VECTOR FIELDS ON SPHERES

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ABSTRACT. This paper presents a solution to the problem of finding the maximum number of linearly independent vector fields that can be placed on a sphere. To produce the correct upper bound, we make use of K-theory. After briefly recapitulating the basics of K-theory, we introduce Adams operations and compute the K-theory of the complex and real projective spaces. We then define the characteristic class ρ^k and develop some of its properties. Next, we recast the question of the upper bound into a question about fiber homotopy equivalent bundles over $\mathbb{R}P^n$, whose resolution reduces to a calculation in K-theory. Finally, we give the purely algebraic proof that this upper bound is realized.

CONTENTS

1. INTRODUCTION

Let S^n be the n-dimensional sphere. A vector field v on S^n is a continuous assignment $x \mapsto v(x)$ of a tangent vector $v(x)$ at x for every point $x \in S^n$. It is a consequence of the degree of a map between spheres being a homotopy invariant that a non-zero vector field on S^n exists if and only if n is odd (one uses the vector field to construct a homotopy between the identity and the antipodal map). This paper will be concerned with a far-reaching generalization of that result, namely, what is the maximum number of linearly independent vector fields one can put on a sphere?

To approach this problem, we will make use of a generalized cohomology theory called topological K-theory. Let X be a compact topological space and let $Vect_\Lambda(X)$ be the set of isomorphism classes of real or complex vector bundles over X, $\Lambda = \mathbb{R}$ or $\mathbb{C}$. $Vect_\Lambda(X)$ can be granted the structure of a commutative ring without negatives under direct sum and tensor product. Adjoining formal additive

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inverses via the Grothendieck group construction forms the ring $K_\Lambda(X)$. K_Λ is a contravariant functor from the category of compact spaces to the category of rings, with maps $f : X \to Y$ giving homomorphisms $f^* : K_\Lambda(Y) \to K_\Lambda(X)$ via pullback of bundles. Since our spaces are compact, homotopic maps induce isomorphic pullbacks of bundles[1] and K_Λ descends to a functor on the homotopy category of compact spaces.

Now suppose that our spaces are based and let $\tilde{K}_\Lambda(X) = ker(K_\Lambda(X) \to K_\Lambda(*))$ and $\tilde{K}_\Lambda^{-n} = \tilde{K}_\Lambda(\Sigma^n X)$ for $n > 0$, where Σ^n denotes n-fold reduced suspension. The Bott periodicity theorem allows us to extend $\tilde{K}_\Lambda^n$ to all n by defining $\tilde{K}_\Lambda^n(X) = \tilde{K}_\Lambda^{n'}(X)$, $n \equiv n' \bmod d$ and $-d < n' \leq 0$, where $d = 2$ if $\Lambda = \mathbb{C}$ and $d = 8$ if $\Lambda = \mathbb{R}$. One can check that the $\tilde{K}_\Lambda^n$ so defined satisfy the Eilenberg-Steenrod axioms and so give a reduced cohomology theory.

Remark 1.1. In fact, the $\tilde{K}_\Lambda^n$ constitute a sequence of represented functors. Suppose $\Lambda = \mathbb{C}$. Complex n-plane bundles are classified by the space $BU(n)$, in the sense that every complex n-plane bundle E over X may be realized as the pullback of a map $f : X \to BU(n)$. We have inclusions $i_n : BU(n) \to BU(n+1)$ for all n, and we may define $BU = \mathrm{colim}_{n\to\infty} BU(n)$. If X is compact and nondegenerately based, then $\tilde{K}_\mathbb{C}(X) = [X, BU \times \mathbb{Z}]$, where the brackets denote based homotopy classes of maps. By definition, $\tilde{K}_\mathbb{C}^{-n}(X) = [X, \Omega^n(BU \times \mathbb{Z})]$ for $n > 0$, and Bott periodicity is the statement that there is a homotopy equivalence $BU \times \mathbb{Z} \simeq \Omega^2(BU \times \mathbb{Z})$. We may then use the represented definition to extend the functors $\tilde{K}_\mathbb{C}^n(X)$ to all spaces of the homotopy type of CW-complexes. Details may be found in May [9, Ch. 24]. The story is similar for $\Lambda = \mathbb{R}$, with the classifying space BO in place of BU.

Following Karoubi [8], we proceed with the derivation of the upper bound on the number of linearly independent vector fields that can be placed on a sphere. All base spaces will be assumed to be compact and connected.

2. Adams operations

A cohomology operation in K-theory is a natural transformation from K_Λ to itself. The following theorem asserts the existence of certain operations ψ_Λ^k, termed Adams operations.

Theorem 2.1. *There exist natural ring homomorphisms $\psi_\Lambda^k : K_\Lambda(X) \longrightarrow K_\Lambda(X)$, defined for all integers k, which satisfy the following properties:*

- *ψ_Λ^1 and $\psi_\mathbb{R}^{-1}$ are the identity. $\psi_\mathbb{C}^{-1}$ is complex conjugation. ψ_Λ^0 assigns to a bundle over X the trivial bundle with fibers of the same dimension.*
- *If ξ is a line bundle, then $\psi_\Lambda^k(\xi) = \xi^k$.*
- *$\psi_\Lambda^k \psi_\Lambda^l = \psi_\Lambda^{kl}$.*
- *$\psi_\Lambda^p(x) = x^p \bmod p$ for any prime p.*
- *If $x \in \tilde{K}_\Lambda(S^{2n})$, then $\psi_\Lambda^k(x) = k^n x$.*

[1]To capture this property, it suffices to consider paracompact spaces.

- *Let β denote the periodicity isomorphism in K-theory. The following two diagrams commute.*

$$\begin{array}{ccc} \tilde{K}_{\mathbb{C}}(X) & \xrightarrow{\beta} & \tilde{K}_{\mathbb{C}}(\Sigma^2 X) \\ \downarrow{\scriptstyle \psi^k_{\mathbb{C}}} & & \downarrow{\scriptstyle \psi^k_{\mathbb{C}}} \\ \tilde{K}_{\mathbb{C}}(X) & \xrightarrow[k\beta]{} & \tilde{K}_{\mathbb{C}}(\Sigma^2 X) \end{array} \qquad \begin{array}{ccc} \tilde{K}_{\mathbb{R}}(X) & \xrightarrow{\beta} & \tilde{K}_{\mathbb{R}}(\Sigma^8 X) \\ \downarrow{\scriptstyle \psi^k_{\mathbb{R}}} & & \downarrow{\scriptstyle \psi^k_{\mathbb{R}}} \\ \tilde{K}_{\mathbb{R}}(X) & \xrightarrow[k^4\beta]{} & \tilde{K}_{\mathbb{R}}(\Sigma^8 X) \end{array}$$

- *Let $ch : K_{\mathbb{C}}(X) \to H^{even}(X;\mathbb{Q})$ be the Chern character and define $\psi^k_H(x) = k^r x$ for $x \in H^{2r}(X;\mathbb{Z})$. The following diagram commutes.*

$$\begin{array}{ccc} K_{\mathbb{C}}(X) & \xrightarrow{ch} & H^{even}(X;\mathbb{Q}) \\ \downarrow{\scriptstyle \psi^k_{\mathbb{C}}} & & \downarrow{\scriptstyle \psi^k_{H}} \\ K_{\mathbb{C}}(X) & \xrightarrow[ch]{} & H^{even}(X;\mathbb{Q}) \end{array}$$

- *Let $c : K_{\mathbb{R}}(X) \to K_{\mathbb{C}}(X)$ be given by complexification of bundles. The following diagram commutes.*

$$\begin{array}{ccc} K_{\mathbb{R}}(X) & \xrightarrow{c} & K_{\mathbb{C}}(X) \\ \downarrow{\scriptstyle \psi^k_{\mathbb{R}}} & & \downarrow{\scriptstyle \psi^k_{\mathbb{C}}} \\ K_{\mathbb{R}}(X) & \xrightarrow[c]{} & K_{\mathbb{C}}(X) \end{array}$$

The Adams operations derive from the exterior power operations λ^k, which we now construct. We must extend the usual exterior power construction on bundles to virtual bundles. For any bundle ξ over X let

$$\lambda(\xi) = 1 + \lambda(\xi)t + \lambda^2(\xi)t^2 + ... + \lambda^k(\xi)t^k + ... \in K_\Lambda(X)[[t]]^\times.$$

By the formula $\lambda^k(\xi \oplus \eta) = \bigoplus_{i+j=k} \lambda^i(\xi) \otimes \lambda^j(\eta)$, we have $\lambda(\xi \oplus \eta) = \lambda(\xi)\lambda(\eta)$. Hence λ extends to a homomorphism $\lambda : K_\Lambda(X) \to K_\Lambda(X)[[t]]^\times$. For $x \in K_\Lambda(X)$ define $\lambda^k(x)$ to be the kth coefficient in $\lambda(x)$.

Let σ_i be the ith symmetric polynomial and let $\pi_k = x_1^k + ... + x_n^k$ be the kth power sum. By the theory of symmetric polynomials, there exists a polynomial Q_k, independent of n for $n \geq k$, such that

$$\pi_k = Q_k(\sigma_1, ..., \sigma_k).$$

Now define $\psi^k_\Lambda(x) = Q_k(\lambda^1(x), ..., \lambda^k(x))$ for $k > 0$. Theorem 2.1 mandates the definition of ψ^0_Λ and ψ^{-1}_Λ, and by the relation $\psi^{-k}_\Lambda = \psi^{-1}_\Lambda \psi^k_\Lambda$ we define Adams operations for all integers k. The proof of Theorem 2.1 can be found in numerous sources, such as Adams [1].

3. K-theory of complex and real projective spaces

In this section we compute the complex K-theory of the complex and real projective spaces and the real K-theory of the real projective spaces. In the course of the computation we will make use of the Atiyah-Hirzebruch spectral sequence, whose definition and properties are given below.

Theorem 3.1. *Let X be a finite CW-complex and let X^p be its p-skeleton. Let $K^n_\Lambda(X)$ be filtered by the groups $K^n_{\Lambda,p}(X) = ker(K^n_\Lambda(X) \to K^n_\Lambda(X^{p-1}))$. There exists a multiplicative spectral sequence arising from this filtration that converges to $K_\Lambda(X)$, such that*

- $E_1^{p,q}(X) \cong C^p(X, K^q_\Lambda(*))$;
- $E_2^{p,q}(X) \cong H^p(X, K^q_\Lambda(*))$;
- $E_\infty^{p,q}(X) \cong G_p K_\Lambda^{p+q}(X) = K^{p+q}_{\Lambda,p}(X)/K^{p+q}_{\Lambda,p+1}(X)$.

Here $*$ denotes a point. The differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ shifts degree by $(r, -r+1)$. The multiplication on the E_2 page is given by the cup product in ordinary cohomology.

For a proof of most of this theorem, see Atiyah and Hirzebruch [5]. There the identification of the multiplication on the E_2 page is only asserted; a proof of this assertion may be found in Dugger [7].

By Bott periodicity, the groups $K^q_\Lambda(*)$ are periodic with period 2 for $\Lambda = \mathbb{C}$ and 8 for $\Lambda = \mathbb{R}$, and they are given as follows.

q	0	1	2	3	4	5	6	7
$K_\mathbb{C}^{-q}(*)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$K_\mathbb{R}^{-q}(*)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0

With these preliminaries in hand, we proceed to compute. Let η be the canonical complex line bundle over $\mathbb{C}P^{n-1}$ and ξ be the canonical real line bundle over $\mathbb{R}P^{2n-1}$.

Theorem 3.2. $K_\mathbb{C}(\mathbb{C}P^{n-1}) = \mathbb{Z}[t]/t^n$, *where the generator t is given by $\eta - 1$. The operation $\psi^k_\mathbb{C}$ is given by $\psi^k_\mathbb{C}((\eta-1)^s) = (\eta^k - 1)^s$.*

Proof. It is a theorem (Atiyah [3], Proposition 2.7.1, p. 102) that for any decomposable vector bundle $E = \Sigma L_i$ over X (the L_i being line bundles), $K_\mathbb{C}(P(E))$ is generated as a $K_\mathbb{C}(X)$-algebra by the tautological line bundle H subject to the single relation

$$\prod (H - L_i) = 0.$$

Apply this theorem to the case X a point, $E = \mathbb{C}^n$ to obtain the indicated description of $K_\mathbb{C}(\mathbb{C}P^{n-1})$. By Theorem 2.1, the operation $\psi^k_\mathbb{C}$ is a ring homomorphism and is the kth power map on line bundles. The given formula follows immediately. □

Let $\pi : \mathbb{R}P^{2n-1} \to \mathbb{C}P^{n-1}$ be the standard projection given by sending a real line to the complex line on which it lies. The next lemma relates η to ξ in terms of π^* and the complexification homomorphism c.

Lemma 3.3. $c\xi = \pi^*\eta$, *and this common element is non-trivial if $n > 1$.*

Proof. The case $n = 1$ being trivial, suppose $n > 1$. Complex line bundles are classified by their first Chern class c_1, and $H^2(\mathbb{R}P^{2n-1}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. It is a fact that π^* on cohomology is nonzero in degree two, so $c_1\pi^*\eta = \pi^* c_1 \eta \neq 0$. It therefore suffices to show that the bundle $c\xi$ is non-trivial. Letting $r : K_\mathbb{C}(X) \to K_\mathbb{R}(X)$ be the map defined by forgetting the complex structure, we have $rc = 2$. $rc\xi = \xi \oplus \xi$ has non-trivial Stiefel-Whitney classes and so is non-trivial, hence $c\xi$ is non-trivial. □

Let $\nu = c(\xi - 1) = \pi^*(\eta - 1) \in K_{\mathbb{C}}(\mathbb{R}P^{2n-1})$ and let $\nu = i^*\nu \in K_{\mathbb{C}}(\mathbb{R}P^{2n-2})$, $i : \mathbb{R}P^{2n-2} \to \mathbb{R}P^{2n-1}$ the inclusion.

Theorem 3.4. *Let f be the integer part of $\frac{1}{2}n$. Then $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n) = \mathbb{Z}/2^f\mathbb{Z}$ and is generated by ν subject to the two relations*

$$\nu^2 = -2\nu, \qquad \nu^{f+1} = 0.$$

The operation $\psi^k_{\mathbb{C}}$ is given by $\psi^k_{\mathbb{C}}(\nu^s) = \begin{cases} 0 & k \text{ even} \\ \nu^s & k \text{ odd} \end{cases}.$

Proof. The case $n = 1$ being trivial, suppose $n > 1$. To prove that the two relations hold, by naturality it suffices to consider n odd. The relation $\nu^2 = -2\nu$ is equivalent to $(1 + \nu)^2 = (c\xi)^2 = 1$, so it suffices to prove that $\xi^2 = 1$. But real line bundles are classified by their first Stiefel-Whitney class and $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, so either $\xi^2 = 1$ or $\xi^2 = \xi$. Since all line bundles are invertible, the second possibility would imply that $\xi = 1$, a contradiction.

The relation $\nu^{f+1} = 0$ follows from the relation $(\eta - 1)^{f+1} = 0$ in $\mathbb{C}P^f$ and naturality. Note as well that by Lemma 3.2, $\nu \neq 0$ for the odd case, and since $c_1 i^* \pi^*(\eta)$ is non-zero, $\nu \neq 0$ for the even case as well.

The spectral sequence in complex K-theory for $X = \mathbb{R}P^n$, n even has $E_2^{p,q}$ term $H^p(X; K^q_{\mathbb{C}}(*))$ equal to $\mathbb{Z}/2\mathbb{Z}$ for q and p even such that $0 < p \leq n$, equal to $\mathbb{Z}$ for q even and $p = 0$, and equal to 0 otherwise, while for n odd the spectral sequence has in addition non-zero $E_2^{p,q}$ terms equal to $\mathbb{Z}$ for q even and $p = n$. As for any space, the non-zero terms on the $p = 0$ column are permanent cycles[2], as may be shown by considering the map of spectral sequences induced by inclusion of the basepoint. For the other terms, the possible $p = n$ column of $\mathbb{Z}$'s are cycles on every page, and any map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$ is trivial, hence those possible $\mathbb{Z}$ terms are also permanent cycles. Thus any $\mathbb{Z}/2\mathbb{Z}$ term can only map non-trivially to another $\mathbb{Z}/2\mathbb{Z}$. But the differentials on each page of the spectral sequence shift the parity of the total degree of the E_r term, hence the spectral sequence is trivial and the associated graded algebra $E_\infty^{p,-p}(X) \cong G_p K_{\mathbb{C}}(X)$ is given by the f copies of $\mathbb{Z}/2\mathbb{Z}$ on the E_2 page. By the commutative diagram

$$\begin{array}{ccc} \mathbb{R}P^3 & \xrightarrow{\pi} & \mathbb{C}P^1 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \mathbb{R}P^{2n-1} & \xrightarrow[\pi]{} & \mathbb{C}P^n \end{array}$$

the element ν generates the $E_2^{2,-2}$ term, hence its powers ν^i generate the successive $E_2^{2i,-2i}$ terms since the multiplication on the E_2 page is given by cup product. Then the group extensions are all of the form

$$0 \to \mathbb{Z}/2^j\mathbb{Z} \to \mathbb{Z}/2^{j+1}\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

as may be shown by use of the relation $\nu^{i+1} = -2\nu^i$ and induction. This completes the description of $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$. It remains to calculate the Adams operations on

[2]A permanent cycle is a class that is both a cycle and not a boundary on every page of the spectral sequence.

$\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$. We showed above that $(\pi^*\eta)^2 = 1$. Since $\nu = \pi^*\eta - 1$, this implies that $\psi^k_{\mathbb{C}}(\nu^s) = \begin{cases} 0 & k \text{ even} \\ \nu^s & k \text{ odd} \end{cases}$. □

Theorem 3.5. *Let f be the number of integers i such that $0 < i \leq n$ and $i \equiv 0$, 1, 2 or 4 mod 8. Then $\tilde{K}_{\mathbb{R}}(\mathbb{R}P^n) = \mathbb{Z}/2^f\mathbb{Z}$ and is generated by $\lambda = \xi - 1$ subject to the two relations*

$$\lambda^2 = -2\lambda, \qquad \lambda^{f+1} = 0.$$

The operation $\psi^k_{\mathbb{R}}$ is given by $\psi^k_{\mathbb{R}}(\lambda^s) = \begin{cases} 0 & k \text{ even} \\ \lambda^s & k \text{ odd} \end{cases}$.

Proof. We first examine the spectral sequence in real K-theory for $X = \mathbb{R}P^n$. The group $\tilde{H}^p(X;\mathbb{Z}/2\mathbb{Z})$ is $\mathbb{Z}/2\mathbb{Z}$ for $0 < p \leq n$ and 0 otherwise, while $\tilde{H}^p(X;\mathbb{Z})$ is $\mathbb{Z}/2\mathbb{Z}$ for p even, $0 < p \leq n$ and 0 otherwise with the exception of $\tilde{H}^n(X;\mathbb{Z}) = \mathbb{Z}$ if n is odd. Real Bott periodicity goes '$\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$' starting at $q = -1$ for $K^q_{\mathbb{R}}(*)$ and going downwards. Thus, on the E_2 page the non-zero terms of total degree 0 apart from the term at $(0, 0)$ consist of f copies of $\mathbb{Z}/2\mathbb{Z}$. It follows that there are at most 2^f elements in the group $\tilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$.

Now consider the complexification homomorphism $c : \tilde{K}_{\mathbb{R}}(\mathbb{R}P^n) \to \tilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$. Since $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$ is generated by ν and $\nu = c\lambda$, c is an epimorphism for all n. Additionally for $n \equiv 6$, 7, or 8 mod 8, by Theorem 3.4 $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$ contains 2^f elements, so c is an isomorphism. In detail, if $n - 8t = 6$ or 7, then $f = 4t+3$ and $\lfloor \frac{1}{2}n \rfloor = 4t+3$; if $n = 8t$, then $f = 4t$ and $\lfloor \frac{1}{2}n \rfloor = 4t$. Thus, all the non-zero E_2 terms of total degree 0 persist to the E_∞ page in those cases. However, the inclusion $i : \mathbb{R}P^n \to \mathbb{R}P^m$ induces a map of spectral sequences for all $n \leq m$, so in fact the same conclusion holds for all n. Thus $\tilde{K}_{\mathbb{R}}(\mathbb{R}P^n) = \mathbb{Z}/2^f\mathbb{Z}$ is cyclic of order 2^f with generator λ. We showed the relation $\lambda^2 = -2\lambda$ in the proof of Theorem 3.4, and the relation $\lambda^{f+1} = 0$ follows from the fact that $2^f\lambda = 0$.

The calculation of the Adams operations is the same as in the proof of Theorem 3.4. □

4. The characteristic class ρ^k

In order to define the characteristic class ρ^k, we first recall the Thom isomorphism theorem in K-theory (an illuminating treatment is given in Atiyah, Bott, and Shapiro [4]). Given a vector bundle E over X, define the Thom complex $T(E)$ of E by taking the one-point compactification of each fiber E_x and then identifying together all the points at infinity. Equivalently, we may choose a metric on E and form the unit disc bundle $D(E)$ and the unit sphere bundle $S(E)$; then $T(E) = D(E)/S(E)$. $\tilde{K}_\Lambda(T(E)) = K_\Lambda(D(E), S(E))$ is a $K_\Lambda(X)$-algebra by way of the projection $\pi : D(E) \to X$ and multiplication $K_\Lambda(D(E)) \otimes K_\Lambda(D(E), S(E)) \to K_\Lambda(D(E), S(E))$. In complex K-theory, there exists a natural isomorphism $\phi : K_{\mathbb{C}}(X) \to \tilde{K}_{\mathbb{C}}(T(E))$ defined by $x \mapsto \lambda_E x$, where $\lambda_E \in K_{\mathbb{C}}(T(E))$ is a distinguished element, termed the Thom element. In real K-theory the same isomorphism exists, but only for E a $Spin(8n)$-bundle.

We observe that λ_E enjoys the following compatibility property with respect to direct sums of bundles. Let E, E' be vector bundles over X, X' respectively, and form their external direct sum $E \times F$ over $X \times X'$. Then $\lambda_{E\times F} = \lambda_E \lambda_F$, where this product is from $\tilde{K}(T(E)) \times \tilde{K}(T(F))$ to $\tilde{K}(T(E \times F))$. By naturality, if $X = X'$ then this holds for $E \oplus E'$ as well.

We define $\rho^k_\Lambda : Vect_\Lambda(X) \to K_\Lambda(X)$ by $\rho^k_\Lambda(E) = \phi^{-1}\psi^k_\Lambda(\lambda_E)$, implicitly restricting the domain of definition to $Spin(8n)$-bundles for $\Lambda = \mathbb{R}$. It is immediate from the definition that ρ^k_Λ is natural and $\phi^{-1}\psi^k\phi(x) = \psi^k_\Lambda(x)\rho^k_\Lambda(E)$. By the multiplicative property of λ_E listed above,

$$\rho^k_\Lambda(E \oplus E') = \rho^k_\Lambda(E)\rho^k_\Lambda(E'). \tag{4.1}$$

We say that ρ^k_Λ is exponential. We make a first step towards calculating ρ^k_Λ with the following proposition.

Proposition 4.2. *$\rho^k_{\mathbb{C}}(L) = 1 + L + ... + L^{k-1}$ for L a line bundle. In particular, $\rho^k_{\mathbb{C}}(n) = k^n$.*

Proof. Since the space $\mathbb{C}P^\infty$ classifies complex line bundles, by naturality it suffices to determine $\rho^k_{\mathbb{C}}$ on the canonical line bundle η over $\mathbb{C}P^n$. We claim that the Thom complex $T(\eta)$ may be identified with $\mathbb{C}P^{n+1}$, with the $K_{\mathbb{C}}(\mathbb{C}P^n) = \mathbb{Z}[t]/t^{n+1}$-module structure given by the usual multiplication in the ideal $\tilde{K}_{\mathbb{C}}(\mathbb{C}P^{n+1}) = (t) \subset \mathbb{Z}[t]/t^{n+2}$. Let η be explicitly given as $p : E = S^{2n+1} \times_{U(1)} \mathbb{C} \to \mathbb{C}P^n$, where S^{2n+1} is the complex unit sphere in $\mathbb{C}^{n+1}$. Using homogeneous coordinates for complex projective space and choosing $[0 : \dots : 0 : 1]$ as the basepoint for $\mathbb{C}P^{n+1}$, we have a homeomorphism of based spaces from $T(\eta) = E \cup \{\infty\}$ to $\mathbb{C}P^{n+1}$ defined by $(x_0, \dots, x_n, \lambda) \mapsto [x_0 : \dots : x_n : \bar{\lambda}]$ and $\infty \mapsto [0 : \dots : 0 : 1]$. Note that the inclusion of $\mathbb{C}P^n$ into E via the zero section corresponds under this homeomorphism to the inclusion $i : \mathbb{C}P^n \to \mathbb{C}P^{n+1}$ defined by $[x_0 : \dots : x_n] \mapsto [x_0 : \dots : x_n : 0]$.

To identify the Thom multiplication, we use commutativity of the following diagram.

$$\begin{array}{ccccc}
\tilde{K}(\mathbb{C}P^n_+) \otimes \tilde{K}(\mathbb{C}P^{n+1}) & \longrightarrow & \tilde{K}(\mathbb{C}P^n_+ \wedge \mathbb{C}P^{n+1}) & \xrightarrow{\Delta^*} & \tilde{K}(\mathbb{C}P^{n+1}) \\
\uparrow{\scriptstyle i^*\otimes id} & & \uparrow{\scriptstyle (i\wedge id)^*} & \nearrow{\scriptstyle d^*} & \\
\tilde{K}(\mathbb{C}P^{n+1}_+) \otimes \tilde{K}(\mathbb{C}P^{n+1}) & \longrightarrow & \tilde{K}(\mathbb{C}P^{n+1}_+ \wedge \mathbb{C}P^{n+1}) & &
\end{array}$$

Here the upper row is the Thom multiplication with the Thom diagonal Δ given by $[x_0 : \dots : x_{n+1}] \mapsto [x_0 : \dots : x_n] \wedge [x_0 : \dots : x_{n+1}]$, and the map d is the usual diagonal (all maps being based). One must only check commutativity of the right triangle, which holds since $(i \wedge id)\Delta$ is homotopic to d via $d_t : [x_0 : \dots : x_{n+1}] \mapsto [x_0 : \dots : x_n : tx_{n+1}] \wedge [x_0 : \dots : x_{n+1}]$. Now let η' be the canonical line bundle over $\mathbb{C}P^{n+1}$ and observe that $i^*(\eta') = \eta$. Thus by the diagram, $\Delta^*(\eta \otimes (\eta' - 1)) = d^*(\eta' \otimes (\eta' - 1)) = \eta'(\eta' - 1)$, proving the claim.

Since the Thom element must be a generator of $\tilde{K}_{\mathbb{C}}(\mathbb{C}P^{n+1})$, it must be equal to $\pm(\eta' - 1)$. Therefore,

$$\rho^k_{\mathbb{C}}(\eta) = \phi^{-1}\psi^k_{\mathbb{C}}(\pm(\eta' - 1)) = (\eta^k - 1)/(\eta - 1) = 1 + \eta + ... + \eta^{k-1}.$$

∎

The complexification homomorphism furnishes a relation between $\rho^k_{\mathbb{R}}$ and $\rho^k_{\mathbb{C}}$.

Proposition 4.3. *Let E be an oriented real vector bundle of rank $4n$. Then $F = E \oplus E$ may be thought of as both the complex bundle $F_{\mathbb{C}} = cE$ and the spin bundle $F_{\mathbb{R}} = rcE$, and we have the relation $\rho^k_{\mathbb{C}}(F_{\mathbb{C}}) = c\rho^k_{\mathbb{R}}(F_{\mathbb{R}})$. In particular, if E is a spin bundle of rank $8n$, then $\rho^k_{\mathbb{C}}(F_{\mathbb{C}}) = c(\rho^k_{\mathbb{R}}(E)^2)$, and if E is the trivial bundle $4n$, then $\rho^k_{\mathbb{R}}(8n) = k^{4n}$.*

Proof. The proposition is a consequence of the commutativity of both the Thom isomorphism and the Adams operations with complexification. Precisely, we have the commutative diagram

$$\begin{array}{ccc} K_{\mathbb{R}}(X) & \xrightarrow{\phi} & \tilde{K}_{\mathbb{R}}(T(F_{\mathbb{R}})) \\ \downarrow c & & \downarrow c \\ K_{\mathbb{C}}(X) & \xrightarrow{\phi} & \tilde{K}_{\mathbb{C}}(T(F_{\mathbb{C}})) \end{array}$$

together with the relevant diagram for ψ^k_{Λ} in Theorem 2.1. See Karoubi [8, Proposition 7.27, p. 261] for the proof of the commutativity of the first diagram. □

We next wish to extend ρ^k_{Λ} to an operation on all of $K_{\Lambda}(X)$ in such a way as to preserve the exponentiation property (4.1). To accomplish this we must invert k in the ring $K_{\Lambda}(X)$. Let $Q_k = \mathbb{Z}[\frac{1}{k}]$ be the subring of $\mathbb{Q}$ consisting of fractions with denominator a power of k. Now specialize to $\Lambda = \mathbb{C}$. Elements of $K_{\mathbb{C}}(X)$ may be written in the form $E - n$ with E an actual bundle. Define $\rho^k_{\mathbb{C}} : K_{\mathbb{C}}(X) \to K_{\mathbb{C}}(X) \otimes Q_k$ by $\rho^k_{\mathbb{C}}(E - n) = \rho^k_{\mathbb{C}}(E)/k^n$. To see that $\rho^k_{\mathbb{C}}$ is well-defined, suppose $E - n = F - m$. Then $E + m = F + n$ implies that

$$\rho^k_{\mathbb{C}}(E + m) = \rho^k_{\mathbb{C}}(E)\rho^k_{\mathbb{C}}(m) = \rho^k_{\mathbb{C}}(F)\rho^k_{\mathbb{C}}(n) = \rho^k_{\mathbb{C}}(F + m)$$

and since by Proposition 4.2 $\rho^k_{\mathbb{C}}(n) = k^n$, the claim follows. For $\Lambda = Spin(8n)$, the extension of ρ^k_{Λ} is a more delicate problem; the interested reader is referred to Adams [2].

The next two propositions make some explicit calculations of ρ^k_{Λ} on $\mathbb{R}P^n$. Recall that in our notation ξ is the canonical line bundle over $\mathbb{R}P^n$.

Proposition 4.4. *Let $\nu = c\xi - 1$ be the generator of $\tilde{K}_{\mathbb{C}}(\mathbb{R}P^n)$ as described in Theorem 3.4. For k odd, the operation $\rho^k_{\mathbb{C}} : K_{\mathbb{C}}(\mathbb{R}P^n) \to K_{\mathbb{C}}(\mathbb{R}P^n) \otimes Q_k$ is given by $\rho^k_{\mathbb{C}}(l\nu) = 1 + \frac{k^l - 1}{2k^l}\nu$.*

Proof. In the proof of Theorem 3.4 we showed that $c\xi^2 = 1$. By Proposition 4.2, $\rho^k_{\mathbb{C}}(c\xi) = 1 + c\xi + ... + (c\xi)^{k-1} = \frac{k+1}{2} + \frac{k-1}{2}c\xi$, so $\rho^k_{\mathbb{C}}(\nu) = \frac{1}{k}\rho^k_{\mathbb{C}}(c\xi) = 1 + \frac{k-1}{2k}\nu$. The exponential property of $\rho^k_{\mathbb{C}}$ along with the relation $\nu^2 = -2\nu$ establishes by induction on l that $\rho^k_{\mathbb{C}}(l\nu) = 1 + \frac{k^l - 1}{2k^l}\nu$. □

Proposition 4.5. *For k odd, $\rho^k_{\mathbb{R}}(4l\xi \oplus 4l) = k^{4l}\left(1 + \frac{k^{2l} - 1}{2k^{2l}}\lambda\right)$, where $\lambda = \xi - 1$.*

Proof. The homomorphism $K_{\mathbb{R}}(\mathbb{R}P^m) \to K_{\mathbb{R}}(\mathbb{R}P^n)$ induced by inclusion for $n \leq m$ is surjective, so by naturality it suffices to consider $n \equiv 0 \bmod 8$. Since $2\xi \oplus 2$ is an oriented real bundle of rank 4, by Proposition 4.3 we have $c\rho^k_{\mathbb{R}}(4l\xi \oplus 4l) = \rho^k_{\mathbb{C}}(2lc\xi + 2l)$. Thus by Proposition 4.4,

$$c\rho^k_{\mathbb{R}}(4l\xi \oplus 4l) = \rho^k_{\mathbb{C}}(2l\nu + 4l) = k^{4l}\left(1 + \frac{k^{2l}-1}{2k^{2l}}\nu\right).$$

In the proof of Theorem 4.4 we showed that if $n \equiv 0 \bmod 8$, then c is an isomorphism with $c\lambda = \nu$. Applying c^{-1}, the desired formula follows. □

To conclude this section, we turn to an analysis of fiber homotopy equivalent bundles. Two vector bundles E and E' over a common base space X are said to be *fiber homotopy equivalent* if there exists a map of bundles $\theta : E \to E'$, such that the map $\theta : S(E) \to S(E')$ is a homotopy equivalence over X (that is, the homotopies in question are through maps that send fibers to fibers). Necessarily this implies that for each point $p \in X$, the map $\theta_p : S(E)_p \to S(E')_p$ is a homotopy equivalence. Call E and E' *fiberwise homotopy equivalent* if they are equivalent in this weaker sense. It is a theorem of Dold [6] that the converse implication holds (for X any CW-complex). We will develop the theory below for fiberwise homotopy equivalent bundles and only use Dold's result in section 5 to obtain agreement with how the theorems are presented in the literature.

Since for every point $p \in X$ $T(E_p)$ is the suspension of $S(E)_p$, it follows that the map θ giving the fibrewise homotopy equivalence yields a map $\tilde{\theta} : T(E) \to T(E')$ whose restriction $T(E_p) \to T(E'_p)$ for each $p \in X$ is a homotopy equivalence, and we have the commutative diagram

$$\begin{array}{ccc} \tilde{K}_{\mathbb{R}}(T(E')) & \xrightarrow{\tilde{\theta}^*} & \tilde{K}_{\mathbb{R}}(T(E)) \\ \downarrow & & \downarrow \\ \tilde{K}_{\mathbb{R}}(T(E'_p)) & \xrightarrow[\cong]{\tilde{\theta}^*_p} & \tilde{K}_{\mathbb{R}}(T(E_p)). \end{array}$$

Suppose now that E and E' are real vector bundles possessed of a spin structure with rank divisible by 8. Then $\tilde{\theta}^* : \tilde{K}_{\mathbb{R}}(T(E')) \to \tilde{K}_{\mathbb{R}}(T(E))$ maps $\lambda_{E'}$ to $x\lambda_E$ for some $x \in K_{\mathbb{R}}(X)$. It is the characterizing property of a Thom element λ_E that λ_E restricts to a generator of $\tilde{K}_{\mathbb{R}}(T(E_p))$ for each point $p \in X$. Hence by restriction to all points $p \in X$, we see that $x\lambda_E$ may serve as a Thom element for E. Consequently, x is invertible, and we may write $x = \pm(1+y)$ for $y \in \tilde{K}_{\mathbb{R}}(X)$. We then have the following proposition relating $\rho^k_{\mathbb{R}}(E)$ to $\rho^k_{\mathbb{R}}(E')$.

Proposition 4.6. *Suppose E and E' are fiberwise homotopy equivalent spin bundles with rank divisible by 8. Then there exists an element $y \in \tilde{K}_{\mathbb{R}}(X)$ such that for each k we have the relation*

$$\rho^k_{\mathbb{R}}(E) = \frac{\psi^k_{\mathbb{R}}(1+y)}{1+y}\rho^k_{\mathbb{R}}(E').$$

Proof. In the setup of the preceding discussion, the proof is the computation

$$\rho^k_{\mathbb{R}}(E) = \phi^{-1}\psi^k_{\mathbb{R}}(\lambda_E) = \frac{\psi^k_{\mathbb{R}}(\lambda_E)}{\lambda_E} = \frac{\psi^k_{\mathbb{R}}(x)\psi^k_{\mathbb{R}}(\lambda'_E)}{x\lambda'_E} = \frac{\psi^k_{\mathbb{R}}(1+y)}{1+y}\rho^k_{\mathbb{R}}(E').$$

□

Corollary 4.7. *Let E be a spin bundle of rank $8l$ over $\mathbb{R}P^{n-1}$, such that the bundles $8l$ and E are fiberwise homotopy equivalent. Then $\rho^k_{\mathbb{R}}(E) = k^{4l}$ if k is odd.*

Proof. For any $y \in \tilde{K}_{\mathbb{R}}(\mathbb{R}P^n)$, by Theorem 3.5 $\psi^k_{\mathbb{R}}(y) = y$ for k odd. Hence by Propositions 4.6 and 4.3, $\rho^k_{\mathbb{R}}(E) = \rho^k_{\mathbb{R}}(8l) = k^{4l}$. □

5. The upper bound

Let $O_{n,m}$ denote the Stiefel variety of n-tuples of orthonormal vectors in $\mathbb{R}^m$, $n \leq m$; $O_{n,m}$ may be identified with $O(m)/O(m-n)$. The Stiefel fibering $\pi : O_{n,m} \to S^{m-1}$ is defined by sending a n-tuple $\omega = (w_1, ..., w_n)$ to its first vector w_1. By the Gram-Schmidt orthonormalization process, the existence of $n-1$ linearly independent vector fields on S^{m-1} is equivalent to the existence of a section $s : S^{m-1} \to O_{n,m}$.

We reduce the question of existence of a section s to a question about fiber homotopy equivalent bundles over $\mathbb{R}P^{n-1}$ in the following way. Define a map $\phi : O_{n,m} \times \mathbb{R}^n \to \mathbb{R}^m$ by $(\omega, v) \to v_1 w_1 + ... + v_n w_n$. Now suppose that a section s exists and define a map $\theta : S^{n-1} \times S^{m-1} \to S^{n-1} \times S^{m-1}$ by $\theta(v, b) = (v, \phi(s(b), v))$. Observe that for any $v \in S^{n-1}$, $\theta_v : b \to \phi(s(b), v)$ is homotopic to θ_{e_1} where e_1 is the first basis vector in $\mathbb{R}^n$, by path-connectedness of S^{n-1}. But by definition $\theta_{e_1} = \mathrm{id}$, hence each θ_v is a homotopy equivalence from S^{m-1} to itself.

Under the antipodal $\mathbb{Z}/2\mathbb{Z}$ action we see that θ descends to give a map $\theta : \mathbb{R}P^{n-1} \times S^{m-1} \to (S^{n-1} \times S^{m-1})/(\mathbb{Z}/2\mathbb{Z})$. These spaces may be identified with the sphere bundles $S(m)$ and $S(m\xi)$ over $\mathbb{R}P^{n-1}$, respectively, and then θ is a fiber homotopy equivalence. Moreover, by radial extension θ may be extended to a map from the trivial bundle m to the bundle $m\xi$. We thus have the following proposition.

Proposition 5.1. *Suppose that S^{m-1} admits $n-1$ linearly independent vector fields. Then there is a map of vector bundles $\theta : m \to m\xi$ over $\mathbb{R}P^{n-1}$, which is a fiber homotopy equivalence.*

We now apply Proposition 4.5 and Corollary 4.7 to prove an upper bound for the number of linearly independent vector fields that can be placed on a sphere.

Theorem 5.2. *Let a_n be the order of the group $\tilde{K}_{\mathbb{R}}(\mathbb{R}P^{n-1})$, so that by Theorem 3.5, $a_n = 2^f$ where f is the number of integers i such that $0 < i < n$ and $i \equiv 0$, 1, 2 or 4 mod 8. Then S^{m-1} admits $n-1$ linearly independent vector fields only if m is a multiple of a_n.*

Proof. By Proposition 5.1 it suffices to show that m and $m\xi$ are fiber homotopy equivalent only if m is a multiple of a_n. It is a fact that fiber homotopy equivalent bundles have the same Stiefel-Whitney classes; this is immediate from the definition of the Stiefel-Whitney classes in terms of the Thom isomorphism and the total Steenrod squaring operation ([10]). Let $x \in H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z})$ be the generator of $H^*(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z})$, so that $1 + x$ is the total Stiefel-Whitney class of ξ. Then we require $(1+x)^m \equiv 1 \bmod 2$. Observe that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \dots .$$

Thus if $n = 2$ then m must be a multiple of $2 = a_2$, and if $n > 2$ then m must be a multiple of 4. Let $m = 4l$. By Corollary 4.7, $\rho^k_{\mathbb{R}}(4l\xi \oplus 4l) = k^{4l}$ for k odd. By Proposition 4.5, this implies that

$$k^{2l}\frac{1}{2}(k^{2l} - 1)\lambda = 0 \text{ for all } k \text{ odd.}$$

This equation is in turn equivalent to

$$k^{2l} \equiv 1 \bmod 2^{f+1} \text{ for all } k \text{ odd.}$$

The group $(\mathbb{Z}/2^{f+1}\mathbb{Z})^\times$ is equal to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{f-1}\mathbb{Z}$, which has an element of order 2^{f-1}. Hence we have $2^{f-1}|2l$, or $2^f = a_n|4l = m$, completing the proof. □

The following corollary is now immediate.

Corollary 5.3. *Let us write each integer m in the form $m = (2\alpha - 1)2^\beta$, where $\beta = 4\delta + \gamma$ with $0 \leq \gamma \leq 3$. Then S^{m-1} admits at most $8\delta + 2^\gamma - 1$ linearly independent vector fields.*

6. Realizing the upper bound

In this brief section, we use the theory of Clifford algebras to construct $n - 1$ linearly independent vector fields on S^{m-1}, provided that m is a multiple of a_n as defined in Theorem 5.2. The Clifford algebra C_k is defined to be the $\mathbb{R}$-algebra generated by 1 and $e_1, \ldots, e_k$ subject to the relations

$$e_i^2 = -1, \qquad e_ie_j + e_je_i = 0 \text{ for } i \neq j.$$

We have the following table identifying the Clifford algebras C_k for $0 < k \leq 8$, as given in Atiyah, Bott, and Shapiro [4, p. 11].

k	C_k
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
8	$M_{16}(\mathbb{R})$

Moreover, the Clifford algebras C_k are periodic with period 8, in the sense that $C_{k+8} = C_k \otimes C_8$. It follows that if $C_k = M_r(\mathbb{F})$, then $C_{k+8} = M_{16r}(\mathbb{F})$. We now prove the converse to Theorem 5.2.

Theorem 6.1. *The sphere S^{m-1} admits $n - 1$ linearly independent vector fields if m is a multiple of a_n.*

Proof. Suppose m is a multiple of a_n. By the definition of a_n, we see that $\mathbb{R}^m$ may be provided with a C_{n-1}-module structure. This means that there exist $n - 1$ automorphisms $e_1, \ldots, e_{n-1}$ of $\mathbb{R}^m$, such that $e_i^2 = -1$ and $e_ie_j + e_je_i = 0$ for $i \neq j$. Let $e_0 = I$ and let G be the multiplicative finite group of order 2^n generated by $\pm e_i$, $0 \leq i < n$. Then we may choose a metric on $\mathbb{R}^m$ such that G preserves the metric, i.e. the e_i are orthogonal transformations, so $e_i^t = -e_i$ for $1 \leq i < n$. Now for each vector $v \in S^{m-1}$, observe that for $i \neq j$,

$$\langle e_i v, e_j v \rangle = v^t e_i^t e_j v = -v^t e_j^t e_i v = -\langle e_j v, e_i v \rangle = -\langle e_i v, e_j v \rangle ,$$

hence $\langle e_i v, e_j v \rangle = 0$. Thus, the vectors $e_i v$ give $n-1$ linearly independent vectors tangent to v, and varying v gives the desired $n-1$ linearly independent vector fields. □

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References

[1] J. F. Adams. Vector fields on spheres. Ann. Math. **75** (1962), 603-632.
[2] J. F. Adams. On the groups $J(X)$ II. Topology. **3** (1965), 137-171.
[3] M. F. Atiyah. K-Theory. Addison-Wesley Publishing Co. 1989.
[4] M. F. Atiyah, R. Bott and A. Shapiro. Clifford modules. Topology. **3** (Suppl. 1) (1964), 3-38.
[5] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. Proc. Sympos. Pure Math., Amer. Math. Soc. **3** (1961), 7-38.
[6] A. Dold. Über fasernweise Homotopieaquivalenz von Faserraumen. Math. Zeit. **62** (1955), 111-126.
[7] D. Dugger. Multiplicative structures on homotopy spectral sequences II. http://hopf.math.purdue.edu/Dugger/mult2DD.pdf.
[8] M. Karoubi. K-Theory: An Introduction. Springer-Verlag. 1978.
[9] J. P. May. A Concise Course in Algebraic Topology. University of Chicago Press. 1999.
[10] R. Thom. Espaces fibrés en sphères et carrés de Steenrod. Ann. Ecole Norm. Sup. **69** (1952), 109-182.