H^i(D) \to 0$$

To complete the proof, it suffices to find a left inverse to j^*. Note that every class in $H^i(E,\mathbb{Z})$ is given uniquely as a linear combination of $\{1, \mu, \ldots, \mu^{r-1}\}$ with coefficients in $f^*H^*(D)$. Classes in $H^i(E)/f^*H^i(D)$ are then given by a linear combination of $\{\mu, \ldots, \mu^{r-1}\}$. Hence there is an isomorphism

$$s : g^*H^{i-2}(D,\mathbb{Z}) \oplus \cdots \oplus g^*H^{i-2r+2}(D,\mathbb{Z}) \smile \mu^{r-2} \xrightarrow{\sim} H^i(E)/f^*H^i(D)$$

given by taking the cup product with $-\mu$. This gives the short exact sequence

$$0 \to H^i(X) \xrightarrow{f^*} H^i(\tilde{X}) \xrightarrow{s \circ j^*} g^* H^{i-2}(D,\mathbb{Z}) \oplus \cdots \oplus g^* H^{i-2r+2}(D,\mathbb{Z}) \smile \mu^{r-2} \to 0$$

Since the composition $j^* j_*$ is equivalent to taking the cup product with $-\mu$, j_* is a left inverse to $s \circ j^*$. Hence,

$$H^i(\tilde{X},\mathbb{Z}) \simeq f^* H^i(X,\mathbb{Z}) \oplus j_*(g^* H^{i-2}(D,\mathbb{Z}) \oplus \cdots \oplus g^* H^{i-2r+2}(D,\mathbb{Z}) \smile \mu^{r-2}).$$

□

The most interesting case, at least for our purposes, is when $i = 2 \cdot \dim X - 2$, in which case Proposition 2.4.1 reduces to the statement comprising Corollary 2.4.2. The birational invariance of IHC_1, proved in Corollary 2.4.3, follows from Corollary 2.4.2.

Corollary 2.4.2. *Suppose* $\dim X = n$, *and let* $\tilde{X}$ *be a blowup of* X *over a smooth center* D *of codimension* $r \geq 2$. *Then*

$$H^{2n-2}(\tilde{X},\mathbb{Z}) \simeq f^* H^{2n-2}(X,\mathbb{Z}) \oplus \mathbb{Z} \cdot j_*(g^* H^{2n-2r}(D,\mathbb{Z}) \smile \mu^{r-2}).$$

Corollary 2.4.3. *If* X *and* Y *are smooth, complex, connected, projective varieties that are birationally equivalent, then* X *satisfies* IHC_1 *iff* Y *satisfies* IHC_1.

Proof. Using the theorem on elimination of indeterminacies twice (see Question E in [18] and the proof of Lemma 1.3.1. in [1]), there exist smooth, complex, connected, projective varieties X_1 and Y_1 such that X_1 and Y_1 are obtained from X, Y via blowups with smooth centers, respectively, and such that there exists an isomorphism $X \xrightarrow{\sim} Y$. Hence, in order to show that failure IHC_1 is a birational invariant, we only need to show X fails IHC_1 iff $\tilde{X}$ fails IHC_1, where $\tilde{X}$ is a blowup of X with smooth center D of codimension $r \geq 2$.

Suppose that X fails IHC_1, and let $\alpha \in H^{2n-2}(X, \mathbb{Z})$ be a class of Hodge type $(n-1, n-1)$ that is not algebraic. Then $f_* f^* \alpha = \alpha$. Since f_* maps algebraic classes to algebraic classes, $f^*\alpha$ cannot be algebraic. Hence $\tilde{X}$ fails IHC_1.

Now suppose $\tilde{X}$ fails IHC_1. By Corollary 2.4.2, there exists a non-algebraic class $\alpha \in H^{2n-2}(\tilde{X}, \mathbb{Z})$ of the form $f^*\beta + c \cdot j_*(g^*[x] \smile \mu^{r-2})$ that has Hodge type $(n-1, n-1)$. Since $j_*(g^*[x] \smile \mu^{r-2})$ is algebraic, it has Hodge type $(n-1, n-1)$, and hence $f^*\beta$ is a non-algebraic class of Hodge type $(n-1, n-1)$. Since f^* is injective, $\beta \in H^{2n-2}(X, \mathbb{Z})$ has Hodge type $(n-1, n-1)$, and since $f^*\beta$ is not algebraic, β is not algebraic. Hence X fails IHC_1. □

Corollaries 2.4.2 and 2.4.3 carry over to the case of cohomology modulo torsion. In particular, NHC_1 is a birational invariant.

Corollary 2.4.4. *Let* $\dim X = n$*, and let* $\tilde{X}$ *be the blowup of* X *with smooth center* D *of codimension* $r \geq 2$.*. Then*

1. $H^{2n-2}(\tilde{X}, \mathbb{Z})/H^{2n-2}(\tilde{X}, \mathbb{Z})_{tors} \simeq f^*(H^{2n-2}(X, \mathbb{Z})/H^{2n-2}(X, \mathbb{Z})_{tors})$
   $\oplus\, \mathbb{Z} \cdot j_*(g^* H^{2n-2r}(D, \mathbb{Z}) \smile \mu^{r-2})$
2. *If* X *and* Y *are smooth, complex, connected, projective varieties that are birationally equivalent, then* X *satisfies* NHC_1 *iff* Y *satisfies* NHC_1.

Remark 2.4.5. *The first part of Corollary 2.4.4, stated in terms of rational cohomology, can be found in Proposition 13.1. of [24].*

Example 2.4.6. *([30], End of Section 2) Let $X \subset \mathbb{P}^4$ be a very general hypersurface failing IHC_1 obtained through Kollár's construction in Section 2.2. By choosing an embedding $\mathbb{P}^4 \hookrightarrow \mathbb{P}^5$, we also obtain an embedding $X \hookrightarrow \mathbb{P}^5$. Let $\tilde{\mathbb{P}}^5$ be the blowup of $\mathbb{P}^5$ with center X. Using the notation from Section 2.4, let $E \subset \tilde{\mathbb{P}}^5$ be the exceptional locus of $\tilde{\mathbb{P}}^5$, and let $j : E \to X$ and $g : X \hookrightarrow \mathbb{P}^5$ be the projection map and the embedding of X in $\mathbb{P}^5$, respectively.*

If $\alpha \in H^4(X, \mathbb{Z})$ is a non-algebraic class of Hodge type $(2,2)$, then consider $j_*(g^*\alpha) \in H^6(\tilde{\mathbb{P}}^5, \mathbb{Z})$. This class has Hodge type $(3,3)$, since an integral multiple of $j_*(g^*\alpha)$ is the class associated to a 2-cycle. Now apply g_*j^* to $j_*(g^*\alpha)$ to obtain :

$$g_*j^*(j_*(g^*\alpha)) = g_*(-g^*\alpha \smile \mu) = -\alpha.$$

The class $-\alpha$ is not algebraic, and g_* and j^* map algebraic classes to algebraic classes. Therefore, the class $j_*(g^*\alpha)$ cannot be algebraic.

Remark 2.4.7. *This example works for dimensions beyond 5; in fact, the original example by Soulé and Voisin does not restrict the dimension. Note that this argument does not apply to varieties of dimension 4, since a blowup of such a variety will have a center of dimension less than or equal to 2, and the integral Hodge conjecture is true for such a center by the Lefschetz-$(1,1)$ Theorem.*

2.5 Specialization

In this section, we consider families of smooth varieties and how the integral Hodge conjecture behaves in these families. The main result of this section is Corollary 2.5.7, which states that, assuming some additional conditions on the degrees, very general hypersurfaces in smooth, complex, connected, projective varieties will fail NHC_1. This follows from Corollary 2.5.6, which gives conditions under which failure of IHC_k for a member of a family of smooth, complex, connected, projective varieties of dimension n implies failure for very general members of the family, namely that the non-algebraic, integral class α of Hodge type $(n-k, n-k)$ specializes to integral classes of Hodge type $(n-k, n-k)$ in other varieties in the family.

In this section, let $\mathscr{X}$ and B be smooth, complex, connected, varieties, and let $\pi : \mathscr{X} \to B$ be a surjective, projective morphism, with the fibers of π being smooth varieties of dimension n. We let $X_t := \pi^{-1}(t)$.

Definition 2.5.1. *Let $s, t \in B$, and let $\gamma \subset B$ be a path from s to t. Then there exists a well-defined homeomorphism $\sigma_{s,t,\gamma}$, called the* <u>*specialization map along γ*</u>*, which is defined as follows : By Lemma A.0.11, there exists a local specialization map $\sigma_{s,t}$ for all points s, t lying in a sufficiently small neighborhood of a point $y \in B$. Given $y \in \gamma(I)$, let U_y be such a trivializing neighborhood of y. The set $\{U_y\}_{y \in \gamma(I)}$ is an open cover of $\gamma(I)$, and since $\gamma(I)$ is compact, there exists a finite subcover $\{U_{y_1}, \ldots, U_{y_m}\}$. Without loss of generality, assume that the y_i are in ascending order, in the sense that if $\gamma(p) = y_i$ and $\gamma(q) = y_{i+1}$, then $p < q$. Now choose a collection of points $j_0 = s, j_{2m} = t, j_{2k+1} = y_k$ for $1 \leq k \leq m-1$, $j_{2k} \in U_{y_k} \cap U_{y_{k+1}}$ for $1 \leq k \leq m-1$. Again, by Lemma A.0.11, the specialization map $\sigma_{j_i, j_{i+1}} : X_{j_i} \xrightarrow{\sim} X_{j_{i+1}}$ is defined. Define $\sigma_{s,t,\gamma} : X_s \xrightarrow{\sim} X_t$ to be the following composition :*

$$\sigma_{j_{2m-1}, j_{2m}} \circ \ldots \circ \sigma_{j_1, j_2} \circ \sigma_{j_0, j_1}.$$

Remark 2.5.2. *The above map $\sigma_{s,t,\gamma}$ does not depend on the choice of the U_y comprising the subcover of $\gamma(I)$, but it does depend on the choice of path.*

Proposition 2.5.3. *There exists a countable union of proper subvarieties $W \subset B$ so that, if $s \in B \setminus W$ and $t \in B$, then, for all $k = 0, \ldots, d$, $\sigma^*_{t,s,\gamma^{-1}} : H^{2k}(X_s, \mathbb{Z}) \to H^{2k}(X_t, \mathbb{Z})$ maps classes associated to effective algebraic cycles to classes associated to effective algebraic cycles.*

Proof. By Appendix B, there exists $\mathscr{H}_{\mathscr{X}/B}$, the relative Hilbert scheme associated to $\pi : \mathscr{X} \to B$. Given an irreducible component $\mathscr{C} \subset \mathscr{H}_{\mathscr{X}/B}$, let $\rho_{\mathscr{C}} : \mathscr{C} \to B$ be as in Appendix B. Set

$$W := \left(\bigcup_{\mathscr{C} \in \mathscr{I}} \rho_{\mathscr{C}}(\mathscr{C}) \right),$$

where $\mathscr{I}$ is the set of all irreducible components $\mathscr{C}$ of $\mathscr{H}_{\mathscr{X}/B}$ such that $\rho_{\mathscr{C}}(\mathscr{C}) \neq B$. Note that, since $\mathscr{H}_{\mathscr{X}/B}$ contains countably many irreducible components, $\mathscr{I}$ is a countable set.

Suppose $s \in B \setminus W$. If $A \subset X_s$ is a subvariety of dimension k, then there exists an irreducible component $\mathscr{A} \subset \mathscr{H}_{\mathscr{X}/Y}$ such that A is a fiber of $\pi_{\mathscr{A}}$ and $\rho_{\mathscr{A}}(\mathscr{A}) = B$, where $\pi_{\mathscr{A}}$ is defined in Appendix B. Let $\phi : \mathscr{A}' \to \mathscr{A}$ be a desingularization of $\mathscr{A}$. Define $\chi'_{\mathscr{A}} := \mathscr{A}' \times_{\mathscr{A}} \chi_{\mathscr{A}}$, where $\chi_{\mathscr{A}}$ is defined in Appendix B.

$$\begin{array}{ccc} \chi'_{\mathscr{A}} & \longrightarrow & \chi_{\mathscr{A}} \\ \big\downarrow & & \big\downarrow {\scriptstyle \chi_{\mathscr{A}}} \\ \mathscr{A}' & \xrightarrow{\phi} & \mathscr{A} \end{array}$$

Define $\mathscr{X}' := \mathscr{A}' \times_B \mathscr{X}$, and let π' be the induced map $\mathscr{X}' \to \mathscr{A}'$.

$$\begin{array}{ccc} \mathscr{X}' & \longrightarrow & \mathscr{X} \\ {\scriptstyle \pi'} \big\downarrow & & \big\downarrow {\scriptstyle \pi} \\ \mathscr{A}' & \xrightarrow{\rho_{\mathscr{A}} \circ \phi} & B \end{array}$$

Then $\chi_{\mathscr{A}'} \hookrightarrow \mathscr{X}'$. Furthermore, associated to $\chi_{\mathscr{A}'}$ is an element

$$[\chi_{\mathscr{A}'}] \in H^0(\mathscr{A}', R^{2n-2k}\pi'_*\mathbb{Z}).$$

To finish the proof, it remains to relate specialization over π with specialization over π'. Let $t \in B$, and let $\gamma \subset B$ be a path from s to t.

Lemma 2.5.4. *There exists a path $\gamma' \subset \mathscr{A}'$ so that $\sigma_{s,t,\gamma} = \sigma_{s',t',\gamma'}$ for some $s', t' \in \mathscr{A}'$ such that $\rho_{\mathscr{A}} \circ \phi(s') = s$ and $\rho_{\mathscr{A}} \circ \phi(t') = t$.*

Proof. As in Definition 2.5.1, let $j_0, \ldots, j_{2m} \in \gamma(I)$ be points determining subpaths of γ such that each subpath lies in an open set in B given by Lemma A.0.11. Let $\gamma_1, \ldots, \gamma_{2m}$ be the subpaths determined by $j_0, \ldots, j_{2m}$. For $i = 1, \ldots, 2m$, we will construct γ' by joining subpaths γ'_i that satisfy $\sigma_{s_i,t_i,\gamma_i} = \sigma_{s'_i,t'_i,\gamma'_i}$ when $\rho_{\mathscr{A}} \circ \phi(s'_i) = s_i$ and $\rho_{\mathscr{A}} \circ \phi(t'_i) = t_i$.

Each γ_i lies in an open neighborhood $U_i \subset B$ so that $\pi^{-1}(U_i)$ is diffeomorphic to the product of U_i and a fiber. Set $V_i := (\rho_{\mathscr{A}} \circ \phi)^{-1}(U_i)$. Since π' is obtained through a fiber product, π' has a local product structure over V_i. Now choose a point $s_i \in (\rho_{\mathscr{A}} \circ \phi)^{-1}(j_{i-1})$. Since $(\rho_{\mathscr{A}} \circ \phi)^{-1}(j_{i-1}) \hookrightarrow (\rho_{\mathscr{A}} \circ \phi)^{-1}(U_i)$ is a homotopy equivalence, there exists a path γ'_i from s'_i to a point $s'_{i+1} \in (\rho_{\mathscr{A}} \circ \phi)^{-1}(j_i)$. The image $(\rho_{\mathscr{A}} \circ \phi)(\gamma'_i)$ lies in U_i and has the same endpoints as γ_i. Since π' is locally a product, this implies the specialization map $\sigma_{s'_i,s'_{i+1},\gamma'_i}$ is identical to $\sigma_{j_{i-1},j'_i,\gamma_i}$.

By the above, define subpaths $\gamma'_1, \ldots, \gamma'_{2m}$ that patch together to form a path γ'. Since specialization agrees on subpaths, the lemma follows. □

The class $[A] \in H^{2n-2k}(X_{s'}, \mathbb{Z})$ is also the image of $[\chi_{\mathscr{A}'}]$ by the restriction map to the stalk at s'. Hence, there is no monodromy, and $\sigma^*_{t',s',\gamma'^{-1}}$ maps $[A]$ to the restriction of $[\chi_{\mathscr{A}'}]$ to t'. The restriction $\chi_{\mathscr{A}'} \cap \pi'^{-1}(t')$ is effective, and thus Lemma 2.5.4 implies that $\sigma^*_{t,s,\gamma^{-1}}$ maps $[A]$ to an effective class. The subvariety A was arbitrary, and so the proposition follows. $\square$

Remark 2.5.5. *Note that the set difference $B \setminus W$ is not empty by Remark 2.1.7.*

Corollary 2.5.6. *Let $\pi : X \to B$ be as in Proposition 2.5.3. Suppose in addition that there exists a smooth fiber X_t with a class $\alpha \in H^{2n-2k}(X_t, \mathbb{Z})$ of Hodge type $(n-k, n-k)$ such that:*

1. *α is not algebraic, and*
2. *for every $t \in B$ and all paths $\gamma \subset B$ from t to s, $\sigma^*_{s,t,\gamma}(\alpha) \in H^{2n-2k}(X_s, \mathbb{Z})$ has Hodge type $(n-k, n-k)$.*

Then very general fibers of π fail IHC_k.

Proof. If $[\alpha_t] \in H^{2n-2k}(X_t, \mathbb{Z})$ is a class of Hodge type $(n-k, n-k)$ that is not algebraic, and if $\sigma^*_{t,s,\gamma}([\alpha_t])$ has Hodge type $(n-k, n-k)$, then

$$\sigma^*_{s,t,\gamma^{-1}}(\sigma^*_{t,s,\gamma}([\alpha_t])) = [\alpha_t].$$

If $s \in B \setminus W$, then, by Proposition 2.5.3, $\sigma^*_{t,s,\gamma}$ maps an algebraic class to an algebraic class, and thus $\sigma^*_{t,s,\gamma}([\alpha_t])$ can't be algebraic. Since $\sigma^*_{t,s,\gamma}([\alpha_t])$ has Hodge type $(n-k, n-k)$, X_s fails IHC_k. $\square$

Corollary 2.5.7. *Let (X, L) be a smooth, complex, connected, projective variety X of dimension 4 with a very ample divisor L. If very general hypersurfaces $Y \subset \mathbb{P}^4$ of degree r fail NHC_1 and if L^4 is not a multiple of r, then very general hypersurfaces $Y' \subset X$ in the family $\mathbb{P}H^0(X, \mathscr{O}_X(r \cdot L))$ fail NHC_1.*

Proof. Let $i : X \hookrightarrow \mathbb{P}^N$ be an embedding induced by global sections of L. Now choose a linear subspace $Q \subset \mathbb{P}^N$ of codimension 5 that is disjoint from $i(X)$. Define $j : X \to \mathbb{P}^4$ to be the map given by projection from Q. Now consider the linear system $\mathfrak{d} \subset H^0(X, \mathscr{O}_X(r \cdot L))$ of hypersurfaces in X obtained as inverse images of hypersurfaces in $\mathbb{P}^4$ under j. The system $\mathfrak{d}$ is comprised of hyperplane sections obtained via hyperplanes containing Q. Hence, the base points of $\mathfrak{d}$ lie in Q, but since $i(X) \cap Q = \emptyset$, $\mathfrak{d}$ has no base points. By the generalization of Bertini's Theorem given in Chapter 3, Corollary 10.9. of [16], the general member of $\mathfrak{d}$ is non-singular. As a consequence, there exists a hypersurface $Y \subset \mathbb{P}^4$ of degree r failing NHC_1 and such that $Y' := j^{-1}(Y)$ is smooth. The restriction of j gives a finite, surjective map between smooth 3-folds $Y' \to Y$, and the degree of this restriction is equal to L^4. By Lemma 2.3.2, Y' fails NHC_1. In particular, note that $j^*(\frac{1}{r}H^2)$ is not algebraic, where $H \in H^2(Y, \mathbb{Z})$ is the class associated to a hyperplane section. Note also that Y' is hypersurface in X corresponding to the divisor $r \cdot L$, and that the class $j^*(\frac{1}{r}H^2)$ remains of Hodge type $(2,2)$ under specialization in the family $\mathbb{P}H^0(X, \mathscr{O}_X(r \cdot L))$. Hence, Corollary 2.5.6 implies that very general members of $\mathbb{P}H^0(X, \mathscr{O}_X(r \cdot L))$ fail NHC_1. □

Chapter 3

Some 3-Folds Not Of General Type

In this chapter, we prove the integral Hodge conjecture for various types of 3-folds not of general type. We prove IHC_1 and NHC_1 for abelian 3-folds and for Fano 3-folds. In addition, we prove NHC_1 for conic bundles satisfying certain additional conditions.

3.1 Abelian 3-Folds

The goal of this section is to prove that all abelian 3-folds satisfy the integral Hodge conjecture, and this is proved in Corollary 3.1.9. We actually prove a more general result in Proposition 3.1.8, namely that we know IHC_1 is true for abelian varieties of dimension g if we know IHC_1 is true for abelian varieties of dimension g that are principally polarized. Principally polarized abelian varieties are defined in Definition 3.1.1. Note that the cohomology of abelian varieties is torsion-free, and hence NHC_1 and IHC_1 are equivalent.

Background material pertaining to abelian varieties can be found in Appendix C.

Definition 3.1.1. *Let A be an abelian variety equipped with an ample line bundle L. We say that L is a* <u>polarization</u> *of A. If, in addition, $\dim A = g$ and $L^g = g!$, we say that L is a* <u>principal polarization</u>. *Equivalently, a principal polarization is a polarization with type $(1, 1, \ldots, 1)$ (see Appendix C for the definition of the type of a polarization). If A has a principal polarization θ, we say that (A, θ) is a* <u>principally polarized abelian variety,</u> *and we will use the abbreviation* <u>ppav</u>.

Notation 3.1.2. *Denote by $CH(A)$ the Chow ring of an abelian variety A, the ring of cycles on A modulo rational equivalence. For a definition, see Section 8.3. of [11]. Let $CH_{\mathbb{Q}}(A) := CH(A) \otimes \mathbb{Q}$.*

Definition 3.1.3. 1. *Let A be an abelian variety of dimension g, and let $p : A \times \hat{A} \to A$ and $q : A \times \hat{A} \to \hat{A}$ be projection maps . Then the Fourier transformation $\mathscr{F}_A : CH_{\mathbb{Q}}(A) \to CH_{\mathbb{Q}}(\hat{A})$ is the map*

$$\mathscr{F}_A(y) = q_*(p^*y.e^{[\mathscr{L}]}),$$

where $\mathscr{L}$ is the Poincare bundle on $A \times \hat{A}$ and where $e^{[\mathscr{L}]} = \sum_{k=0}^{g} \frac{[\mathscr{L}]^k}{k!}$.

2. *There is also a Fourier transformation $\mathscr{F}_{h,A} : H^*(A, \mathbb{Q}) \to H^*(\hat{A}, \mathbb{Q})$ defined as the map $\mathscr{F}_{h,A}(z) = q_*(p^*z \smile e^{[\mathscr{L}]})$.*

Remark 3.1.4. *Given an isogeny $f : A \to B$, there is a dual isogeny $\hat{f} : \hat{B} \to \hat{A}$. Since $\hat{A} \simeq Pic^0(A)$, the map $f^* : Pic^0(B) \to Pic^0(A)$ induces an isogeny $\hat{f} : \hat{B} \to \hat{A}$ by II.4.3 of [23].*

The main tool used in this section is the next statement, Proposition 3.1.5, which describes the behavior of the Fourier transformation on integral cohomology and on Hodge classes.

Proposition 3.1.5. *([5], Section 1, Prop. 1) Let A be an abelian variety of dimension g. If $[c] \in H^p(A, \mathbb{Z})$, then $\mathscr{F}_{h,A}([c]) \in H^{2g-p}(\hat{A}, \mathbb{Z})$, and the map $\mathscr{F}_{h,A} : H^p(A, \mathbb{Z}) \to H^{2g-p}(\hat{A}, \mathbb{Z})$ is surjective. In addition, the Fourier transformation extends to complex cohomology classes, so that $\mathscr{F}_{h,A}(H^{r,s}(A, \mathbb{C})) = H^{g-s,g-r}(\hat{A}, \mathbb{C})$.*

Remark 3.1.6. *If $cl : CH_{\mathbb{Q}}(A) \to H^*(A, \mathbb{Q})$ is the map taking cycles modulo rational equivalence to cycles modulo homological equivalence, then the following diagram is commutative:*

$$\begin{array}{ccc} CH_{\mathbb{Q}}(A) & \xrightarrow{cl} & H^*(A, \mathbb{Q}) \\ {\scriptstyle \mathscr{F}_A}\downarrow & & \downarrow{\scriptstyle \mathscr{F}_{h,A}} \\ CH_{\mathbb{Q}}(\hat{A}) & \xrightarrow{cl} & H^*(\hat{A}, \mathbb{Q}) \end{array}$$

Consequently, results for the Fourier transformation on Chow groups carry over to results for the Fourier transformation on algebraic cohomology classes. Specifically, we have the following :

Lemma 3.1.7. 1. *If $[\theta] \in H^2(A, \mathbb{Z})$ is the first Chern class of a principal polarization, then*

$$\mathscr{F}_{h,A}\left(\left[\frac{\theta^d}{d!}\right]\right) = (-1)^{g-d} \cdot \left[\frac{\theta^{g-d}}{(g-d)!}\right].$$

2. *Let $f : A \to B$ be an isogeny of abelian varieties. If $[c]$ and $\mathscr{F}_{h,B}([c])$ are algebraic classes, then*

$$\mathscr{F}_{h,A}(f^*([c])) = \hat{f}_*(\mathscr{F}_{h,B}([c])).$$

Proof. Both statements follow from similar results in [5] applied to the Fourier transformation on Chow groups. The first statement follows from Lemma 1 of Section 3 in [5], and the second follows from Proposition 3iii. in Section 2 of [5]. □

Proposition 3.1.8. *Let (A, L) be a polarized abelian variety of dimension g. Suppose that, for all ppav (S, θ) of dimension g, the minimal class*

$$\left[\frac{\theta^{g-1}}{(g-1)!}\right] \in H^{2g-2}(S, \mathbb{Z}) \tag{3.1}$$

is algebraic. Then A satisfies IHC_1.

Proof. Let (A, L) be a polarized abelian variety of dimension g, and let $[\alpha] \in H^2(A, \mathbb{Z})$ be a class of Hodge type $(1, 1)$. We claim that $[\alpha]$ is equal to a sum

$$[\alpha] = \sum_{i=1}^{M} c_i[L_i],$$

where each $[L_i]$ is the first Chern class associated to a polarization L_i on A and where $c_1, \ldots, c_M \in \mathbb{Z}$. Indeed, note that $m[L] + [\alpha]$ has Hodge type $(1, 1)$ for all $m \in \mathbb{Z}$, and hence is the Chern class of a line bundle. By Chap. 4, Cor. 3.3 in [23], a line bundle L_0 is a polarization iff $L^\nu.L_0^{g-\nu} > 0$ for all $\nu = 0, \ldots, g$. If L_0 is a line bundle with first Chern class given by $m[L] + [\alpha]$, then for each ν the intersection $L^\nu.L_0^{g-\nu}$ is a polynomial in $\mathbb{Z}[m]$ with leading coefficient $L^g > 0$. Hence, we can choose m sufficiently large so that the intersection $L^\nu.L_0^{g-\nu}$ is positive for all ν, so that $m[L] + [\alpha]$ is the first Chern class of a polarization. Since $[\alpha] = (m[L] + [\alpha]) - m[L]$, the claim follows.

According to IV.1.2. of [23], for every polarized abelian variety (A, L), there exists a ppav (S, θ) and an isogeny $f : A \to S$ with $L = f^*\theta$. For each polarization L_i in (3.1), let f_i: $(A, L_i) \to (S_i, \theta_i)$ be an isogeny to a ppav S_i. Then we have :

$$\mathscr{F}_{h,A}([\alpha]) = \mathscr{F}_{h,A}\left(\sum_{i=1}^{M} c_i[L_i]\right)$$

$$= \sum_{i=1}^{M} c_i \cdot \mathscr{F}_{h,A}([L_i])$$

$$= \sum_{i=1}^{M} c_i \cdot \mathscr{F}_{h,A}([f_i^* \theta_i])$$

$$= \sum_{i=1}^{M} c_i \cdot \hat{f}_{i*}(\mathscr{F}_{h,S_i}(\theta_i)), \text{ by Lemma 3.1.7.1.}$$

$$= \sum_{i=1}^{M} c_i \cdot \hat{f}_{i*}\left(\left[\frac{\theta_i^{g-1}}{(g-1)!}\right]\right), \text{ by Lemma 3.1.7.2.}$$

Since $\hat{f}_i$ maps algebraic classes to algebraic classes, the above sum is an integral linear combination of algebraic classes, and hence it is algebraic. By Proposition 3.1.5, $\mathscr{F}_{h,A}$ maps classes in $H^2(A, \mathbb{Z})$ of Hodge type $(1,1)$ surjectively onto classes in $H^{2g-2}(\hat{A}, \mathbb{Z})$ of Hodge type $(g-1, g-1)$, and since $[\alpha]$ was arbitrary, it follows that $\hat{A}$ satisfies IHC_1. Every abelian variety A is the dual to its dual $\hat{A}$, and hence the proposition follows. □

Corollary 3.1.9. *Abelian 3-folds satisfy IHC_1.*

Proof. (See also proof of XI.8.2a. in [23].) The corollary follows from the fact that the minimal class associated to a ppav of dimension 3 is algebraic. Indeed, the moduli space of genus three curves has dimension $3 \cdot 3 - 3 = 6$, and the moduli space of ppav of dimension 3 has dimension 6. There exists a well-defined map from the moduli space of curves to the moduli space of ppav given by mapping a curve C to its Jacobian $(J(C), \theta)$. By Torelli's Theorem, if $(J(C), \theta) \simeq (J(C'), \theta')$, then $C \simeq C'$. Consequently, the moduli space of genus three curves maps injectively into the moduli space of ppav of dimension 3. Both moduli spaces are irreducible, and hence the general ppav of dimension 3 is a Jacobian. As a corollary to the Criterion of Matsusaka-Ran ([23], XI.8.3.), it follows that every ppav of dimension 3 is isomorphic to a product of Jacobian varieties :

$$(A, L) \simeq (J(C_1), \theta_1) \times \dots (J(C_N), \theta_N),$$

where the product on the right has principal polarization $L \simeq p_1^*\theta_1 \otimes \cdots \otimes p_N^*\theta_N$, with the $p_j : J(C_1) \times \cdots \times J(C_N) \to J(C_j)$ being projection maps.

If $N = 1$, $(A, L) \simeq (J, \theta)$ is a principally polarized Jacobian, and the minimal class $\frac{1}{2}\theta^2$ is algebraic by Poincaré's Formula (XI.2.1. in [23]). If $N = 2$, then $(A, L) \simeq (J(C_1), \theta_1) \times (J(C_2), \theta_2)$, and without loss of generality assume that $g(C_1) = 1$ and $g(C_2) = 2$. We then obtain an equality of cohomology classes :

$$\frac{[L]^2}{2} = \frac{(p_1^*[\theta_1] + p_2^*[\theta_2])^2}{2} = p_1^*[\theta_1] \smile p_2^*[\theta_2] + p_2^*\left(\left[\frac{\theta_2^2}{2}\right]\right).$$

Since $(J(C_2), \theta_2)$ is a Jacobian of dimension 2, the class $\frac{1}{2}[\theta_2^2]$ is Poincaré dual to a point, and hence its pullback via p_2^* is algebraic. The cup product $p_1^*[\theta_1] \smile p_2^*[\theta_2]$ of algebraic classes is algebraic, and thus the class $\frac{1}{2}[L]^2$ is algebraic.

If $N = 3$, then $(A, L) \simeq (J(C_1), \theta_1) \times (J(C_2), \theta_2) \times (J(C_3), \theta_3)$, in which case the minimal class associated to the principal polarization is given by

$$p_1^*[\theta_1] \smile p_2^*[\theta_2] + p_1^*[\theta_1] \smile p_3^*[\theta_3] + p_2^*[\theta_2] \smile p_3^*[\theta_3],$$

and this class is algebraic. Since $1 \leq N \leq 3$, it follows that, for every ppav (A, L) of dimension 3, the class $\frac{1}{2}[L]^2$ is algebraic. □

3.2 Conic Bundles

Definition 3.2.1. *([26], Definition 6.1.) A morphism $f : X \to S$ from a smooth variety X onto a smooth surface S is a conic bundle if every fiber is isomorphic to a conic, i.e., a scheme of zeros of a nonzero homogeneous form of degree 2 on $\mathbb{P}^2$. The set $\{s \in S \mid f^{-1}(s) \text{ is not smooth}\}$ is called the discriminant locus of f and denoted by Δ_f.*

Proposition 3.2.2. *([26], Proposition 6.2.) Let $f : X \to S$ be a conic bundle. Then*

1. *f is flat, $f_*\omega_X^{-1}$ is a vector bundle of rank 3, and the natural map $X \to \mathbb{P}(f_*\omega_X^{-1})$ is an embedding. In particular, X is projective if S is projective.*
2. *If Δ_f is non-empty, then it is a curve with only ordinary double points, and $\operatorname{Sing} \Delta_f = \{s \in S \mid f^{-1}(s) \text{ is non-reduced}\}$.*

Notation 3.2.3. *Given a smooth, complex, connected, projective variety X, let $\rho(X)$ denote the Picard number of X, that is the rank of the Picard group Pic X.*

Proposition 3.2.4. *([26], Proposition 6.3.) Let $f : X \to S$ be a conic bundle over a projective surface S. Then*

1. *$\rho(X) - \rho(S) = 1$ iff $f^{-1}(C)$ is irreducible for every irreducible curve C on S.*
2. *Assume that $f^{-1}(C)$ is reducible for an irreducible curve C on S. Then*
   (a) *C is smooth.*
   (b) *$f^{-1}(C)$ is a union of E_1 and E_2 such that $f|_{E_i} : E_i \to C$ is a $\mathbb{P}^1$-bundle for $i = 1, 2$.*

(c) *There are conic bundles* $g_i : Y_i \longrightarrow S$ *and morphisms* $\alpha_i : X \longrightarrow Y_i$ *so that* α_i *is a contraction of all fibers of* $f|_{E_i}$ *and* $g_i \circ \alpha_i = f$ *for both* $i = 1, 2$. *In addition,* $\Delta_{g_1} = \Delta_{g_2}$, $\rho(Y_1) = \rho(Y_2)$, $\Delta_f = \Delta_{g_i} \cup C$, *and* $\rho(X) = \rho(Y_i) + 1$ *for* $i = 1, 2$.

Proposition 3.2.5. *Let* $f : X \longrightarrow S$ *be a conic bundle with* S *and* X *projective, and suppose* $\Delta_f \neq \emptyset$. *Then* X *satisfies* NHC_1.

Proof. First, we may assume that $\rho(X) - \rho(S) = 1$. Indeed, if this is not the case, then 1. of Prop. 3.2.4 implies that there exists a curve $C \subset S$ so that $f^{-1}(C)$ is reducible. By 2c. of Proposition 3.2.4, X is birationally equivalent to a variety Y with a conic bundle $g : Y \longrightarrow S$ such that $\rho(X) - \rho(Y) = 1$. By 1. of Proposition 3.2.2, Y is a smooth, complex projective variety. Since NHC_1 for X is equivalent to NHC_1 for Y by Corollary 2.4.4, we may replace X with Y, and by induction on $\rho(X) - \rho(S)$, we may assume $\rho(X) - \rho(S) = 1$ without loss of generality.

Suppose that $\Delta_f \neq \emptyset$. By 2. of Proposition 3.2.2, Δ_f is a curve. Let $C \subset \Delta_f$ be an irreducible curve. For a general point $s_0 \in C$, $f^{-1}(s_0)$ is the union of two distinct rational curves C_1 and C_2 such that each C_i corresponds to an irreducible component $\mathscr{C}_i$ of the relative Hilbert scheme $Hilb_{f^{-1}(C)/C}$. If $\mathscr{C}_1 \neq \mathscr{C}_2$, then this would imply $f^{-1}(C)$ is reducible. However, since $\rho(X) - \rho(S) = 1$, 1. of Proposition 3.2.4 implies that $f^{-1}(C)$ is irreducible. Hence, C_1 and C_2 are parametrized by the same component of the relative Hilbert scheme, and thus C_1 and C_2 are algebraically equivalent. It follows that C_1 and C_2 are homologically equivalent, and since $[C_1] + [C_2]$ is algebraically equivalent to $f^*[x]$ for a point $x \in S$, $[C_1], [C_2]$, and $\frac{1}{2}f^*[x]$ are all homologically equivalent.

By the proof of Proposition 6.2. of [26], *Pic* S is generated by smooth curves B such that $f^{-1}(B)$ is also smooth and such that B intersects Δ_f transversally. Let

$$\{B_1, \ldots, B_{\rho(S)}\}$$

be a set of such curves generating *Pic* S. Since $f|_{f^{-1}(B_i)} : f^{-1}(B_i) \to B_i$ is a map between smooth varieties with the general fiber a smooth rational curve, there exists a section S_i by Tsen's Theorem (see also Theorem 1.1. of [12]).

Since $\rho(X) = \rho(S) + 1$, $\{\frac{1}{2}f^*[x], S_1, \ldots, S_{\rho(S)}\}$ generate those classes in $H^4(X, \mathbb{Q})$ of Hodge type $(2,2)$. Indeed, if a linear combination of $\{\frac{1}{2}f^*[x], S_1, \ldots, S_{\rho(S)}\}$ is equal to zero, then apply f_*:

$$\begin{aligned} a_0 \cdot \frac{1}{2}f^*[x] + \sum_{\nu=1}^{\rho(S)} a_\nu \cdot S_\nu &= 0 \\ f_*\left(a_0 \cdot \frac{1}{2}f^*[x] + \sum_{\nu=1}^{\rho(S)} a_\nu \cdot S_\nu\right) &= 0 \\ \sum_{\nu=1}^{\rho(S)} a_\nu \cdot B_\nu &= 0 \end{aligned}$$

Since $\{B_1, \ldots, B_{\rho(S)}\}$ are linearly independent, it follows that $a_\nu = 0$ for $\nu = 1, \ldots, \rho(S)$. In addition, a_0 must also equal zero.

Suppose $W := c_0 \cdot \frac{1}{2}f^*[x] + c_1 \cdot S_1 + \cdots + c_{\rho(S)} \cdot S_{\rho(S)} \in H^4(X, \mathbb{Z})/H^4(X, \mathbb{Z})_{tors}$. We claim that, for $i = 0, \ldots, \rho(S)$, $c_i \in \mathbb{Z}$. Note that, by Proposition 3.27. of [17], for $j = 1, \ldots, \rho(S)$, there exists $D_j \in H^2(S, \mathbb{Z})/H^2(S, \mathbb{Z})_{tors}$ so that $D_j \smile B_i = \delta_{ij}$. Then

$$\left(c_0 \cdot \frac{1}{2}f^*[x] + c_1 \cdot S_1 + \cdots + c_{\rho(S)} \cdot S_{\rho(S)}\right) \smile f^*D_j = c_j \in \mathbb{Z}$$

by the projection formula. Also, by the projection formula,

$$\begin{aligned} W \smile -K_X = & \quad c_0 \cdot \frac{1}{2} f^*[x] \smile -K_X + (c_1 \cdot S_1 + \cdots + c_{\rho(S)} \cdot S_{\rho(S)}) \smile -K_X \\ = & \quad c_0 \cdot \frac{1}{2}[x] \smile 2[S] + (c_1 \cdot S_1 + \cdots + c_{\rho(S)} \cdot S_{\rho(S)}) \smile -K_X \\ = & \quad c_0 + (c_1 \cdot S_1 + \cdots + c_{\rho(S)} \cdot S_{\rho(S)}) \smile -K_X \end{aligned}$$

The above quantity is an integer, since it is obtained via intersection of integral classes. Similarly, $(c_1 \cdot S_1 + \cdots + c_{\rho(S)} \cdot S_{\rho(S)}) \smile -K_X \in \mathbb{Z}$. Hence, $c_0 \in \mathbb{Z}$ as well.

It follows that every class in $H^4(X, \mathbb{Z})/H^4(X, \mathbb{Z})_{tors}$ is an integral linear combination of algebraic classes, and hence X satisfies NHC_1. □

Proposition 3.2.6. *Let $f : X \to S$ be a conic bundle with X and S being projective. Suppose that S is simply connected. Then X satisfies NHC_1.*

Proof. Let $f : X \to S$ be as above. By Proposition 3.2.5, we may assume that $\Delta_f = \emptyset$. Let $E_2^{p,q} := H^p(S, R^q f_* \mathbb{Z})$ be the E_2 terms of the Leray spectral sequence associated to $f : X \to S$. Since every fiber is isomorphic to $\mathbb{P}^1$, there exists a tubular neighborhood about each fiber of S. If $U \subset S$ is a sufficiently small neighborhood of a point in S, $f^{-1}(U)$ is homeomorphic to $U \times f^{-1}(s)$, and thus the stalks of $R^k f_* \mathbb{Z}$ are isomorphic to $H^k(f^{-1}(s), \mathbb{Z})$. Since $H^1(\mathbb{P}^1, \mathbb{Z})$ and $H^k(\mathbb{P}^1, \mathbb{Z})$ are trivial for $k \geq 3$, $R^k f_* \mathbb{Z}$ is trivial for $k = 1$ and for $k \geq 3$. Hence, $E_2^{p,q}$ is trivial for $q \geq 3$ and for $q = 1$. In particular, $E_\infty^{1,3}$, $E_\infty^{3,1}$, and $E_\infty^{0,4}$ are trivial, and thus we obtain the following short exact sequence :

$$0 \to E_\infty^{4,0} \to H^4(X, \mathbb{Z}) \to E_\infty^{2,2} \to 0 \qquad (3.2)$$

Since $E_2^{p,q}$ is trivial for $q \geq 3$, no element of $E_2^{2,2}$ will be the image of a differential. Consequently, the elements $E_\infty^{2,2}$ will be the subgroup of $E_2^{2,2}$ for which $d_k = 0$ for

$k \geq 2$. Hence, there exists an inclusion $E_\infty^{2,2} \hookrightarrow E_2^{2,2}$, and since the differentials d_k applied to $E_k^{2,2}$ give the zero map for $k \geq 2$, $E_\infty^{2,2} \simeq E_2^{2,2}$. By assumption, S is simply connected, and thus there is no monodromy. Hence $R^2 f_* \mathbb{Z}$ is isomorphic to the constant sheaf $\bar{\mathbb{Z}}$, and $E_2^{2,2} \simeq H^2(S, \mathbb{Z})$. The sequence (3.2) becomes :

$$0 \to E_\infty^{4,0} \to H^4(X, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \to 0 \tag{3.3}$$

Since the differentials d_k gives the zero map on $E_k^{4,0}$ for $k \geq 2$, we obtain surjective edge homomorphisms $e_k : E_k^{4,0} \to E_\infty^{4,0}$. Moreover, the maps e_k vanish for elements in $E_k^{4,0}$ lying in the image of a differential map d_k, and in this case non-zero elements can only come from the d_3 map. This gives us the following short exact sequence :

$$0 \to E_3^{1,2} \xrightarrow{d_3} E_3^{4,0} \xrightarrow{e_3} E_\infty^{4,0} \to 0 \tag{3.4}$$

Note that the differential d_2 applied to $E_2^{1,2}$ and $E_2^{4,0}$ are the zero map, and no elements lie in the image of another differential map. Hence, $E_2^{1,2} \simeq E_3^{1,2}$ and $E_2^{4,0} \simeq E_3^{4,0}$. Applying these isomorphisms to (3.4) gives :

$$0 \to E_2^{1,2} \xrightarrow{d_2} E_2^{4,0} \xrightarrow{e_2} E_\infty^{4,0} \to 0 \tag{3.5}$$

Again, since $R^2 f_* \mathbb{Z} \simeq \bar{\mathbb{Z}}$, $E_2^{1,2} \simeq H^1(S, \mathbb{Z})$. Since S is simply connected, $H^1(S, \mathbb{Z})$ is trivial. Hence, $E_2^{4,0} \simeq E_\infty^{4,0}$. Note also that $E_2^{4,0} \simeq H^4(S, \mathbb{Z})$. Combining these isomorphisms with (3.3) gives the following :

$$0 \to H^4(S, \mathbb{Z}) \xrightarrow{f^*} H^4(X, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \to 0 \tag{3.6}$$

We now claim that the class $f^*[x] \in H^4(X, \mathbb{Z})$ of a fiber is not divisible. Indeed, the class $f^*[x]$ is the image of the generator of $H^4(S, \mathbb{Z})$ under the left map in (3.6). If $f^*[x]$ were divisible, then the exactness of (3.6) implies $H^2(S, \mathbb{Z})$ has a torsion component. Note that, since S being simply connected, $H_1(S, \mathbb{Z})$ is trivial, and hence

$H^2(S, \mathbb{Z})$ is torsion-free by the Universal Coefficients Theorem for Cohomology. This is a contradiction. Hence, the class of a fiber cannot be divisible.

To finish the proof, we now apply the argument in the proof of Proposition 3.2.5. The difference between the Picard numbers of X and S is 1. It follows from Proposition 3.2.4 that we can choose generators $\{B_1, \ldots, B_{\rho(S)}\}$ of $Pic\ S$ such that B_i and $f^{-1}(B_i)$ are smooth for $i = 1, \ldots, \rho(S)$. Associated to each B_i is a section $S_i \subset X$, and the set $\{f^*[x], [S_1], \ldots, [S_{\rho(S)}]\}$ is a basis for $H^4(X, \mathbb{Q})$.

Now suppose $W := c_0 \cdot f^*[x] + c_1 \cdot [S_1] + \cdots + c_{\rho(S)} \cdot [S_{\rho(S)}] \in H^4(X, \mathbb{Z})/H^4(X, \mathbb{Z})_{tors}$ is a class of Hodge type $(2,2)$. As in Proposition 3.2.5, the goal is to show that $c_i \in \mathbb{Z}$ for all $i = 0, \ldots, \rho(S)$. For $j = 1, \ldots \rho(S)$, let $D_j \in H^2(S, \mathbb{Z})/H^2(S, \mathbb{Z})$ be classes such that $D_j \smile [B_i] = \delta_{ij}$. Then $W \smile f^*D_j = c_j \in \mathbb{Z}$. Since $c_1, \ldots, c_{\rho(S)}$ are integers, and since W is an integral class, $c_0 \cdot f^*[x] \in H^4(X, \mathbb{Z})/H^4(X, \mathbb{Z})_{tors}$. We know that $f^*[x]$ is non-divisible, and thus c_0 must be an integer. In particular, W, being an integral linear combination of algebraic classes, is algebraic. Since W was an arbitrary class of Hodge type $(2,2)$, X satisfies NHC_1. □

3.3 Fano 3-Folds

In [30], Soulé and Voisin ask if Fano varieties, or more generally, rationally connected varieties, of dimension n satisfy the integral Hodge conjecture for 1-cycles. In this section, we show that Fano 3-folds do satisfy the non-torsion version of the integral Hodge conjecture. The proof itself uses the results from the previous section on conic bundles, along with the work of Mori and Mukai classifying Fano 3-folds with second Betti number greater than or equal to 2 and the work of Iskovskih and Sokurov classifying Fano 3-folds with second Betti number equal to 1.

Definition 3.3.1. *A smooth, complex, connected, projective variety X is Fano if its anticanonical sheaf $-K_X$ is ample.*

Lemma 3.3.2. *Every class in $H^2(X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})$ is of Hodge type.*

Proof. The first claim is Proposition 1.15i. of [20]. The proof uses the long exact sequence of cohomology associated to the exponential sequence and the fact that $h^i(\mathcal{O}_X) = 0$ for $i > 0$, which follows from the Kodaira Vanishing Theorem. The second claim of the lemma then follows from the Hard Lefschetz Theorem. □

Remark 3.3.3. *By Lemma 3.3.2, the rank of the Picard group of X is equal to the 2nd Betti number $B_2 := h^2(X,\mathbb{Z})$.*

Lemma 3.3.4. *([22], Exercise 4.10.3.) Let $V_{d_1,\ldots,d_k} \subset \mathbb{P}^n$ be a smooth, connected, complete intersection of hypersurfaces of degrees $d_1,\ldots,d_k$. $V_{d_1,\ldots,d_k}$ contains a line if*

$$\sum_{i=1}^{k}(d_i+1) \leq 2n-2.$$

In particular, $V_{d_1,\ldots,d_k}$ satisfies NHC_1 and IHC_1 if $\dim V_{d_1,\ldots,d_k} \geq 3$.

Definition 3.3.5. *Let X be a Fano variety. The index of X is the largest positive integer r such that there exists an $H \in Pic\,X$ with $rH \simeq -K_X$.*

Definition 3.3.6. *([25], Definition 3) A Fano 3-fold X is said to be imprimitive if X is isomorphic to the blowup of a Fano 3-Fold along a smooth, irreducible curve. X is said to be primitive if it is not imprimitive.*

Theorem 3.3.7. *(Theorem 5 of [25]) Let X be a primitive Fano 3-fold. Then*

1. $B_2 \leq 3$.

2. *If $B_2 = 2$, then X is a conic bundle over $\mathbb{P}^2$.*

3. *If $B_2 = 3$, then X is a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proposition 3.3.8. *([26], Proposition 5.12.) An imprimitive Fano 3-fold X with $B_2 = 2$ satisfies one of the following conditions :*

1. *X is isomorphic to the blowup of $\mathbb{P}^3$ along a smooth irreducible curve which is a scheme-theoretic intersection of cubics.*

2. *X is isomorphic to the blowup of a smooth quadric $Q \subset \mathbb{P}^4$ along a smooth irreducible curve which is a scheme-theoretic intersection of members of $|\mathscr{O}_Q(2)|$.*

3. *X is isomorphic to the blowup of a Fano 3-fold Y with index 2 and $1 \leq (-\frac{1}{2}K_Y)^3 \leq 5$ along an elliptic or rational curve.*

Lemma 3.3.9. *Let X be a Fano 3-fold with index $r \geq 2$. Then X satisfies NHC_1.*

Proof. Let r be the index of X, and let $H \in Pic\ X$ satisfy $r \cdot H \simeq -K_X$. By Theorem 6 of [21], $2 \leq r \leq 4$ and $1 \leq H^3 \leq 7$. In addition, there are only two cases if $r \geq 3$. If $r = 4$, then $X \simeq \mathbb{P}^3$, and hence IHC_1 and NHC_1 hold. If $r = 3$, then X is isomorphic to a smooth quadric $V_2 \subset \mathbb{P}^4$. By Lemma 3.3.4, V_2 satisfies IHC_1 and NHC_1.

Now suppose that $r = 2$. There are 7 possible cases :

Suppose that $H^3 = 7$. By the proof of Theorem 4.2iii. in [20], X is then obtained from $\mathbb{P}^3$ by blowing up a point. Hence X is birational to $\mathbb{P}^3$, and since $\mathbb{P}^3$ satisfies NHC_1 and NHC_1 is a birational invariant, X satisfies NHC_1.

Suppose that $H^3 = 6$. Then, by Theorem 6iv. of [21], X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. X then satisfies NHC_1 and IHC_1 by the Künneth formula.

Suppose $H^3 = 5$. Then, by Theorem 6iv. of [21], X is birational to a quadric in $\mathbb{P}^4$. By Lemma 3.3.4, a quadric in $\mathbb{P}^4$ satisfies IHC_1 and NHC_1, and any 3-fold birational to a quadric satisfies IHC_1 and NHC_1.

Suppose $H^3 = 4$. Then, by Theorem 6iv. of [21], X is isomorphic to the complete intersection of two quadrics in $\mathbb{P}^5$. By Lemma 3.3.4, NHC_1 holds in this case.

Suppose $H^3 = 3$. Then, by Theorem 6iv. of [21], X is isomorphic to a cubic in $\mathbb{P}^4$, and thus NHC_1 and IHC_1 are true by Lemma 3.3.4.

Suppose $H^3 = 2$. Then X is isomorphic to a double cover of $\mathbb{P}^3$ with branch locus $B \subset \mathbb{P}^3$ being a smooth hypersurface of degree 4. Let $f : X \to \mathbb{P}^3$ be the double cover. X may also be represented as a degree 4 hypersurface in weighted projective space $\mathbb{P}(1,1,1,1,2)$. By [7], $Pic\, X \simeq \mathbb{Z}$, and hence $Pic\ X$ is generated by a rational multiple of f^*H, where $H \in H^2(\mathbb{P}^3, \mathbb{Z})$ is the class associated to a hyperplane. In addition, the Hard Lefschetz Theorem implies that the submodule of classes in $H^4(X,\mathbb{Z})/H^4(X,\mathbb{Z})_{tors}$ of Hodge type $(2,2)$ has rank 1. By V.4.11.2. of [4], through every point of X, there exists a rational curve C such that $\mathcal{O}_X(1) \cdot C \leq 1$. This implies that $[C]$ generates classes in $H^4(X,\mathbb{Z})/H^4(X,\mathbb{Z})_{tors}$ of Hodge type $(2,2)$. Therefore, X satisfies NHC_1.

Finally, suppose $r = 2$ and $H^3 = 1$. Theorem 6vi. of [21] implies that $Pic\ X \simeq \mathbb{Z}$.

Since H generates those classes in $H^2(X,\mathbb{Z})/H^2(X,\mathbb{Z})_{tors}$ of Hodge type $(1,1)$ and $H^3 = 1$, H^2 generates those classes in $H^4(X,\mathbb{Z})/H^4(X,\mathbb{Z})_{tors}$ of Hodge type $(2,2)$. Therefore, NHC_1 holds for X.

In all cases, X satisfies NHC_1, and hence the lemma follows.

□

Proposition 3.3.10. *Let X be a Fano 3-fold. Then X satisfies NHC_1.*

Proof. Let X be a Fano 3-fold, and suppose $B_2 \geq 2$. We will distinguish between the cases when X is primitive and when X is imprimitive. Suppose that X is primitive. By Theorem 3.3.7, X is then isomorphic to a conic bundle over $\mathbb{P}^2$ or to a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. NHC_1 is true in both cases by Proposition 3.2.5, and thus X satisfies NHC_1.

Now suppose that X is imprimitive. By Proposition 3.3.8, there are three cases to consider. Both IHC_1 and NHC_1 hold in the first two cases, since X is either birational to $\mathbb{P}^3$ or birational to a quadric in $\mathbb{P}^4$. In the third case, X is birational to a Fano 3-fold Y with index 2 and $B_2 = 1$. By Lemma 3.3.9, X satisfies NHC_1. Hence X satisfies NHC_1 if $B_2 \geq 2$.

Now suppose that $B_2 = 1$. By Lemma 3.3.9, we may assume that the index of X is 1. We now say a Fano 3-fold X with index $r = 1$ is <u>*hyperelliptic*</u> if the map $X \to \mathbb{P}H^0(X, -K_X)$ is a morphism of degree 2.

Lemma 3.3.11. *Let X be a hyperelliptic Fano 3-fold with $B_2 = 1$. Then X satisfies NHC_1.*

Proof. By Theorem 9 of [21], there are two cases to consider :

1. X is isomorphic to a double cover of $\mathbb{P}^3$ with the branch locus being a smooth hypersurface of degree 6.

2. X is isomorphic to a double cover of a smooth quadric $W \subset \mathbb{P}^4$ with the branch locus D being a smooth complete intersection of W with a smooth quartic in $\mathbb{P}^4$.

Suppose that X satisfies the first case, and there exists a double cover $f : X \longrightarrow \mathbb{P}^3$ with branch locus D a smooth hypersurface of degree 6. X may also be represented as a degree 6 hypersurface in weighted projective space $\mathbb{P}(1,1,1,1,3)$. By [7], $Pic\ X \simeq \mathbb{Z}$, and hence $Pic\ X$ is generated by a rational multiple of f^*H, where $H \in H^2(\mathbb{P}^3, \mathbb{Z})$ is the class associated to a hyperplane.

We may assume without loss of generality that D does not contain a line. Indeed, if there exists a line $l \subset D$, then there exists a line $l' \subset X$ mapping 1:1 onto l via f, in which case, by the projection formula, $f^*H \smile [l'] = H \smile [l] = 1$. Since $H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$, $[l']$ is a generator, and thus X satisfies NHC_1.

Let $Gr(\mathbb{P}^1, \mathbb{P}^3)$ be the Grassmanian of lines in $\mathbb{P}^3$. The goal is to show that there exists a line $l \in Gr(\mathbb{P}^1, \mathbb{P}^3)$ so that, when D restricts to l, the local equations of the intersection are given by a square.

Choose coordinates $x_0, \ldots, x_3$ for $\mathbb{P}^3$, so that D is the zero locus of a homogeneous form $F_D \in \mathbb{C}[x_0, \ldots, x_3]$ of degree 6. Let $L' \subset \mathbb{P}^3$ be the line for which $x_0 = x_1 = 0$, and choose local coordinates y_0 and y_1 on L'. Let

$$g_{L'} : \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(6)) \dashrightarrow \mathbb{P}H^0(L', \mathcal{O}_{L'}(6))$$

be the rational map given by $g_{L'}(\mathbb{V}(F(x_0, \ldots, x_3)) = \mathbb{V}(F(0,0,y_0,y_1))$. Let $B_{L'} \subset$

$\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$ be the locus of hypersurfaces for which $g_{L'}$ is not defined. Hypersurfaces in $B_{L'}$ are those hypersurfaces that are the zero locus of a homogeneous degree 6 form F whose monomials all contain an x_0 term or an x_1 term. These include all monomials except those 7 monomials having only x_2 and x_3 terms, and hence $codim_{\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))} B_{L'} = 7$.

Let

$$\mathbb{P}H^0(L', \mathscr{O}_{L'}(3)) \to \mathbb{P}H^0(L', \mathscr{O}_{L'}(6))$$

be the squaring map, and let $S \subset \mathbb{P}H^0(L', \mathscr{O}_{L'}(6))$ be its image. Note that

$$codim_{\mathbb{P}H^0('L, \mathscr{O}_{L'}(6))} S = 3.$$

Since $codim_{\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))} = 7$, the subvariety $S_{L'} := B_{L'} \cup \overline{g_{L'}^{-1}(S)}$ has codimension 3 in $\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$. Note that $S_{L'}$ consists of those hypersurfaces in $\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$ that either contain L' or restrict to L' to give the square of a cubic form in $\mathbb{C}[x_2, x_3]$.

Given $L \in Gr(\mathbb{P}^1, \mathbb{P}^3)$, define $S_L \subset \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$ to be the locus of hypersurfaces that either contain L or restrict to L to give a square of a cubic form in $\mathbb{C}[x_2, x_3]$. By applying a projective linear tranformations on $\mathbb{P}^3$ taking L to L', we have $codim_{\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))} S_L = 3$.

Consider the union

$$S := \bigcup_{L \in Gr(\mathbb{P}^1, \mathbb{P}^3)} S_L.$$

The above union S is closed, since it is equal to the projection to $\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$ of the incidence variety

$$\{(Z, l) \in \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6)) \times Gr(\mathbb{P}^1, \mathbb{P}^3) \mid Z \cap l \text{ is the zero locus of a square}\}.$$

In fact, S is equal to $\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$. Indeed, let $PSL(4)$ be the group of automorphisms on $\mathbb{P}^3$. The claim then follows if the orbit of $PSL(4) \cdot S_{L'}$ is equal to $\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$. The morphism $\varphi : PSL(4) \times S_{L'} \to \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$ is surjective if the induced map on tangent spaces is surjective. Indeed, by the Implicit Function Theorem, there exists an open neighborhood of a point in $(Id, \mathbb{V}(G))$ mapping to an open neighborhood of $\varphi(Id, \mathbb{V}(G))$ of maximal dimension. Since the image of φ is closed, the image of φ must contain the open neighborhood of $\varphi(Id, \mathbb{V}(G))$, and thus the image must have the same dimension as $\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$, since $\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$ is irreducible.

Consider the restriction of $\varphi : PSL(4) \times B_{L'} \to \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6))$. Note that $\varphi : \{Id\} \times B_{L'} \to B_{L'}$, and, since $B_{L'}$ is a linear subspace, $B_{L'}$ is isomorphic to its tangent space. Therefore, on vectors in the tangent space to $S_{L'}$ corresponding to $B_{L'} \subset S_{L'}$, the map on tangent spaces induced by φ has maximal rank.

Let $\mathbb{V}(G) \in S_{L'}$ be general. $G(x_0, \dots, x_3)$ may be written in the form

$$x_0 \cdot g_0(x_2, x_3) + x_1 \cdot g_1(x_2, x_3) + g_{23}(x_2, x_3) + n(x_0, x_1, x_2, x_3),$$

where $n(x_0, x_1, x_2, x_3)$ is the sum of all monomials with x_0 and x_1 terms whose total degree is greater than or equal to 2. Note that, for G general, the polynomial g_{23} is a square in $\mathbb{C}[x_2, x_3]$. If $r^2 = g_{23}$, then a deformation

$$(r + \epsilon h)^2 \sim r^2 + \epsilon \cdot 2rh$$

allows us to describe elements of $\mathbb{T}_{\mathbb{V}(G)}S_{L'}$ is terms of cubics $h \in \mathbb{C}[x_2, x_3]$. Let $h_1, \dots, h_4$ be linearly indepedent cubics so that elements of $\mathbb{T}_{\mathbb{V}(G)}S_{L'}$ are given by $2rh_1, \dots, 2rh_4$. These four vectors, plus the subspace of vectors coming from $B_{L'}$, generate $\mathbb{T}_{\mathbb{V}(G)}S_{L'}$, and, as above, by the restriction of φ to $\{Id\} \times S_{L'} \to S_{L'}$, φ has maximal rank on these vectors.

It remains to show that, via a defommation of the identity mapping on $PSL(4)$, φ induces a surjective map on tangent spaces. We now apply the automorphism $\sigma_\epsilon \in PSL(4)$ given by $x_0 \mapsto x_0 - \epsilon_1 x_2 - \epsilon_2 x_3$, $x_1 \mapsto x_1 - \epsilon_3 x_2 - \epsilon_4 x_3$, where $\epsilon_1, \ldots, \epsilon_4$ are sufficiently small.

We apply σ_ϵ to G and mod out by the $\epsilon_i \cdot \epsilon_j$ terms to get :

$$G + \epsilon_1 \cdot x_2 \cdot g_0(x_2, x_3) + \epsilon_2 \cdot x_3 \cdot g_0(x_2, x_3) + \epsilon_3 \cdot x_2 \cdot g_1(x_2, x_3) + \epsilon_4 \cdot x_3 \cdot g_1(x_2, x_3).$$

Now let G be chosen so that it satisfies the following : the polynomials $\{g_{23}, x_2 \cdot g_0, x_3 \cdot g_0, x_2 \cdot g_1, x_3 \cdot g_1\}$ are linearly independent as elements of the $\mathbb{C}$-vector space of homogeneous degree 6 forms in x_2 and x_3, and the intersection of $Span\{x_2 \cdot g_0, x_3 \cdot g_0, x_2 \cdot g_1, x_3 \cdot g_1\}$ with $Span\{2rh_1, \ldots, 2rh_4\}$ is 1-dimensional. It follows that, via $\sigma_\epsilon \times S_{L'} \to \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(6))$, we obtain a 3 dimensional subspace of vectors not coming from $\mathbb{T}_{\mathbb{V}(G)}S_{L'}$, and thus the map on tangent spaces induced by φ is surjective.

In particular, the sextic D either contains a line or there exists a line $l \in Gr(\mathbb{P}^1, \mathbb{P}^3)$ such that $D \cap l$ is finite and the zero locus of the square of a cubic form. D does not contain a line by assumption, and therefore there exists a line $l \in Gr(\mathbb{P}^1, \mathbb{P}^3)$ such that $D \cap l$ is finite and the zero locus of a square of a cubic form.

$f^{-1}(l)$ is isomorphic to a degree 6 hypersurface in weighted projective space $\mathbb{P}(1,1,3)$, given by the zero locus of $y^2 - f(x_0, x_1)$, where $\deg y = 3, \deg x_i = 1$, and f is a homogeneous polynomial of degree 6. Since l is tangent to D at three points, $f^{-1}(l)$ is isomorphic to a curve in $\mathbb{P}(1,1,3)$ that is the zero locus of a polynomial of the form

$$y^2 - f_1^2(x_0, x_1) \cdot f_2^2(x_0, x_1) \cdot f_3^2(x_0, x_1),$$

where the f_i's have degree 2. The above polynomial is a difference of squares, and hence it factors, implying that $f^{-1}(l)$ is reducible. This implies that there exists a curve $l' \subset X$ such that $f_*[l'] = [l]$. By the projection formula, $f^*L \smile [l'] = L \smile [l] = 1$, and hence $[l']$ generates $H^4(X, \mathbb{Z})$. In particular, X satisfies NHC_1.

Now consider the second case, for which there exists a surjective map $f : X \to W$ of degree two, with $W \subset \mathbb{P}^4$ a quadric and branch locus $D \subset W$ a smooth, complete intersection of W with a smooth quartic $Q \subset \mathbb{P}^4$. The goal now is to find a line $l \subset W$ bitangent to Q. By Example 15.21. of [15], the locus $B \subset Gr(\mathbb{P}^1, \mathbb{P}^4)$ of lines bitangent to D has dimension 4. The locus $L_W \subset Gr(\mathbb{P}^1, \mathbb{P}^4)$ of lines in $\mathbb{P}^4$ contained in W has dimension 3 (in fact, $L_W \simeq \mathbb{P}^3$ by Exercise 22.6. of [15]). We need to show that $L_W \cap B \neq \emptyset$.

By a Proposition in Chapter 1, Section 5 of [13], $H^4(Gr(\mathbb{P}^1, \mathbb{P}^4), \mathbb{Z}) \simeq \mathbb{Z}^2$ and is generated by the Schubert classes A_1 and A_2, where A_1 is the locus of lines contained in a $\mathbb{P}^3$ and where A_2 is the locus of lines meeting a $\mathbb{P}^1$. Additionally, $H^8(Gr(\mathbb{P}^1, \mathbb{P}^4), \mathbb{Z})$ is generated by the Schubert classes α_1 and α_2, where α_1 is the locus of lines in a $\mathbb{P}^2$ and where α_2 is the locus of lines containing a point and meeting a $\mathbb{P}^2$. By Example 14.7.4. of [11], $A_i \smile \alpha_j = \delta_{ij}$.

We now claim that the cohomology class $[B] \in H^4(Gr(\mathbb{P}^1, \mathbb{P}^4), \mathbb{Z})$ may be written in the form $c_1 \cdot A_1 + c_2 \cdot A_2$, where c_1 and c_2 are positive integers. Since $Gr(\mathbb{P}^1, \mathbb{P}^4)$ is a homogeneous space, Theorem 10.8 in Chapter 3 of [16] implies that, for any two effective cycles C_1 and C_2 in $Gr(\mathbb{P}^1, \mathbb{P}^4)$, we may apply a projective linear transformation on $Gr(\mathbb{P}^1, \mathbb{P}^4)$ to C_1 to have the support of the cycles either be disjoint or intersecting in the expected dimension. Consequently, if C_1 and C_2 are effective cycles of complimentary dimension, then $C_1.C_2 \geq 0$. If we now set $[B] = c_1 \cdot A_1 + c_2 \cdot A_2$, then, since α_1 and α_2 are classes associated to effective cycles,

$$0 \leq [B] \smile \alpha_1 = c_1,$$

$$0 \leq [B] \smile \alpha_2 = c_2.$$

Hence, the class $[B] \in H^4(Gr(\mathbb{P}^1, \mathbb{P}^4), \mathbb{Z})$ may be written in the form $c_1A_1 + c_2A_2$, where $c_1, c_2 \geq 0$ are integers.

Now consider $[L_W] \smile A_1$ and $[L_W] \smile A_2$. The intersection $[L_W] \smile A_1$ will be the class associated to the locus of lines contained in the intersection of W with a general hyperplane. A quadric in $\mathbb{P}^3$ contains a 1-parameter family of lines by Chapter 1, Section 6.4. of [29], and hence this class is associated to an effective 1-cycle. Also, $[L_W] \smile A_2$ is the class associated to the locus of lines in W meeting a line. W intersects a line in two points, and thus this is the locus of lines in W contained in one of two points. By Lemma 3.3.4, there exists a line through every point of W, and thus the class $[L_W] \smile A_2$ is effective.

In fact, the locus of lines through a point of W is 1-dimensional. Indeed, a line contained in $w \in W$ also lies in $\mathbb{T}_w W$, and hence in $W \cap \mathbb{T}_w W$, which is isomorphic to a singular quadric in $\mathbb{P}^3$ with a singularity at w. A singular quadric in $\mathbb{P}^3$ is isomorphic to a cone over a quadric curve in $\mathbb{P}^2$, and thus the intersection $W \cap \mathbb{T}_w W$ is a 1-parameter family of lines.

It follows that $[L_W] \smile A_2$ is an effective class associated to a 1-cycle. Hence, $[L_W] \smile [B] = c_1 \cdot [L_W] \smile A_1 + c_2 \cdot [L_W] \smile A_2$ is the class associated to an effective

1-cycle. In particular $L_W \cap B \neq \emptyset$.

Let l then be a bitangent to Q contained in W. As in the previous case, we consider the inverse image of l in W via f. Since the ramification locus of f is the intersection of W with a quartic, $f^{-1}(l)$ is isomorphic to a hypersurface in $\mathbb{P}(1,1,2)$ given by the zero locus of $y^2 - g(x_0, x_1)$, where $\deg y = 2$, $\deg x_i = 1$, and $\deg g = 4$. Since l is a bitangent, the polynomial $g(x_0, x_1)$ is a square, and thus $f^{-1}(l)$ in X is reducible. As in the first case, it follows that X satisfies NHC_1. □

By Lemma 3.3.11, we may now assume that X is not hyperelliptic. The following lemma will allow us to assume in addition that $|-K_X|$ has no base points.

Lemma 3.3.12. *Let X be a Fano 3-fold with the property that $|-K_X|$ has a basepoint. Then X satisfies IHC_1.*

Proof. By Prop. 5 of [21], if $|-K_X|$ has a basepoint, then X is birationally equivalent to a Fano 3-fold with index $r = 2$. Therefore, X satisfies NHC_1 by Lemma 3.3.9. □

By Lemmas 3.3.11 and 3.3.12, we may now assume the following about X : it has Picard number 1, it has index 1, it is not hyperelliptic, and $|-K_X|$ has no base points. By Proposition 11i. of [21], $-K_X$ is very ample, in which case X is a Fano 3-fold of the *principal series* ([21], Def. 10.). Theorem 16 of [21] implies that X contains a line under the embedding induced by $-K_X$. Since $Pic\ X \simeq \mathbb{Z}$, X must satisfy NHC_1. This exhausts all cases, and hence the proposition follows.

□

Chapter 4

Hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$

This chapter is devoted to determining whether or not smooth hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ satisfy the integral Hodge conjecture. In this context, the integral Hodge conjecture and the non-torsion integral Hodge conjecture are equivalent by the Lefschetz Hyperplane Theorem (Lemma 4.1.3). If $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a hypersurface of bidegree (a, b) (see Notation 4.1.1 for an explanation of bidegree), then there are several cases in which we are able to determine if the integral Hodge conjecture holds. If a and b are relatively prime, then the integral Hodge conjecture holds for W by 1. of Corollary 4.1.5. Additionally, if one of a or b is equal to $0, 1$ or 2, then the integral conjecture holds for W (Lemma 4.1.2, Corollary 4.1.5, and Remark 4.1.6, respectively).

Using an adaptation of the method used by Kollár in [4] for hypersurfaces in $\mathbb{P}^4$, we show that, for strictly positive integers $a, d,$and r satisfying suitable conditions, very general hypersurfaces of bidegree (da, dr) fail the integral Hodge conjecture (see Proposition 4.2.1 for a precise formulation).

The most interesting case not resolved is when W has a bidegree of the form $(3, 3r)$, where $r \geq 1$. In this case, W is a smooth, complex, connected, projective variety of dimension three that is not of general type, and it would be interesting to see if such a 3-fold could fail the integral Hodge conjecture. The only subcase that is resolved is when $r = 1$, in which case W satisfies the integral Hodge conjecture by Lemma 4.3.1.

4.1 Preliminary Results

Notation 4.1.1. *Let $D_{a,b}$ denote the divisor of a non-zero section of the sheaf*

$$p_1^*\mathscr{O}_{\mathbb{P}^2}(a) \otimes p_2^*\mathscr{O}_{\mathbb{P}^2}(b)$$

on $\mathbb{P}^2 \times \mathbb{P}^2$, where $p_1 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ is projection onto the first factor and $p_2 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ is projection onto the second factor. We say that such a divisor has bidegree (a, b).

Lemma 4.1.2. *Let $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth hypersurface of bidegree (a, b) such that one of a or b is equal to 0. Then W satisfies IHC_1 and NHC_1.*

Proof. If one of a or b equals 0, then $W \simeq X \times \mathbb{P}^2$, where $X \subset \mathbb{P}^2$ is a smooth curve. Since X is a curve, $H^k(X, \mathbb{Z})$ is trivial for $k \geq 3$. Hence, by the Künneth formula, we have

$$H^4(W, \mathbb{Z}) \simeq H^0(X, \mathbb{Z}) \otimes H^4(\mathbb{P}^2, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \otimes H^2(\mathbb{P}^2, \mathbb{Z}).$$

Since X is a Riemann surface, all classes in $H^0(X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ are algebraic. Moreover, all classes in $H^4(\mathbb{P}^2, \mathbb{Z})$ are algebraic, since $H^4(\mathbb{P}^2, \mathbb{Z})$ is generated by a class associated to a point. Hence, the first component in the above sum must consist of algebraic classes. Finally, $H^2(\mathbb{P}^2, \mathbb{Z})$ is generated by the class of a line, and hence every class in $H^2(\mathbb{P}^2, \mathbb{Z})$ is algebraic. Thus, every class in $H^4(W, \mathbb{Z})$ is algebraic. □

Lemma 4.1.3. *Let $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a hypersurface of bidegree (a, b) with $a, b > 0$. Then*

1. $\mathbb{Z}^2 \simeq H^2(W, \mathbb{Z})$. *In particular, $H^2(W, \mathbb{Z})$ is torsion-free, and hence IHC_1 and NHC_1 are equivalent.*
2. $\mathbb{Q}^2 \simeq H^2(W, \mathbb{Q}) = H^{1,1}(W, \mathbb{Q}) \cap H^2(W)$.
3. $\mathbb{Q}^2 \simeq H^4(W, \mathbb{Q}) = H^4(W, \mathbb{Q}) \cap H^{2,2}(W)$.

Proof. Let $j : W \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$. To prove 1., note that $H^2(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}^2$ by the Künneth Formula. By the Lefschetz Hyperplane Theorem, $j^* : H^2(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \to H^2(W, \mathbb{Z})$ is an isomorphism, and so part 1. follows.

To prove 2., note that $H^2(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Q}) \simeq \mathbb{Q}^2$ by the Künneth Formula. Note also that $H^2(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Q}) = H^2(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Q}) \cap H^{1,1}(\mathbb{P}^2 \times \mathbb{P}^2)$. By the Lefschetz Hyperplane Theorem, $j^* : H^2(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Q}) \to H^2(W, \mathbb{Q})$ is an isomorphism. Since j^* preserves Hodge type, it follows that $H^2(W, \mathbb{Q}) = H^{1,1}(W, \mathbb{Q})$.

To prove 3., note that the Hard Lefschetz Theorem gives an isomorphism

$$H^2(W, \mathbb{Q}) \xrightarrow{\sim} H^4(W, \mathbb{Q})$$

given by the cup product with a cohomology class associated to an ample divisor. It follows that every element of $H^4(W, \mathbb{Q})$ has Hodge type $(2, 2)$. □

Proposition 4.1.4. *Let $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth hypersurface of bidegree (a, b) with $a, b > 0$. Suppose there exists a non-algebraic class $\alpha \in H^4(W, \mathbb{Z})$ of Hodge type $(2, 2)$ such that $p \cdot \alpha$ is algebraic for some prime p. Then $p \mid a$ and $p \mid b$.*

Proof. Let α be as above. By Proposition 2.3.1, there exists $L \in H^2(W, \mathbb{Z})$ such that $L \smile \alpha = 1$ and $p \mid (L \smile A)$ for all algebraic classes $A \in H^4(W, \mathbb{Z})$. Let $i : W \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be the inclusion map. By Lemma 4.1.3, $H^2(W, \mathbb{Z}) \simeq \mathbb{Z}^2$ is generated by $B_1 := i^*[D_{0,1}]$ and $B_2 := i^*[D_{1,0}]$, where $D_{a,b}$ is a divisor as described in Notation 4.1.1. Hence, L may be written in the form

$$[L] = x_1 \cdot B_1 + x_1 \cdot B_2,$$

where $x_1, x_2 \in \mathbb{Z}$.

It cannot be the case that both x_1 and x_2 are multiples of p. Indeed, since $p \nmid [L] \smile [\alpha]$, or

$$p \nmid (x_1 \cdot B_1 \smile [\alpha] + x_2 \cdot B_2 \smile [\alpha]),$$

and since $B_1 \smile [\alpha], B_2 \smile [\alpha] \in \mathbb{Z}$, at least one of x_1 and x_2 must not be a multiple of p.

Since p does not divide both x_1 and x_2, assume that $p \nmid x_1$. The class B_2^2 is algebraic, implying $p \mid (L \smile B_2^2)$. Since $L \smile B_2^2 = x_1 \cdot a$, $p \mid (x_1 \cdot a)$. Since $p \nmid x_1$ by assumption and since p is prime, $p \mid a$. By a similar argument, $p \nmid x_2$ implies that $p \mid b$. If $p \mid a$ and $p \mid b$, the proposition holds. We therefore assume that p divides one of x_1 and x_2.

Now suppose $p \mid x_2$ and $p \nmid x_1$. Consider

$$[L] \smile B_1 \smile B_2 = x_1 \cdot b + x_2 \cdot a.$$

This expression is a multiple of p, and since $p \mid x_2$, $p \mid (x_2 \cdot a)$. It follows that $p \mid (x_1 \cdot b)$. Since $p \nmid x_1$, it must be the case that $p \mid b$. Also, we already saw that $p \nmid x_1$ implies $p \mid a$. Thus $p \mid a$ and $p \mid b$. Using a similar argument, we also see that $p \nmid x_2$ and $p \mid x_1$ implies that $p \mid a$ and $p \mid b$. Hence, $p \mid a$ and $p \mid b$ in all cases. □

From Proposition 4.1.4, we are able to show IHC_1 holds in a number of cases, as given in the following Corollary :

Corollary 4.1.5. *Let $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth hypersurface of bidegree (a, b), where $a, b > 0$.*

1. *If a and b are relatively prime, then W satisfies IHC_1 and NHC_1.*
2. *If one of a or b is equal to 1, then W satisfies IHC_1 and NHC_1.*

Remark 4.1.6. *Let W be as in Proposition 4.1.4. If a or b is equal to 2, then $W \to \mathbb{P}^2$ is a conic bundle, and hence W satisfies IHC_1 and NHC_1 by Proposition 3.2.5.*

4.2 Hypersurfaces Failing IHC_1

The following proposition is used to show that there exist very general hypersurfaces of $\mathbb{P}^2 \times \mathbb{P}^2$ failing IHC_1. These hypersurfaces W have bidegree (da, dr), where $d \geq 4$, $r \geq 3$, $r < d$, and $\binom{a+2}{2} \leq \binom{r+3}{3}$.

The proof is an adaptation of the argument Kollár in 2.2. In this case, we choose a very general hypersurface $X \subset \mathbb{P}^3$ of degree $d \geq 4$. By the Noether-Lefschetz Theorem, we may assume that every curve $C \subset X$ has degree divisible by d. We then choose a family $Y \subset \mathbb{P}^2 \times X$ of intersections of X with hypersurfaces in $\mathbb{P}^2$ of degree r. Given Y, there exists a projection map $q : Y \to X$, and if H is a hyperplane section of X, then, for all curves $C' \subset Y$, $d \mid [C'] \smile q^*[H]$ by the projection formula.

Given a suitable Y, we then choose a point $p \in \mathbb{P}^3$ and project from this point to obtain a map $\varphi_p : Y \to \mathbb{P}^2 \times \mathbb{P}^2$. As in Kollár's example, the image Y' will be highly singular, but φ_p is generically 1:1, 2:1 on a divisor, 3:1 on a curve, and 4:1 or higher on points. Proving this fact about the map φ_p is the most difficult part of the proof.

Proposition 4.2.1. *Let a, r, d be strictly positive integers satisfying the following conditions :*

1. $d \geq 4$ *and* $d \neq 6$
2. $r \geq 3$
3. $r < d$

Then very general hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (da, dr) fail IHC_1.

Proof. To prove the proposition, we will prove the following statement :

($*$) Let r, d, and a be strictly positive integers satisfying the following conditions : $r \geq 3$, $d \geq 4$, $r < d$. There exists a hypersurface $Y \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (da, dr) so that, for all curves $C \subset Y$, $d | 6 \cdot ([D_{0,1}] \frown [C])$.

Lemma 4.2.2. *Statement ($*$) implies the proposition.*

Proof. Hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (da, dr) are zero loci of bihomogeneous polynomials of bidegree (da, dr) in

$$\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2].$$

Hence, hypersurfaces of bidegree (da, dr) in $\mathbb{P}^2 \times \mathbb{P}^2$ are parametrized by $\mathbb{P}^Q$, where

$$Q := \binom{da+2}{2} \cdot \binom{dr+2}{2} - 1.$$

Associated to $\mathbb{P}^Q$ is a universal hypersurface $\mathscr{X} \subset \mathbb{P}^Q \times \mathbb{P}^2 \times \mathbb{P}^2$ and a projection map $\pi : \mathscr{X} \longrightarrow \mathbb{P}^Q$. By statement $(*)$, there exists a fiber $\pi^{-1}(p_0)$ such that for all curves $C \subset \pi^{-1}(p_0)$, $d | 6 \cdot ([D_{0,1}] \frown [C])$. Let $\mathscr{H}$ be the relative Hilbert scheme of curves supported on fibers of $\pi : \mathscr{X} \longrightarrow \mathbb{P}^Q$. Using the notation from Appendix B, define $W \subset \mathbb{P}^Q$ to be

$$\left(\bigcup_{A \in \mathscr{I}} \rho_A(A) \right) \cup \{t \in \mathbb{P}^Q \mid \pi^{-1}(t) \text{ is not smooth}\},$$

where $\mathscr{I}$ is the set of all irreducible components A of $\mathscr{H}$ such that $\rho_A(A) \neq \mathbb{P}^Q$. Let $s \in \mathbb{P}^Q \setminus W$. If $C \subset \pi^{-1}(s)$ is a curve, then there exists an irreducible component $\mathscr{C} \subset \mathscr{H}$ such that $\rho_{\mathscr{C}}(\mathscr{C}) = \mathbb{P}^Q$ and, for some point $c \in \mathscr{C}$, $\pi'^{-1}_{\mathscr{C}}(c) \simeq C$. Also, there exists a point $c_0 \in \mathscr{C}$ such that $\rho(c_0) = p_0$. Thus, $C_0 := \pi'^{-1}_{\mathscr{C}}(c_0)$ is a curve lying on Y.

There exists a class $H_{\mathscr{C}} \in H^2(\mathscr{C} \times \mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z})$ obtained by pulling back the class $[D_{0,1}]$ via the projection map $\mathscr{C} \times \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$, and let $\Sigma \in H^6(\mathscr{C} \times \mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z})$ be the class associated to $\chi_{\mathscr{C}}$. By the Leray spectral sequence, there are the following maps :

$$H^2(\mathscr{C} \times \mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \longrightarrow H^0(\mathscr{C}, R^2 pr_{\mathscr{C}*}\mathbb{Z})$$
$$H^6(\mathscr{C} \times \mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \longrightarrow H^0(\mathscr{C}, R^6 pr_{\mathscr{C}*}\mathbb{Z})$$

We now have the cup product map :

$$H^0(\mathscr{C}, R^2 pr_{\mathscr{C}*}\mathbb{Z}) \otimes H^0(\mathscr{C}, R^6 pr_{\mathscr{C}*}\mathbb{Z}) \longrightarrow H^0(\mathscr{C}, R^8 pr_{\mathscr{C}*}\mathbb{Z}).$$

Since $R^8 pr_{\mathscr{C}*}\mathbb{Z}$ is isomorphic to the constant sheaf $\mathbb{Z}$ on $\mathscr{C}$, the cup product will be constant on fibers, which implies that, in $\mathbb{P}^2 \times \mathbb{P}^2$, the products $[D_{0,1}] \times C$ and $[D_{0,1}] \times C_0$ are equal. Since $d \mid 6 \cdot [D_{0,1}] \times C_0$, $d \mid 6 \cdot [D_{0,1}] \times C$. The curve C and the point s was arbitrary, and thus, for very general hypersurfaces $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (da, dr), curves $C \subset W$ satisfy

$$d \mid 6 \cdot [D_{0,1}] \times C. \tag{4.1}$$

Finally, W is an ample subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$, and hence every class in $H^2(W, \mathbb{Z})$ has Hodge type $(1,1)$ by the Lefschetz Hyperplane Theorem. By the Hard Lefschetz Theorem, every class in $H^4(W, \mathbb{Z})$ has Hodge type $(2,2)$. There exists a class $\alpha \in H^4(W, \mathbb{Z})$ such that $\alpha \smile [D_{0,1}] = 1$ by Proposition 3.37 of [17], and α has Hodge type $(2,2)$. It can't be algebraic, however, by (4.1). Therefore, W fails IHC_1. □

Let $a, r,$and d be integers satisfying the conditions : $r \geq 3$, $d \geq 4$, $r < d$. Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree d that is not contained in the Noether-Lefschetz locus. Then $Pic\ X \simeq \mathbb{Z}$ and is generated by a hyperplane section. Let $Y' \subset \mathbb{P}^2 \times X$ be a general member of

$$|pr_{\mathbb{P}^2}^* \mathscr{O}_{\mathbb{P}^2}(a) \times pr_X^* \mathscr{O}_{\mathbb{P}^3}(r)|,$$

where $pr_{\mathbb{P}^2}$ and pr_X are the projection maps $\mathbb{P}^2 \times X \to \mathbb{P}^2$ and $\mathbb{P}^2 \times X \to X$, respectively. Note that Y' is smooth by Bertini's Theorem. Also, curves on Y' satisfy a property given by the following lemma :

Lemma 4.2.3. *For every curve $\tilde{C} \subset Y'$, $d \mid \deg(pr_X^* \mathscr{O}_X(1)|_{\tilde{C}})$.*

Proof. Let $C' \subset Y'$ be a curve. Then, by the projection formula,

$$pr_X^* c_1(\mathscr{O}_X(1)) \smile [\tilde{C}] = pr_{X*}(pr_X^* c_1(\mathscr{O}_X(1)) \smile [\tilde{C}]) = c_1(\mathscr{O}_X(1)) \smile pr_{X*}[\tilde{C}]$$

Since $Pic\ X$ is generated by complete intersections with hypersurfaces in $\mathbb{P}^3$, the above cup product $c_1(\mathscr{O}_X(1)) \smile pr_{X*}[\tilde{C}]$ is a multiple of the degree of X, and hence

$$pr_X^* c_1(\mathscr{O}_X(1)) \smile [\tilde{C}] \in d \cdot \mathbb{Z}.$$

□

Now fix a point $p \in \mathbb{P}^3 \setminus X$, and define $(\mathbb{P}^2)^\vee$ to be the space of lines in $\mathbb{P}^3$ containing p. We then obtain a map $\varphi_p : \mathbb{P}^2 \times X \to \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ given by the identity map on the first factor and projection from p on the second factor.

$$\begin{array}{ccc} Y' & \longrightarrow & \mathbb{P}^2 \times X \\ {\scriptstyle\varphi_p}\downarrow & & {\scriptstyle\varphi_p}\downarrow \\ Y & \longrightarrow & \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \end{array}$$

We define Y to be the image of Y' under φ_p. We claim the (highly singular) variety Y will satisfy condition $(*)$, and this follows if we prove the following :

$(**)$ The map $\varphi_p : Y' \to Y$ is generically 1:1, 2:1 on a divisor of Y', 3:1 on a curve of Y', and $4:1$ or higher on points of Y'.

Lemma 4.2.4. *Condition $(**)$ implies that Y satisfies condition $(*)$.*

Proof. Let $C \subset Y$ be a curve, and consider $[\varphi_p^{-1}(C)] \in H_2(Y', \mathbb{Z})$. Since φ_p is 1:1, 2:1, or 3:1 on $\varphi_p^{-1}(C)$ by condition $(**)$, $\varphi_{p*}([\varphi_p^{-1}(C)])$ is an element of $\{[C], 2[C], 3[C]\} \subset H_2(Y, \mathbb{Z})$. By the projection formula,

$$\begin{aligned} \varphi_p^*[D_{0,1}] \frown [\varphi_p^{-1}(C)] &= \varphi_{p*}(\varphi_p^*[D_{0,1}] \frown [\varphi_p^{-1}(C)]) \\ &= [D_{0,1}] \frown \varphi_{p*}[\varphi_p^{-1}(C)] \end{aligned}$$

$$= [D_{0,1}] \frown M \cdot [C],$$

where $M = 1, 2, 3$. Since $pr_X^* c_1(\mathscr{O}_X(1) = \varphi_p^*[D_{0,1}]$, $d | (\deg \varphi_p^*[D_{0,1}] \frown [\varphi_p^{-1}(C)])$. Since M might be 2 or 3, $d | 6 \cdot \deg[D_{0,1}] \frown [C]$.

It remains to show that Y is a hypersurface of bidegree (da, dr). Note first that $\dim Y = 3$, since φ_p is generically 1:1 by condition (**). The components of the bidegree of Y are given by the intersection numbers $[Y] \smile [D_{0,1}]^2 \smile [D_{1,0}]$ and $[Y] \smile [D_{1,0}]^2 \smile [D_{0,1}]$ on $\mathbb{P}^2 \times \mathbb{P}^2$. Since φ_p is generically 1:1, it suffices to compute the intersection numbers $\varphi_p^*[D_{1,0}]^2 \smile \varphi_p^*[D_{0,1}]$ and $\varphi_p^*[D_{1,0}]^2 \smile \varphi_p^*[D_{0,1}]$ on Y'.

Since Y' is a family of curves of degree rd in X, $\varphi_p^*[D_{1,0}]^2$ is the class associated to a degree rd curve lying in $\{pt\} \times \mathbb{P}^2$. Hence, $\varphi_p^*[D_{1,0}]^2 \smile \varphi_p^*[D_{0,1}] = rd$.

The class $\varphi_p^*[D_{0,1}]^2$ is represented by the inverse image of $\mathbb{P}^2 \times \{pt\}$ under $\varphi_p : Y' \to Y$. A point in the second component of $\mathbb{P}^2 \times \mathbb{P}^2$ is associated via φ_p to a line $l \in (\mathbb{P}^2)^\vee$, and hence a point $(s_1, s_2) \in Y'$ will lie in the inverse image if $s_2 \in X \cap l$. Since $\deg X = d$, $X \cap l$ consists of d distinct points for general l. For each of these d points, s_1 must lie on a degree a curve in $\mathbb{P}^2$, and when we intersect these d curves of degree a with $\varphi_p^*[D_{1,0}]$, we get da points. Hence, Y is a hypersurface of bidegree (da, dr) in $\mathbb{P}^2 \times \mathbb{P}^2$.

□

It remains to prove condition (**), and to do so, we will work with the universal family of degree r curves in X lying in $\mathbb{P}H^0(X, \mathscr{O}_X(r)) \times X$. Since $r < d$, there exists an isomorphism

$$\phi_r : \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) \to \mathbb{P}H^0(X, \mathscr{O}_X(r)),$$

and hence we may replace $\mathbb{P}H^0(X, \mathcal{O}_X(r))$ with $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))$.

Let $\mathcal{Y}_r' \subset \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r)) \times \mathbb{P}^3$ be the universal family of hypersurfaces of degree r in $\mathbb{P}^3$, and let $\mathcal{Y}_r \subset \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r)) \times X$ be $\mathcal{Y}_r' \times_{\mathbb{P}^3} X$, which is the universal family of intersections of X with degree r hypersurfaces in $\mathbb{P}^3$. Let $\Pi : \mathcal{Y}_r \to \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))$ and $q : \mathcal{Y}_r \to X$ be the projection maps.

$$\begin{array}{ccc} \mathcal{Y}_r & \xrightarrow{q} & X \\ \Pi \big\downarrow & & \\ \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r)) & & \end{array}$$

Lemma 4.2.5. *$Y' \subset \mathbb{P}^2 \times X$ is isomorphic to $\mathbb{P}^2 \times_i \mathcal{Y}_r$ for a general choice of morphism $i : \mathbb{P}^2 \to \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))$ such that $i^*\mathcal{O}_{\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))}(1) \simeq \mathcal{O}_{\mathbb{P}^2}(a)$.*

Proof. Let $\mathbb{P}^2 \times \mathbb{P}^3$ have coordinates $\{y_0, \dots, y_2, x_0, \dots, x_3\}$. An element

$$Z \in |pr_1^*\mathcal{O}_{\mathbb{P}^2}(a) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^3}(r)|$$

will be the zero locus of a bihomogeneous polynomial of bidegree (a, r) of the form

$$\sum_{|I|=r} b_I(y_0, \dots, y_2) \cdot x^I.$$

Now let $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r)) \times \mathbb{P}^3$ have coordinates $\{\{a_I\}_{|I|=r}, x_0, \dots, x_3\}$. The universal family $\mathcal{Y}_r'$ is then the zero locus of the polynomial

$$\sum_{|I|=r} a_I \cdot x^I.$$

We then have a natural map $i : \mathbb{P}^2 \to \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))$ given by

$$[y_0 : y_1 : y_2] \mapsto [\dots : b_I(y_0, \dots, y_2) : \dots].$$

The *inverse image* via i of a hyperplane in $\mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r))$ will be the zero locus of a degree a curve in $\mathbb{P}^2$. Therefore $i^*\mathscr{O}_{\mathbb{P}H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(r))}(1) \simeq \mathscr{O}_{\mathbb{P}^2}(a)$. Also, note that Z is isomorphic to $\mathbb{P}^2 \times_i \mathscr{Y}_r'$. A general $Y' \subset \mathbb{P}^2 \times X$ is isomorphic to $Z \cap \mathbb{P}^2 \times X$, and hence $Y' \simeq \mathbb{P}^2 \times_i \mathscr{Y}_r$. □

We will now prove (**) by considering a general choice of

$$i : \mathbb{P}^2 \to \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)).$$

Define $\bar{\mathscr{Y}}_r \subset \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) \times (\mathbb{P}^2)^\vee$ to be the image of $\mathscr{Y}_r \subset \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) \times X$ under the map that is the identity on the first factor and projection from p on the second factor :

$$\begin{array}{ccc} \mathscr{Y}_r & \longrightarrow & \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) \times X \\ {\scriptstyle \xi}\downarrow & & \downarrow \\ \bar{\mathscr{Y}}_r & \xrightarrow{j} & \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) \times (\mathbb{P}^2)^\vee \end{array}$$

Let

$$\mathscr{X}_{r,s} := \{(z,l) \in \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) \times (\mathbb{P}^2)^\vee \mid \text{length}((\xi \circ j)^{-1}(z,l)) \geq s\},$$

and let $\pi_1 : \mathscr{X}_{r,s} \to \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r))$ and $\pi_2 : \mathscr{X}_{r,s} \to (\mathbb{P}^2)^\vee$ be projection maps.

$$\begin{array}{ccc} \mathscr{X}_{r,s} & \xrightarrow{\pi_2} & (\mathbb{P}^2)^\vee \\ {\scriptstyle \pi_1}\downarrow & & \\ \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) & & \end{array}$$

Define $Z_{r,s} := \pi_1(\mathscr{X}_{r,s})$, and define

$$O_{r,s} := \{y \in \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) : \dim(\pi_2 \circ \pi_1^{-1}(y) \geq 1)\}.$$

$Z_{r,s}$ is the locus of degree r hypersurfaces $Z \subset \mathbb{P}^3$ so that there exists an $l \in (\mathbb{P}^2)^\vee$ satisfying $length(l \cap (X \cap Z)) \geq s$. $O_{r,s}$ is the locus of degree r hypersurfaces $Z \subset \mathbb{P}^3$ with a 1-dimensional subvariety of lines $l \in (\mathbb{P}^2)^\vee$ satisfying $length(l \cap (X \cap Z)) \geq s$. If $z \in i(\mathbb{P}^2)$ also lies in $Z_{r,s}$, then φ_p will be $s:1$ on $\Pi^{-1}(z)$, and if $z \in i(\mathbb{P}^2)$ also lies in $O_{r,s}$, then φ_p will be $s:1$ for a 1-dimensional subvariety of $\Pi^{-1}(z)$.

Lemma 4.2.6. *If, for* $2 \leq s \leq 4$,

1. $codim_{\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))} O_{r,s} \geq s-1$, *and*

2. $codim_{\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))} Z_{r,s} \geq s-2$,

then condition $(**)$ *holds.*

Proof. Suppose $\text{codim}_{\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))} O_{r,s} \geq s-1$ and $\text{codim}_{\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))} Z_{r,s} \geq s-2$ for $2 \leq s \leq 4$. By Lemma 4.2.5, we may represent a general $Y' \subset \mathbb{P}^2 \times X$ by choosing a morphism $i : \mathbb{P}^2 \to \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))$ with $Y' \simeq \Pi^{-1}(i(\mathbb{P}^2))$. We choose i so that $\Pi^{-1}(i(\mathbb{P}^2))$ is smooth and so that $i(\mathbb{P}^2)$ intersects $Z_{r,s}$ and $O_{r,s}$ with the expected dimension for $2 \leq s \leq 4$.

Note that, given $z \in i(\mathbb{P}^2)$, φ_p is $s:1$ for at most a 1-dimensional subvariety of $\Pi^{-1}(z)$, since $\dim \Pi^{-1}(z) = 1$. The locuus of points on Y' for which φ_p is $s:1$ is determined by $(\mathbb{P}^2) \cap Z_{r,s}$ and $(\mathbb{P}^2) \cap O_{r,s}$. $\dim(\mathbb{P}^2) \cap O_{r,s} \leq 3-s$, and over each point of $(\mathbb{P}^2) \cap O_{r,s}$ is a 1-dimensional family for which φ_p is $s:1$. Therefore, there exists a subvariety $O_s \subset \Pi^{-1}((\mathbb{P}^2) \cap O_{r,s})$ of points for which φ_p is $s:1$, and $\dim O_s \leq 3-s+1 = 4-s$.

Similarly, $\dim(\mathbb{P}^2) \cap Z_{r,s} \leq 4-s$, and over a genral point of $(\mathbb{P}^2) \cap Z_{r,s}$ is a 0-dimensional family for which φ_p is $s:1$. Therefore, there exists a subvariety

$Z_s \subset \Pi^{-1}((\mathbb{P}^2) \cap Z_{r,s})$ of points for which φ_p is $s:1$, and $\dim Z_s \leq 4-s+0 = 4-s$.

The $s:1$ locus is contained in $Z_s \cup O_s$, which has dimension $4-s$. Hence, on Y', φ_p is generically 1:1, 2:1 on a divisor, 3:1 on a curve, and 4:1 or higher on points.

□

Let $l \in (\mathbb{P}^2)^\vee$, and let $P \subset \mathbb{P}^3$ be a plane containing p. If $\#(X \cap l) \geq s$, let $x_1, \ldots, x_{\#(X\cap l)}$ be the points of $X \cap l$, and let $Q_{s,l}$ be the set of subsets of $\{x_1, \ldots, x_{\#(X\cap l)}\}$ with cardinality s, and define

$$Z_{r,s,P,l} := \{Z \in \mathbb{P}H^0(P, \mathscr{O}_P(r)) : \exists T \in Q_{s,l} \text{ such that } T \subset Z\}.$$

Define

$$Z_{r,s,P} := \overline{\bigcup_{\{l\in(\mathbb{P}^2)^\vee : l\subset P \text{ and } \#(X\cap l)\geq s\}} Z_{r,s,P,l}}.$$

The proof of the proposition is complete if we can establish hypotheses 1. and 2. of Lemma 4.2.6, which we do in the following lemma :

Lemma 4.2.7. *Let $2 \leq s \leq 4$, let $l \in (\mathbb{P}^2)^\vee$ satisfy $\#(X \cap l) \geq s$, and let $P \subset \mathbb{P}^3$ be a plane containing p. Then,*

1. *$codim_{\mathbb{P}H^0(P,\mathscr{O}_P(r))} Z_{r,s,P,l} \geq s$.*
2. *$codim_{\mathbb{P}H^0(P,\mathscr{O}_P(r))} Z_{r,s,P} \geq s-1$.*
3. *$codim_{\mathbb{P}H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(r))} O_{r,s} \geq s-1$.*
4. *$codim_{\mathbb{P}H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(r))} Z_{r,s} \geq s-2$.*

Proof. Let us prove the first assertion. There are two cases to consider, namely when $s \leq r$ and when $r < s$. Suppose $s \leq r$. Then, for $l \in (\mathbb{P}^2)^\vee$ such that $l \subset P$,

$X \cap l$ consists of d points up to multiplicity. $Z_{r,s,P,l}$ is a finite union of at most $\binom{d}{s}$ subvarieties of $\mathbb{P}H^0(P, \mathscr{O}_P(r))$, and each subvariety is the collection of curves of degree r in P containing s points. To compute the dimension of $Z_{r,s,P,l}$, it suffices to compute the dimension of the collection of curves of degree r in P containing s points.

Suppose $s \leq r$. To prove the first part of the lemma, it suffices to show that the linear system of degree r curves in a plane containing r points has no unassigned base points. If $r = 1$, then the linear system of lines in a plane through a point has no unassigned base points. Now suppose that a linear system of degree k curves containing k points has no unassigned base points. Let $q_1, \ldots, q_{k+1}$ be $k+1$ points in the plane, and let x be another point distinct from the q_i. Then there is a degree k curve C_1 containing $q_1, \ldots, q_k$ but not x, and there is a line C_2 containing q_{k+1} but not x. $C_1 \cup C_2$ is a degree $k+1$ curve containing $q_1, \ldots, q_{k+1}$ but not x. The point x was arbitrary, and hence the linear system of $k+1$ curves in the plane containing $k+1$ points has no unassigned base points. It follows that $\operatorname{codim}_{\mathbb{P}H^0(P,\mathscr{O}_P(r))} Z_{r,s,P,l} \geq s$.

Now suppose that $s > r$. This occurs only if $s = 4, r = 3$. Note that a cubic in P containing 4 collinear points on a line l must contain l. If $\mathbb{V}(f)$ is a cubic containing l, then $F = F_l \cdot G$, where F_l is the degree 1 form such that $l = \mathbb{V}(F_l)$ and G is a homogeneous polynomial of degree 2. Now

$$\dim \mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^3}(r)) = \binom{n+r}{r} - 1,$$

and hence the collection of all cubics in P containing l is a subvariety of the space

$\mathbb{P}H^0(P, \mathscr{O}_P(3))$ whose codimension is equal to

$$\binom{3+2}{2} - \binom{2+2}{2} = 4.$$

In this case, $\mathrm{codim}_{\mathbb{P}H^0(P,\mathscr{O}_P(r))} Z_{r,s,P,l} \geq 4 = s$. This proves the first claim.

To prove the second claim, note that $Z_{r,s,P}$ is a 1-parameter family of varieties of codimension at least s. Therefore, $Z_{r,s,P}$ must have codimension at least $s-1$.

To prove the remaining two claims, consider the rational map

$$g_P : \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r)) \dashrightarrow \mathbb{P}H^0(P, \mathscr{O}_P(r))$$

given by restriction of a hypersurface to the plane P. Let $B_P \subset \mathbb{P}H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(r))$ be the subvariety for which g_P is not defined. First we note that the subvariety B_P has large codimension :

Lemma 4.2.8. *$\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(r))} B_P \geq 6$ for $r \geq 2$.*

Proof. B_P is comprised of hypersurfaces of degree r in $\mathbb{P}^3$ containing P. If $P = \mathbb{V}(f_P)$, then $x \in B_P$ if $x = \mathbb{V}(h \cdot f_P)$, where h is a homogeneous polynomial of degree $r-1$. Since

$$\dim \mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^3}(r)) = \binom{n+r}{r} - 1,$$

this implies that

$$\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(r))} B_P = \binom{r+3}{3} - \binom{r+2}{3} = \frac{1}{2}(r+2)(r+1)$$

If $r = 2$, then $\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(r))} B_P = 6$. Since

$$\frac{d}{dr}\left(\frac{1}{2}(r+2)(r+1)\right) = \frac{1}{2}(2r+3) > 0,$$

$\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(r))} B_P \geq 6$ for $r \geq 2$. $\square$

Now suppose $y \in O_{r,s}$. Then $\dim \pi_2 \circ \pi^{-1}(y) = 1$. Given a plane $P \subset \mathbb{P}^3$ containing p, let $L_P \subset (\mathbb{P}^2)^\vee$ be the locus of lines contained in P, and note that $\dim L_P = 1$. Then $(\pi_2 \circ \pi^{-1}(y)) \cap L_P \neq \emptyset$, and hence there exists a line $l \subset P$ through p such that $\#(l \cap \Pi^{-1}(y)) \geq s$ up to multiplicity. It follows that $y \in \overline{g_P^{-1}(Z_{r,s,P})} \cup B_P$. By Lemma 4.2.8, $\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(r))} B_P \geq 6$, and we already proved that $\mathrm{codim}_{\mathbb{P}H^0(P,\mathcal{O}_P(r))} Z_{r,s,P} \geq s-1$. Hence, $\overline{g_P^{-1}(Z_{r,s,P})} \cup B_P$ is a subvariety of $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))$ of codimension $\geq s-1$, and since $O_{r,s} \subset \overline{g_P^{-1}(Z_{r,s,P})} \cup B_P$, $\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(r))} O_{r,s} \geq s-1$. This proves the third part of the lemma.

It remains to show that $\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(r))} Z_{r,s} \geq s-2$. Let $y \in Z_{r,s}$. Choose a pencil W of hyperplanes in $\mathbb{P}^3$ such that every hyperplane in the family contains p. Then every line through p is contained in a hyperplane in the pencil. Therefore, a line $l \subset P$ through p with $\#(l \cap \Pi^{-1}(y)) \geq s$ must lie in one of the hyperplanes in the pencil. Hence,

$$Z_{r,s} \subseteq \bigcup_{P \subset W} \overline{g_P^{-1}(Z_{r,s,P})} \cup B_P.$$

We proved that $\mathrm{codim}_{\mathbb{P}H^0(P,\mathcal{O}_P(r))} Z_{r,s,P} \geq s-1$, and by Lemma 4.2.8,

$$\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(r))} B_P \geq 6.$$

Thus $\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(r))} \overline{g_P^{-1}(Z_{r,s,P})} \cup B_P \geq s-1$. Since $Z_{r,s}$ is contained in a 1-dimensional family of subvarieties of $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r))$ of codimension $s-1$,

$$\mathrm{codim}_{\mathbb{P}H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(r))} Z_{r,s} \geq s-2.$$

$\square$

□

Remark 4.2.9. *Using Corollary 2.5.7, we can get a result analogous to Proposition 4.2.1 by noting that a divisor of bidegree (a, r) is very ample when both a and r are strictly positive. If $X := \mathbb{P}^2 \times \mathbb{P}^2$ and L is a very ample divisor of bidegree (a, r), then, by Corollary 2.5.7, very general hypersurfaces $Y \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (da, dr) fail IHC_1 when IHC_1 fails for very general hypersurfaces of degree d in $\mathbb{P}^4$ and when d does not divide $L^4 = 6a^2r^2$.*

4.3 Unresolved Cases

One case left unresolved is the case where $a = 1$ and $d = 3$, or hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(3, 3r)$, where $r \geq 1$. Such hypersurfaces are not of general type, and it would be interesting to find a smooth, complex, connected, projective variety of dimension 3 not of general type that fails IHC_1. In the case where $r = 1$, we can show that the integral Hodge conjecture holds:

Lemma 4.3.1. *Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth hypersurface of bidegree $(3, 3)$. Then X satisfies IHC_1.*

Proof. By 2.5.6, it suffices to show that X satisfies IHC_1 when X is very general. A hypersurface $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(3, 3)$ has two elliptic fibrations $\pi_1 : X \longrightarrow \mathbb{P}^2$ and $\pi_2 : X \longrightarrow \mathbb{P}^2$ given by projection onto the components of $\mathbb{P}^2 \times \mathbb{P}^2$. The fibers of π_1 and π_2 are degree 3 curves in $\mathbb{P}^2$. Let $N := \binom{3+2}{2} - 1$, let $\mathbb{P}^N$ parametrize cubic curves in $\mathbb{P}^2$, and let $\mathscr{X} \subset \mathbb{P}^N \times \mathbb{P}^2$ be the universal hypersurface of cubics in $\mathbb{P}^2$. If $p : \mathscr{X} \longrightarrow \mathbb{P}^N$ is the projection map, then X is isomorphic to $p^{-1}(S)$, where $S \subset \mathbb{P}^N$ is isomorphic to $\mathbb{P}^2$ and the fibers of p are the fibers of π_1.

Let $\mathscr{L} \subset \mathbb{P}^N$ parametrize the family of cubics in $\mathbb{P}^2$ containing a line. Such cubics are isomorphic to $\mathbb{V}(F \cdot G)$, where F is a homogeneous linear form in 3 variables and G is a degree 2 form in 3 variables. The dimension of $\mathscr{L}$ is then equal to

$$\left(\binom{1+2}{2} - 1\right) + \left(\binom{2+2}{2} - 1\right) = 7,$$

since $\binom{i+2}{2} - 1$ is the dimension of the space of i forms in 3 variables modulo scalars. In particular, $\mathscr{L}$ is codimension 2 subvariety on $\mathbb{P}^N$, and hence $\mathscr{L}$ intersects S. This implies that there exists a fiber of π_1 containing a line. By symmetry, there also exists a fiber of π_2 containing a line.

Now let H be a line in $\mathbb{P}^2$. By above, there exists a curve C_1 such that $\pi_{1*}[C_1] \smile [H] = [C_1] \smile \pi_1^*[H] = 1$, and there exists a curve C_2 such that $\pi_{2*}[C_2] \smile [H] = [C_2] \smile \pi_2^*[H] = 1$. Proposition 4.1.4 implies then that X must satisfy IHC_1. $\square$

Appendix A

Algebraic Neighborhoods

Remark A.0.2. *All of this material is found in [8].*

Definition A.0.3. *An algebraic set in $\mathbb{P}^n$ is the zero locus of a finite set of homogeneous polynomials in $n+1$ variables.*

Definition A.0.4. *([8], Definition 2.1) Let M be an algebraic set in $\mathbb{P}^n$, and let X be an algebraic subset of M so that $M \setminus X$ is nonsingular. An (algebraic) rug function for X in M is a rational function $\alpha : M \to \mathbb{R} \cup \{\infty\}$ such that $\alpha(x) \geq 0$ for all $x \in M$ and $\alpha^{-1}(0) = X$.*

Remark A.0.5. *Two notable facts about rug functions are :*

1. *Any set X as above has a rug function.*
2. *A rug function has only finitely many critical values. This is used in the next definition.*

Definition A.0.6. *Let M and X be as above. A subset T with $X \subset T \subset M$ is an* *algebraic neighborhood of X in M if $T = \alpha^{-1}([0,\delta])$ for some rug function α and some positive real number δ that is less than all critical values of δ.*

Proposition A.0.7. *Let T be an algebraic neighborhood of X in M. Then the inclusion $i : X \hookrightarrow T$ is a homotopy equivalence.*

Remark A.0.8. *If X and M are smooth, then every algebraic neighborhood of X in M is a smooth tubular neighborhood of X in M (cf. [8], Corollary 1.9).*

Proposition A.0.9. *Let T_1 and T_2 be algebraic neighborhoods of X in M. Then there is a continuous family of homeomorphisms $h : I \times M \to M$ such that*

1. $h_0 = id_M$
2. $h_t|_X = id_X$ *for all* $t \in I$
3. $h_1(T_1) = T_2$, *and h_1 is a smooth diffeomorphism of $T_1 \setminus X$ onto $T_2 \setminus X$.*

Remark A.0.10. *For the following lemma, let $\pi : X \to B$ be a morphism between smooth, complex, connected varieties whose fibers are smooth, complex, connected, projective varieties of dimension d. We let $X_t := \pi^{-1}(t)$.*

Lemma A.0.11. *Let X_s be smooth. Then there exists an open neighborhood $U \subset B$ of s so that, for all $t \in U$, there is a homeomorphism $\sigma_{s,t} : X_s \to X_t$.*

Proof. Since B is smooth, there exists a rug function α for s in B by Remark A.0.5. Hence there is an algebraic neighborhood V of s. Moreover, $\alpha \circ \pi$ is also a rug function, and, after possibly shrinking V, $\pi^{-1}(V)$ is an algebraic neighborhood of

X_s. By Remark A.0.8, $\pi^{-1}(V)$ is a tubular neighborhood of X_s. Taking an open $U \subset V$, we obtain an open tubular neighborhood $\pi^{-1}(U)$ of X_s.

Since $\pi^{-1}(U)$ is a tubular neighborhood, there is a homeomorphism $\phi : \pi^{-1}(U) \to X_s \times U$ preserving fibers. Hence there is an induced homeomorphism $\sigma_{s,t}$. □

Appendix B

Relative Hilbert Scheme

Remark B.0.12. *Two references for the relative Hilbert scheme are VI.2. of [9] and Chapter I of [22].*

Let X and B be smooth, complex, connected, varieties with a surjective, projective morphism $\pi : X \to B$. Associated to each polynomial $P \in \mathbb{Z}[x]$ is a scheme $\mathscr{H}_P$ parametrizing all subvarieties of X that are supported on a fiber and have P as its Hilbert polynomial. There exists a natural map $\rho_P : \mathscr{H}_P \to B$ mapping a subvariety $Z \subset X_y$ to $y \in B$. By Theorem 1.4. of [22], ρ_P is projective. Define $\mathscr{H}_{X/B}$ to be the disjoint union of all $\mathscr{H}_P$ for all $P \in \mathbb{Z}[x]$. Since $\mathbb{Z}[x]$ is countable, $\mathscr{H}_{X/B}$ is a countable union of complex varieties.

Notation B.0.13. *([9], VI.2.) Let $\mathscr{H}_{X/B}$ be the relative Hilbert scheme of subvarieties of X supported on fibers of $\pi : X \to B$. Let $X_t := \pi^{-1}(t)$.*

Definition B.0.14. *Let $\mathscr{C} \subset \mathscr{H}_{X/B}$ be an irreducible component. There exists a flat, universal family $\chi_{\mathscr{C}} := \{(Z, x) \in \mathscr{C} \times X : x \in Z\}$ with a commutative diagram of maps*

$$
\begin{array}{ccc}
X_{\mathscr{C}} & \longrightarrow & X \\
{\scriptstyle \pi'_{\mathscr{C}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\mathscr{C} & \xrightarrow{\rho_{\mathscr{C}}} & B
\end{array}
$$

where $\rho_{\mathscr{C}}(Z) = y$ if Z is supported on X_y.

Definition B.0.15. *Let $Z \subset X_y$ be an effective l-cycle Z. The support of Z, denoted $S_Z \subset B$, is given by :*

$$\bigcup_{\{irreducible\ \mathscr{C} \subset \mathscr{C}_{l,X/Y} : Z \in \mathscr{C}\}} \rho_{\mathscr{C}}(\mathscr{C}).$$

Appendix C

Abelian Varieties

Definition C.0.16. *A complex torus of dimension g is the quotient V/Λ of a complex vector space V of dimension g with a lattice $\Lambda = Span\{\lambda_1, \ldots, \lambda_{2g}\}$.*

Definition C.0.17. *An isogeny between two complex tori X and X' is a surjective homomorphism with a finite kernel.*

Remark C.0.18. *Let $X = V/\Lambda$ be a complex torus.*

1. *Since the quotient map $\pi : V \to X$ is also a covering map, we obtain :*

$$\pi_1(X, 0) = H_1(X, \mathbb{Z}) = \Lambda. \tag{C.1}$$

2. *The Universal Coefficients Theorem then implies that :*

$$H^1(X, \mathbb{Z}) \simeq Hom(\Lambda, \mathbb{Z}). \tag{C.2}$$

3. *Finally, note that X is homeomorphic to the product of $2g$ S^1's, in which case the Kunneth formula implies :*

$$H^n(X, \mathbb{Z}) \simeq \bigwedge^n H^1(X, \mathbb{Z}) \simeq Alt^n Hom(\Lambda, \mathbb{Z}). \tag{C.3}$$

For (C.3), an important case to consider is $n = 2$, in which case elements of $H^2(X, \mathbb{Z})$ are given by wedges of integral forms on Λ. Note that, by tensoring with $\mathbb{R}$, elements of $Hom(\Lambda, \mathbb{Z})$ extend naturally to give elements of $Hom(V, \mathbb{R})$. Using this fact and the following proposition, we can describe those elements of $H^2(X, \mathbb{Z})$ that are the first Chern class of a line bundle.

Proposition C.0.19. *Let $X = V/\Lambda$ be a complex torus.*

1. *([23], Chap. 3, Prop. 1.6.) For an alternating form $E : V \times V \to \mathbb{R}$ the following are equivalent :*

   (a) *There exists a holomorphic line bundle L on X such that E represents the first Chern class $c_1(L)$.*

   (b) *$E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for all $v, w \in V$.*

2. *([23], Chap. 3, Lemma 1.7.) There is a 1-1 correspondence between the set of Hermitian forms H on V and the set of real-valued alternating forms E on V satisfying $E(iv, iw) = E(v, w)$, given by*

$$E(v, w) = Im\ H(v, w) \text{ and } H(v, w) = E(iv, w) + iE(v, w)$$

   for all $v, w \in V$.

Let $E : V \times V \to \mathbb{R}$ be an alternating form associated to a line bundle L. By Chapter 3, Section 1 of [23], there exists a basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ of Λ so that E is given by a matrix of the form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D = diag(d_1, \ldots, d_g)$ and $d_i \mid d_{i+1}$ for $i = 1, \ldots, g-1$. Moreover, the numbers $d_1, \ldots, d_g > 0$ are uniquely determined by E and Λ. The basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ is called a symplectic basis of Λ for L, and the ordered tuple $(d_1, \ldots, d_g)$ is called the type of L.

Given a complex torus $X = V/\Lambda$, let $\bar{\Omega} := Hom_{\bar{\mathbb{C}}}(V, \mathbb{C})$ be the $\mathbb{C}$-vector space of $\mathbb{C}$-antilinear forms on V. $\bar{\Omega}$ is canonically isomorphic to $Hom_{\mathbb{R}}(V, \mathbb{R})$. The isomorphism is $l \mapsto k = Im\ l$, and its inverse is $k \mapsto l(v) = -k(iv) + i \cdot k(v)$. The $\mathbb{R}$-linear pairing $\langle \cdot, \cdot \rangle : \bar{\Omega} \times V \to \mathbb{R}$ given by $\langle l, v \rangle = Im\ l(v)$ is non-degenerate, and thus the set $\hat{\Lambda} := \{l \in \bar{\Omega} \mid \langle l, \Lambda \rangle \subseteq \mathbb{Z}\}$ is a lattice. Consequently, we define $\hat{X} := \bar{\Omega}/\hat{\Lambda}$ to be the dual torus of X. The following proposition makes explicit the correspondence between the dual torus of X and line bundles on X.

Proposition C.0.20. *([23], Chap. 2, Section 4, Prop. 4.1.) There exists an isomorphism $\hat{X} \to Pic^0\, X$.*

Definition C.0.21. *([23], Chapter 2, Section 5) Let X be a complex torus. The Poincaré bundle for X, denoted by $\mathscr{P}$, is a holomorphic line bundle on $X \times \hat{X}$ satisfying :*

1. *$\mathscr{P}|X \times \{L\} \simeq L$ for every $L \in \hat{X}$.*

2. *$\mathscr{P}|\{0\} \times \hat{X}$ is trivial.*

Remark C.0.22. *By Theorem 5.1 in Chapter 2 of [23], Poincaré bundles exist and are unique up to isomorphism.*

Definition C.0.23. *An abelian variety (A, L) is a complex torus V/Λ with a line* *bundle L such that the Hermitian form H associated to L by Proposition C.0.19 is positive definite.*

Appendix D

Hodge Conjecture

Definition D.0.24. *Let X be a smooth, complex, connected, projective variety. The coniveau filtration, denoted $N^pH^l(X,\mathbb{Q})$, is defined as*

$$N^pH^l(X,\mathbb{Q}) := \sum_{codim\ Y \geq p} \ker(H^l(X,\mathbb{Q}) \to H^l(X \setminus Y, \mathbb{Q})).$$

Remark D.0.25. *$N^pH^l(X,\mathbb{Q})$ is a sub-Hodge structure of $H^l(X,\mathbb{Q})$, and*

$$N^pH^l(X,\mathbb{Q}) \subset F^pH^l(X,\mathbb{C}) \cap H^l(X,\mathbb{Q}).$$

It may be the case, though, that $F^pH^l(X,\mathbb{C}) \cap H^l(X,\mathbb{Q})$ is not a Hodge structure. For an example, see [14].

Definition D.0.26. *Let $F_h^pH^l(X,\mathbb{Q})$ be the maximal sub-Hodge structure of*

$$F^pH^l(X,\mathbb{C}) \cap H^l(X,\mathbb{Q}).$$

Conjecture D.0.27. *(General Hodge Conjecture) The General Hodge Conjecture, abbreviated by $GHC(p,l,X)$, says that*

$$N^pH^l(X,\mathbb{Q}) = F_h^pH^l(X,\mathbb{Q})$$

Remark D.0.28. *Note that $GHC(k, 2k, X)$ is equivalent to the original Hodge conjecture.*

Lemma D.0.29. *Let X be a smooth, complex, projective variety with $\dim X = n$. $GHC(p, l, X)$ is false iff there exists a class $[\alpha] \in F_h^p H^l(X, \mathbb{Q})$ and a class $[L] \in H^{2n-2k}(X, \mathbb{Q})$ such that*

1. $[L] \smile [\alpha] \neq 0$.
2. $[L] \smile N^p H^l(X, \mathbb{Q}) = 0$.

Proposition D.0.30. *Let $f : X \to Y$ be a surjective morphism between smooth, complex, projective varieties X, Y of dimension m, n, respectively. If $GHC(p, l, Y)$ is false, then $GHC(p + k, l + 2k, X)$ is false for $k = 0, 1, \ldots, m - n$.*

Proof. By Lemma D.0.29, there exists $[\alpha] \in F_h^p H^l(Y, \mathbb{Q})$ and $[L] \in H^{2n-2k}(Y, \mathbb{Q})$ so that $[L] \smile [\alpha] \neq 0$ and $[L] \smile N^p H^l(Y, \mathbb{Q}) = 0$. Without loss of generality, we may assume $[L] \smile [\alpha] = 1$.

By Proposition 1.1. of [2], N^p and F_h^p are preserved under f^*. Hence $f^*[\alpha] \in F_h^p H^l(X, \mathbb{Q})$. Let $[H] \in H^2(X, \mathbb{Q})$ be a class associated to a very ample divisor. Then, since $f^*[L] \smile f^*[\alpha]$ is the class corresponding to the fiber of a point, $f^*[L] \smile f^*[\alpha] \smile [H]^k \neq 0$ for $k = 0, 1, \ldots, m-n$. In particular, $f^*[\alpha] \smile [H]^k \neq 0$. Let $[\alpha_k] := f^*[\alpha] \smile [H]^k$, and let $[L_k] := f^*[L] \smile [H]^{m-n-k}$. Note that $[\alpha_k] \in F_h^{k+p} H^{2k+l}(X, \mathbb{Q})$, since $[H]$ corresponds to a hyperplane section. Then $[\alpha_k] \smile [L_k] \neq 0$. Also, if $[A] \in N^{k+p} H^{2k+l}(X, \mathbb{Q})$, then

$$[L_k] \smile [A] = f_*([L_k] \smile [A]) = [L] \smile f_*([H]^{m-n-k} \smile [A]).$$

by the projection formula, and since $f_*([H]^{m-n-k} \smile [A]) \in N^pH^l(Y,\mathbb{Q})$ by Proposition 1.1. of [2], $[L] \smile f_*([H]^{m-n-k} \smile [A]) = 0$. It follows that the classes $[\alpha_k]$ are counterexamples to $GHC(p+k, l+2k, X)$. □

Corollary D.0.31. *Let $f : X \to Y$ be a surjective morphism between smooth, complex, connected, projective varieties. If X satisfies the Hodge conjecture, then Y satisfies the Hodge conjecture.*

Bibliography

[1] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002), no. 3, 531–572 (electronic). MR MR1896232 (2003c:14016)

[2] Donu Arapura, *Hodge cycles on some moduli spaces.*

[3] M. F. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1962), 25–45. MR 26 #3091

[4] E. Ballico, F. Catanese, and C. Ciliberto (eds.), *Classification of irregular varieties*, Lecture Notes in Mathematics, vol. 1515, Berlin, Springer-Verlag, 1992, Minimal models and abelian varieties. MR MR1180332 (93d:14005)

[5] A. Beauville, *Quelques remarques sur la transformation de Fourier dans l'anneau de Chow d'une variété abélienne*, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 238–260. MR 86e:14002

[6] Arnaud Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 3, 309–391. MR 57 #12532

[7] Igor Dolgachev, *Weighted projective varieties*, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 34–71. MR 85g:14060

[8] Alan H. Durfee, *Neighborhoods of algebraic sets*, Trans. Amer. Math. Soc. **276** (1983), no. 2, 517–530. MR 84m:32013

[9] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000. MR 2001d:14002

[10] Hélène Esnault and Eckart Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR 94a:14017

[11] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 99d:14003

[12] Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic). MR 2003m:14081

[13] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994, Reprint of the 1978 original. MR 95d:14001

[14] A. Grothendieck, *Hodge's general conjecture is false for trivial reasons*, Topology **8** (1969), 299–303. MR 40 #5624

[15] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course, Corrected reprint of the 1992 original. MR MR1416564 (97e:14001)

[16] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[17] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 2002k:55001

[18] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR MR0199184 (33 #7333)

[19] W. V. D. Hodge, *The topological invariants of algebraic varieties*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1 (Providence, R. I.), Amer. Math. Soc., 1952, pp. 182–192. MR 13,679i

[20] V. A. Iskovskih, *Fano threefolds. I*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 3, 516–562, 717. MR 80c:14023a

[21] V. A. Iskovskih and V. V. Šokurov, *Biregular theory of Fano 3-folds*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, pp. 171–182. MR 81a:14014

[22] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 98c:14001

[23] Herbert Lange and Christina Birkenhake, *Complex abelian varieties*,

Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 1992. MR 94j:14001

[24] James D. Lewis, *A survey of the Hodge conjecture*, second ed., CRM Monograph Series, vol. 10, American Mathematical Society, Providence, RI, 1999, Appendix B by B. Brent Gordon. MR 2000a:14010

[25] Shigefumi Mori and Shigeru Mukai, *Classification of Fano 3-folds with* $B_2 \geq 2$, Manuscripta Math. **36** (1981/82), no. 2, 147–162. MR 83f:14032

[26] ______, *On Fano 3-folds with* $B_2 \geq 2$, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 101–129. MR 85b:14055

[27] Ziv Ran, *The (dimension +2)-secant lemma*, Invent. Math. **106** (1991), no. 1, 65–71. MR 92j:14055

[28] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR MR924157 (88k:00002)

[29] Igor R. Shafarevich, *Basic algebraic geometry. 1*, second ed., Springer-Verlag, Berlin, 1994, Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid. MR MR1328833 (95m:14001)

[30] C. Soule and C. Voisin, *Torsion cohomology classes and algebraic cycles on complex projective manifolds.*

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