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On the bad points of positive semidefinite polynomials

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Abstract

A bad point of a positive semidefinite real polynomial f is a point at which a pole appears in all expressions of f as a sum of squares of rational functions. We show that quartic polynomials in three variables never have bad points. We give examples of positive semidefinite polynomials with a bad point at the origin, that are nevertheless sums of squares of formal power series, answering a question of Brumfiel. We also give an example of a positive semidefinite polynomial in three variables with a complex bad point that is not real, answering a question of Scheiderer.

Introduction

Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial with real coefficients that is positive semidefinite, i.e., that only takes nonnegative values. Its degree d is then even. Sometimes, one may explain the positivity of f by writing it as a sum of squares of polynomials. Such is the case when $n \leq 1$, when $d \leq 2$, and, as Hilbert proved in [16], when $(n, d) = (2, 4)$. For all other values of (n, d), there exist positive semidefinite polynomials that are not sums of squares of polynomials [16].

Hilbert asked in his celebrated 17th problem whether all positive semidefinite polynomials f could however be written as sums of squares of rational functions. This was proven by him [17] when $n = 2$ and by Artin [1, Satz 4] in general.

To understand the possible denominators in a representation of f as a sum of squares in $\mathbb{R}(x_1, \dots, x_n)$, it is natural to introduce the set $B(f) \subset \mathbb{C}^n$ of *bad points* of f: those points at which some denominator vanishes in all possible representations of f as a sum of squares in $\mathbb{R}(x_1, \dots, x_n)$. The existence of a bad point may be thought of as an explanation why f cannot be a sum of squares in $\mathbb{R}[x_1, \dots, x_n]$.

As indicated in [10, p. 20], that bad points may exist was first noted by Straus in a 1956 letter to Kreisel: if $f \in \mathbb{R}[x_1, \dots, x_n]$ is not a sum of squares of polynomials, then its homogenization in $\mathbb{R}[x_1, \dots, x_{n+1}]$ has a bad point at the origin. Such examples only appeared in

Dedicated to Olivier Debarre.

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print twenty years later (see [6, Theorem 4.3], [5, p. 196], [8, Proposition 3.5], [4, Counterexample 9.1] or [10, pp. 59–61]).

The bad locus $B(f) \subset \mathbb{C}^n$ of f always has codimension ≥ 3, as was shown in increasing generality by Choi and Lam [6, Theorem 4.2], by Delzell [10, Proposition 5.1], and by Scheiderer [28, Theorem 4.8]. In particular, bad points never appear when $n = 2$ (which yields examples of polynomials f with no bad points that are nevertheless not sums of squares of polynomials).

Our first theorem shows that a similar phenomenon occurs when $(n, d) = (3, 4)$.

Theorem 0.1 (Theorem 2.4) *Positive semidefinite real polynomials of degree four in three variables have no bad points.*

The $(n, d) = (3, 4)$ case considered in this theorem is the only one for which the question of the existence of bad points is not covered by the above-mentioned results. It was ostensibly left open in [6, Theorem 4.3].

Our proof builds on the works of several authors: Hilbert's classical theorem on quartics in two variables [16], Choi, Lam and Reznick's detailed study of quartics in three variables [7], and Scheiderer's general results on sums of squares in local rings [28]. The argument works over an arbitrary real closed field.

In three variables, all known examples of bad points share striking common features. To begin with, they are all real points. It was asked by Scheiderer [27, Remark 1.4 2] whether a positive semidefinite $f \in \mathbb{R}[x, y, z]$ could have a nonreal bad point. In our second main theorem, we construct such an example.

Theorem 0.2 (Theorem 3.6) *There exists a positive semidefinite polynomial in* $\mathbb{R}[x, y, z]$ *with a bad point that is not real.*

The only bad points of our example are $(0, 0, i)$ and $(0, 0, -i)$ (see Theorem 3.6).

In ≥ 4 variables, examples of nonreal bad points were already known since the bad locus $B(f)$ may have dimension ≥ 1 (see [10, Example 1 p. 59]). However, Theorem 0.2 is the first example in any number of variables where the real bad points of f are not Zariski-dense in $B(f)$.

Additionally, in all existing examples of positive semidefinite $f \in \mathbb{R}[x, y, z]$ with a real bad point, assumed to be the origin, this point is shown to be bad by an analysis of some low degree monomials of f. As a consequence, the polynomial f is not even a sum of squares in the ring $\mathbb{R}[[x, y, z]]$ of formal power series. An old question of Brumfiel appearing in [10, p. 62] asks whether this is a general phenomenon. In our third main theorem, we answer this question in the negative.

Theorem 0.3 (Theorem 3.11) *There exists a positive semidefinite polynomial in* $\mathbb{R}[x, y, z]$ *that has a bad point at the origin, but that is a sum of squares in* $\mathbb{R}[[x, y, z]]$.

Our example does not have other bad points than the origin (see Theorem 3.11).

Brumfiel asked his question in any number of variables. There are however easier examples in ≥ 4 variables, as it may happen that a positive semidefinite $f \in \mathbb{R}[w, x, y, z]$ is a sum of squares in $\mathbb{R}[[w, x, y, z]]$ but not in $\mathbb{R}[w, x, y, z]_{\langle w,x,y,z\rangle}$ for the simple reason that it is not even a sum of squares in some other completion of $\mathbb{R}[w, x, y, z]_{\langle w,x,y,z\rangle}$. We give such an example in Theorem 3.15.

This last remark points to what is difficult in proving Theorems 0.2 and 0.3. Let $\mathfrak{m} \subset \mathbb{R}[x, y, z]$ be the maximal ideal corresponding to the bad point. Under the hypotheses of either theorem, the polynomial f has to be a sum of squares in all the completions of $\mathbb{R}[x,y,z]_{\mathfrak{m}}$ (apply [28, Corollary 2.4 and Theorem 4.8]). We thus need to devise an obstruction to f being a sum of squares in the local ring $\mathbb{R}[x,y,z]_{\mathfrak{m}}$ that is sufficiently global in nature to allow f to be a sum of squares in all the completions of $\mathbb{R}[x,y,z]_{\mathfrak{m}}$. We now briefly explain how to overcome this difficulty (see Sect. 3 for more details).

Let $\Gamma \subset \mathbb{A}^3_{\mathbb{R}} := \mathrm{Spec}(\mathbb{R}[x,y,z])$ be an integral curve through $\mathfrak{m}$ whose real locus $\Gamma(\mathbb{R})$ is Zariski-dense in Γ and such that f vanishes on Γ. It follows from these facts that, in any representation $f = \sum_i f_i^2$ of f as a sum of squares in $\mathbb{R}[x,y,z]_{\mathfrak{m}}$, the f_i must vanish on Γ. As a consequence, one has $f \in (I_\Gamma^2)_{\mathfrak{m}}$, where I_Γ is the ideal defining Γ. It thus suffices to arrange that $f \notin (I_\Gamma^2)_{\mathfrak{m}}$ to ensure that it is not a sum of squares in $\mathbb{R}[x,y,z]_{\mathfrak{m}}$.

This is not easy to achieve. Indeed, since f is positive semidefinite, it belongs to the ideal I_Γ^2 at all smooth real points of Γ, hence generically along Γ. In other words, it belongs to the symbolic square $I_\Gamma^{(2)}$ of I_Γ (see (3.1) for the definition of $I_\Gamma^{(2)}$ and the survey [9] for more information on this topic). We thus need the ideals I_Γ^2 and $I_\Gamma^{(2)}$ to be distinct. The simplest example of this phenomenon, already appearing in [25, Example 3 p. 29], is the ideal of the image of the morphism $t \mapsto (t^3, t^4, t^5)$.

The polynomials we use to prove Theorems 0.2 and 0.3 are both constructed by modifying appropriately this basic example. For Theorem 0.2, this strategy leads to the concrete polynomial of degree ten $x^{10} + x^2y^6 + (z^2+1)^3 - 3x^4y^2(z^2+1)$ (see Theorem 3.6). The proof of Theorem 0.3 is more involved and does not yield an explicit example.

Our strategy actually works on arbitrary smooth varieties over any base field. We thus obtain the following result. Recall that a field is said to be *formally real* if it admits a field ordering (in particular, such a field has characteristic 0).

Theorem 0.4 (Theorem 3.13) *Let X be an affine variety over a field k. Let A be a local ring of X that is regular, with maximal ideal $\mathfrak{m}$. Assume that* $\dim(A) \geq 3$ *and that* $\mathrm{Frac}(A)$ *is formally real. Then there exists $f \in \mathcal{O}(X)$ such that:*

(i) The element f is a sum of squares in the completion $\widehat{A}_{\mathfrak{m}}$ of A at $\mathfrak{m}$.
(ii) For all prime ideals $\mathfrak{p} \neq \mathfrak{m}$ of A, f is a sum of squares in the localization $A_{\mathfrak{p}}$.
(iii) But f is not a sum of squares in A.

Notice that Theorem 0.2 (resp. Theorem 0.3) may be obtained as the particular case of Theorem 0.4 where $k = \mathbb{R}$, $X = \mathbb{A}^3_{\mathbb{R}}$ and $\mathfrak{m}$ has residue field $\mathbb{C}$ (resp. $\mathbb{R}$).

Theorem 0.4 yields the first examples of a regular local ring A with $2 \in A^*$ and of an element $f \in A$ that is a sum of squares in all the completions of A but not in A. Such examples do not exist if $\dim(A) \leq 2$ by [28, Theorem 4.8], or if $\mathrm{Frac}(A)$ is not formally real [28, Corollaries 1.5 and 2.4].

Thanks to Theorem 0.4, we are able to complete the proof of the following result, which is almost entirely due to Scheiderer (the case that was still open is explicitly mentioned in [27, Remark 1.4 2]). To state it, we recall that an element f of a ring A is *positive semidefinite* if it is nonnegative with respect to all the orderings of the residue fields of A.

Theorem 0.5 *Let A be the local ring at a regular point of a variety over a field k of characteristic not 2. The following are equivalent:*

(i) All positive semidefinite elements of A are sums of squares in A.
(ii) Either $\dim(A) \leq 2$ *or* $\mathrm{Frac}(A)$ *is not formally real.*

Proof If $\dim(A) \leq 2$, one may apply [28, Theorem 4.8], and if $\mathrm{Frac}(A)$ is not formally real, the theorem follows from [28, Corollaries 1.5 and 2.4]. The other cases are covered by

Theorem 0.4, but were already known if either $\dim(A) \geq 4$ or if the residue field of A is formally real (see [27, Propositions 1.2 and 1.5]). □

Understanding when assertion (i) of Theorem 0.5 holds is also interesting when A is possibly singular. We refer to [28, Theorem 3.9], to [12, Theorem 3.1] and to [13, Theorem 1.1] for the best known results in dimensions 1, 2 and ≥ 3 respectively.

It is tempting to ask if Theorem 0.5 remains true for arbitrary regular local rings, not necessarily of geometric origin. In our last result, we show that this is not the case, answering a question raised in [28, bottom of p. 209].

Theorem 0.6 (Theorem 4.2) *For all $n \geq 0$, there exists a regular local $\mathbb{R}$-algebra A of dimension n with the following properties:*

(i) All positive semidefinite elements of A are sums of squares in A.
(ii) The field $\mathrm{Frac}(A)$ *is formally real.*

The regular local rings that we consider to prove Theorem 0.6 are actually not far from geometry. When $n \geq 1$, they lie between the local ring of $\mathbb{A}^n_{\mathbb{R}}$ at a closed point with complex residue field and its Henselization.

Notation

If A is ring and $\mathfrak{p} \subset A$ is a prime ideal, we let $A_{\mathfrak{p}}$ and $\widehat{A}_{\mathfrak{p}}$ be the localization and the completion of A at $\mathfrak{p}$, and we denote by $I_{\mathfrak{p}} = IA_{\mathfrak{p}} \subset A_{\mathfrak{p}}$ and $\widehat{I}_{\mathfrak{p}} = I\widehat{A}_{\mathfrak{p}} \subset \widehat{A}_{\mathfrak{p}}$ the ideals generated by an ideal $I \subset A$.

An algebraic variety X over a field k is a separated scheme of finite type over k. If k' is a field extension of k, we denote by $X_{k'} := X \times_k k'$ the extension of scalars, and by $X(k')$ the set of k'-points of X.

1 Generalities on real spectra and sums of squares

The *real spectrum* $\mathrm{Sper}(A)$ of a ring A is the set of pairs $\xi = (\mathfrak{p}, \prec)$, where $\mathfrak{p}$ is a prime ideal of A and $\prec$ is a field ordering of $\mathrm{Frac}(A/\mathfrak{p})$. The element $\xi \in \mathrm{Sper}(A)$ is said to be *supported* at $\mathfrak{p}$. We denote by $\prec_\xi$ the ordering associated with ξ. We endow $\mathrm{Sper}(A)$ with its *spectral topology* [3, Definition 7.1.3], generated by open sets of the form $\{\xi \in \mathrm{Sper}(A) \mid f_i \succ_\xi 0\}$ for $(f_i)_{1 \leq i \leq m} \in A^m$. If $\xi, \zeta \in \mathrm{Sper}(A)$, one says that ξ is a *specialization* of ζ if ξ belongs to the closure of ζ. An element $f \in A$ is *positive semidefinite* (resp. *totally positive*) if it is nonnegative (resp. positive) with respect to all points of $\mathrm{Sper}(A)$. A real polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is positive semidefinite in this sense if and only if it is positive semidefinite in the sense considered in the introduction (see [3, Propositions 7.2.1 and 7.2.2]).

If k is a field, then $\mathrm{Sper}(k)$ coincides with the set of field orderings of k endowed with the Harrison topology (see [20, VIII, §6]). The field k is said to be *formally real* if $\mathrm{Sper}(k)$ is nonempty, i.e., if k admits a field ordering.

We now collect a few known statements that will be used repeatedly in the sequel. We start with two lemmas.

Lemma 1.1 ([27, Lemma 0.1]) *Let A be a regular domain with fraction field K. Then* $\mathrm{Sper}(K)$ *is dense in* $\mathrm{Sper}(A)$.

Lemma 1.2 ([27, Lemma 5.1 a)]) *Let A be a regular local ring with maximal ideal $\mathfrak{m}$. View $\mathfrak{m}/\mathfrak{m}^2$ as an $A/\mathfrak{m}$-vector space. Let $f \in A$ be positive semidefinite. If $f \in \mathfrak{m}^d$, the image of f in $\mathfrak{m}^d/\mathfrak{m}^{d+1} = \mathrm{Sym}^d(\mathfrak{m}/\mathfrak{m}^2)$ is a positive semidefinite polynomial function on the dual vector space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$.*

In particular, if $A/\mathfrak{m}$ is formally real and d is odd, then $f \in \mathfrak{m}^{d+1}$.

The following two theorems are due to Scheiderer.

Theorem 1.3 ([28, Corollary 2.4]) *Let A be a local ring with $2 \in A^*$. If $f \in A$ is totally positive, then f is a sum of squares in A.*

Theorem 1.4 ([28, Theorem 4.8]) *Let A be a regular local ring of dimension two with $2 \in A^*$. If $f \in A$ is positive semidefinite, then f is a sum of squares in A.*

2 Quartics in three variables have no bad points

In this section, we show that positive semidefinite quartic polynomials in three variables have no bad points (Theorem 2.4). We also study quartics in three variables that are nonnegative in a neighbourhood of the origin (see Theorem 2.5).

Throughout, we work over a real closed field R. We start with a series of lemmas.

Lemma 2.1 *Let B be a ring with $2 \in B^*$. Fix $g \in B[[x_1, \ldots, x_n]]$ with only terms of degree ≥ 3. Choose $0 \leq r \leq n$. Then there exist $(a_i)_{1 \leq i \leq r} \in B[[x_1, \ldots, x_n]]^r$ with only terms of degree ≥ 2, and $b \in B[[x_{r+1}, \ldots, x_n]]$, such that*

$$\sum_{i=1}^{r} x_i^2 + g = \sum_{i=1}^{r} (x_i + a_i)^2 + b.$$

Proof By induction on $N \geq 1$, we will construct $(a_{i,N})_{1 \leq i \leq r} \in B[x_1, \ldots, x_n]^r$, $b_N \in B[x_{r+1}, \ldots, x_n]$ and $c_N \in B[[x_1, \ldots, x_n]]$ with the following properties:

(i) One has $\sum_{i=1}^{r} x_i^2 + g = \sum_{i=1}^{r} (x_i + a_{i,N})^2 + b_N + c_N$.
(ii) Only terms of degree ≥ 2 appear in $a_{i,N}$.
(iii) Only terms of degree $\geq N + 2$ appear in c_N.
(iv) Only terms of degree $\geq N + 1$ appear in $a_{i,N+1} - a_{i,N}$ and in $b_{N+1} - b_N$.

To do so, we set $a_{i,1} = b_1 = 0$ and $c_1 = g$. If $a_{i,N}$, b_N and c_N have been constructed, write the degree $N+2$ term of c_N as $\sum_{i=1}^{r} x_i u_i + v$, where $u_i \in B[x_1, \ldots, x_n]$ has degree $N+1$ and $v \in B[x_{r+1}, \ldots, x_n]$ has degree $N+2$. It now suffices to define $a_{i,N+1} = a_{i,N} + u_i/2$, $b_{N+1} = b_N + v$ and $c_{N+1} = c_N - v - \sum_{i=1}^{r} (x_i u_i + a_{i,N} u_i + u_i^2/4)$.

To conclude, define $a_i := \lim_{N \to \infty} a_{i,N}$ and $b := \lim_{N \to \infty} b_N$, where the limits are taken with respect to the $\langle x_1, \ldots, x_n \rangle$-adic topology. □

Lemma 2.2 *Let $f \in R[[x_1, \ldots, x_n]]$ be positive semidefinite. Assume that the lowest degree term of f is a quadratic form of rank $r \geq n-2$. Then f is a sum of squares in $R[[x_1, \ldots, x_n]]$.*

Proof The degree 2 term of f is positive semidefinite by Lemma 1.2. Since it may be diagonalized after a linear change of coordinates, we may assume that $f = \sum_{i=1}^{r} x_i^2 + g$, where all monomials in $g \in R[[x_1, \ldots, x_n]]$ have degree ≥ 3.

By Lemma 2.1 applied with $B = R$, there exist $(a_i)_{1 \leq i \leq r} \in R[[x_1, \ldots, x_n]]^r$ with only terms of degree ≥ 2, and $b \in R[[x_{r+1}, \ldots, x_n]]$, such that $f = \sum_{i=1}^{r} (x_i + a_i)^2 + b$ (by convention, $R[[x_{r+1}, \ldots, x_n]] = R$ when $r = n - 2$).

Since f is positive semidefinite, so is its image in $R[[x_1,\dots,x_n]]/\langle x_i+a_i\rangle_{1\le i\le r}$, showing that $b\in R[[x_{r+1},\dots,x_n]]$ is positive semidefinite. As $r\ge n-2$, Theorem 1.4 shows that b is a sum of squares in $R[[x_{r+1},\dots,x_n]]$. The proof is now complete. □

Lemma 2.3 *A polynomial $f\in R[x,y,z]$ of degree ≤ 4 which is positive semidefinite in $R[[x,y,z]]$ is a sum of squares in $R[[x,y,z]]$.*

Proof By Lemma 1.2, the lowest degree term of f is positive semidefinite of degree 0, 2 or 4. If it has degree 0, then f is a square in $R[[x,y,z]]$, and we are done. If it has degree 2, one may apply Lemma 2.2 to conclude. If it has degree 4, then f is a sum of three squares of quadratic polynomials by Hilbert's theorem [16] (this result, proven over $\mathbb{R}$ in *loc. cit.*, holds over an arbitrary real closed field by the Tarski-Seidenberg principle [3, Proposition 5.2.3]). □

Now comes the main theorem of Sect. 2.

Theorem 2.4 *Let $f\in R[x,y,z]$ be positive semidefinite of degree ≤ 4. For all maximal ideals $\mathfrak{m}\subset R[x,y,z]$, the polynomial f is a sum of squares in $R[x,y,z]_{\mathfrak{m}}$.*

Proof If f has at least twelve real zeros, then it is a sum of six squares of quadratic polynomials, by a theorem of Choi, Lam and Reznick [7, Theorem 5.1] (this fact, proven over the reals in *loc. cit.*, is valid over any real closed field by the Tarski-Seidenberg principle [3, Proposition 5.2.3]).

We may thus assume that f has finitely many real zeros. Using [3, Theorem 7.2.3], we see that $\mathrm{Sper}((R[x,y,z]/\langle f\rangle)_{\mathfrak{m}})$ contains exactly one point (which is supported at $\mathfrak{m}$) if the residue field of $\mathfrak{m}$ is R, and is empty otherwise. In the latter case, the element $f\in R[x,y,z]_{\mathfrak{m}}$ is totally positive, hence a sum of squares by Theorem 1.3.

It remains to deal with the case where the residue field of $\mathfrak{m}$ is R. After changing coordinates by a translation, one may suppose that $\mathfrak{m}=\langle x,y,z\rangle$. By Lemma 2.3, the polynomial f is a sum of squares in $R[[x,y,z]]$. Since $\mathrm{Sper}((R[x,y,z]/\langle f\rangle)_{\mathfrak{m}})$ is supported at $\mathfrak{m}$, a theorem of Scheiderer [28, Corollary 2.7 (ii)$\Rightarrow$(i)] shows that f is a sum of squares in $R[x,y,z]_{\mathfrak{m}}$, as wanted. □

We conclude this section with a more concrete reformulation of Lemma 2.3.

Theorem 2.5 *Let $f\in R[x,y,z]$ be of degree ≤ 4. The following are equivalent:*

(i) The function $f: R^3\to R$ takes only nonegative values in a Euclidean neighbourhood of the origin.
(ii) The polynomial f is a sum of squares in $R[[x,y,z]]$.

Proof It suffices to combine Lemma 2.3 and Lemma 2.6 below. □

Lemma 2.6 *Let A be a finitely generated R-algebra, fix $f\in A$ and let $\mathfrak{m}\subset A$ be a maximal ideal with residue field R. The following assertions are equivalent:*

(i) The element $f\in\widehat{A}_{\mathfrak{m}}$ is positive semidefinite.
(ii) The function $\mathrm{Spec}(A)(R)\to R$ induced by f takes only nonnegative values in a Euclidean neighbourhood of the point of $\mathrm{Spec}(A)(R)$ corresponding to $\mathfrak{m}$.

Proof Let x (resp. $\tilde{x}$) be the point of $\mathrm{Spec}(A)(R)$ (resp. of $\mathrm{Sper}(A)$) associated to $\mathfrak{m}$. By [26, Théorème 1.1], the image of the natural map $\mathrm{Sper}(\widehat{A}_{\mathfrak{m}})\to\mathrm{Sper}(A)$ consists exactly of the elements having $\tilde{x}$ as a specialization.

If (ii) holds, the semi-algebraic subset $\{f \geq 0\}$ of $\mathrm{Spec}(A)(R)$ contains a neighbourhood of x. It follows from [3, Theorem 7.2.3] that the constructible subset $\{f \geq 0\}$ of $\mathrm{Sper}(A)$ contains a neighbourhood of $\tilde{x}$, hence all points having $\tilde{x}$ as a specialization. Consequently, f is positive semidefinite in $\widehat{A}_{\mathfrak{m}}$.

Conversely, assume that (ii) does not hold, hence that the open subset $\{f < 0\}$ of $\mathrm{Spec}(A)(R)$ contains x in its closure. By [3, Theorem 7.2.3], the open subset $\{f < 0\}$ of $\mathrm{Sper}(A)$ contains $\tilde{x}$ in its closure. In view of [3, Proposition 7.1.21], this subset of $\mathrm{Sper}(A)$ contains a point specializing to $\tilde{x}$. This shows that f is not positive semidefinite as an element of $\widehat{A}_{\mathfrak{m}}$. □

Remark 2.7 Scheiderer has shown in [29, Theorem 2.1] the existence of a homogeneous polynomial $f \in \mathbb{Q}[x, y, z]$ of degree 4 that is positive semidefinite, but not a sum of squares in $\mathbb{Q}[x, y, z]$. Using the homogeneity of f, one sees that f is not a sum of squares in $\mathbb{Q}[[x, y, z]]$ either. This shows that Theorems 2.4 and 2.5 cannot be extended to general base fields that are not necessarily real closed.

3 Examples of bad points

In Sect. 3.1, we state a simple criterion for an element of a ring not to be a sum of squares. This criterion is applied in Sects. 3.3 and 3.5 to give examples of real positive semidefinite polynomials in three variables with a nonreal bad point, or with a real bad point that cannot be detected after completion (Theorems 3.6 and 3.11). We apply it again in Sect. 3.6 to give examples of regular bad points on varieties over a field that satisfy minimal hypotheses (Theorem 3.13). Another example of bad point, of a different nature, is presented in Sect. 3.7.

The proofs of Theorems 3.6, 3.11 and 3.13 rely on auxiliary polynomials f_1, f_2, f_3 and f_4, respectively constructed in Lemma 3.5, Propositions 3.7, 3.9 and 3.12. For $1 \leq i \leq 3$, the polynomial f_i is used to construct f_{i+1}. While the polynomial f_1 is simple (see Lemma 3.5), the expression of f_2 is quite complicated (see Remark 3.8), and we do not provide explicit formulas for f_3 and f_4.

In this whole section, we let k be a field of characteristic 0.

3.1 A criterion to be a bad point

We will use the following easy lemma.

Lemma 3.1 *Let I be a radical ideal in a ring A such that the image of* $\mathrm{Sper}(A/I)$ *in* $\mathrm{Spec}(A/I)$ *is Zariski-dense. If $f \in I \setminus I^2$, then f is not a sum of squares in A.*

Proof Assume for contradiction that $f = \sum_i f_i^2$ is a sum of squares in A. Then $\sum_i f_i^2$ vanishes in A/I. It follows that the f_i vanish at all formally real residue fields of A/I. As $\mathrm{Sper}(A/I)$ is Zariski-dense in $\mathrm{Spec}(A/I)$ and as A/I is reduced, the f_i vanish in A/I. The f_i thus belong to I, so that $f \in I^2$, which is absurd. □

Recall that the symbolic square of an ideal I in a Noetherian ring A is

$$I^{(2)} := \{f \in A \mid f \in I_{\mathfrak{p}}^2 \text{ for all associated primes } \mathfrak{p} \text{ of } A/I\}. \tag{3.1}$$

The following lemma will not be used in the sequel, but explains why it may be difficult to apply Lemma 3.1 in practice.

Lemma 3.2 *Under the hypotheses of Lemma 3.1, if the ring A is regular and the element $f \in A$ is positive semidefinite, then $f \in I^{(2)}$.*

Proof Let $\mathfrak{p}$ be an associated prime ideal of A/I. Then $\mathfrak{p}A_{\mathfrak{p}} = I_{\mathfrak{p}}$ because I is radical. As $\mathrm{Sper}(A/I)$ is Zariski-dense in $\mathrm{Spec}(A/I)$, one cannot write -1 as a sum of squares in $\kappa := \mathrm{Frac}(A/\mathfrak{p})$, so that κ is formally real by [3, Theorem 1.1.8]. Since $f \in \mathfrak{p}A_{\mathfrak{p}}$ is positive semidefinite and κ is formally real, Lemma 1.2 shows that $f \in \mathfrak{p}^2 A_{\mathfrak{p}} = I_{\mathfrak{p}}^2$. □

Consequently, to apply Lemma 3.1 to give an example of a positive semidefinite element in a regular ring that is not a sum of squares, we must ensure that $I^{(2)} \neq I^2$. A basic example of an ideal in a regular ring whose square and symbolic square are distinct will be given later, in Lemma 3.5.

3.2 A criterion to be a sum of squares

In our proofs of Theorems 3.6 and 3.11, we also need a way to check that a regular function on a variety over k is a sum of squares in a neighbourhood of a point. This is the role of Proposition 3.3.

Proposition 3.3 *Let X be a smooth affine variety of dimension n over k. Let $f \in \mathcal{O}(X)$ be positive semidefinite, and let $p \in X$ be a closed point. Let $Y \subset X$ be the Zariski closure of $\mathrm{Sper}(\mathcal{O}(X)/\langle f\rangle)$. If $p \in Y$, then the differential of f at p vanishes. If moreover Y is smooth of dimension $n-c$ at p and the Hessian of f at p has rank $\geq c$, then f is a sum of squares in $\mathcal{O}_{X,p}$.*

Proof By Lemma 1.2, the differential of f vanishes at all points $x \in X$ with formally real residue field such that $f(x) = 0$. It follows that the differential of f vanishes on Y, hence at p.

Set $A := \mathcal{O}_{X,p}$ and let $I \subset A$ be the ideal of functions vanishing on $\mathrm{Sper}(A/\langle f\rangle)$. As the generic points of Y are formally real by Lemma 1.1, we see that I is the ideal of the subscheme $Y \times_X \mathrm{Spec}(A)$ of $\mathrm{Spec}(A)$.

Since Y is smooth of dimension $n-c$ at p, the local ring $B := A/I$ is regular of dimension $n-c$. We have seen above that the fraction field of B is formally real. As f is positive semidefinite, it follows from Lemma 1.2 that the image of f in the localization A_I belongs to $I^2 A_I$. We deduce from [30, Appendix 6, Lemma 5] applied with $\mathfrak{a} = I$ (or from [18, (2.1)]) that $f \in I^2$.

As Y is smooth at p, [22, Theorem 30.3 (1)⇒(2)] shows that B is 0-smooth over k in the sense of [22, p. 193], hence that the natural surjections $A/I^n \to B$ admit compatible sections for $n \geq 1$. This yields a section $s : B \to \widehat{A}_I$ of the quotient map $\widehat{A}_I \to \widehat{A}_I/I = B$. Let $x_1, \dots, x_c$ be generators of I. By [11, Theorem 7.16], the section s induces a surjective morphism of B-algebras $B[[x_1, \dots, x_c]] \to \widehat{A}_I$. As $\widehat{A}_I$ is faithfully flat over A by [22, Theorem 8.14], one may apply [22, Theorem 15.1 (ii)] to show that $\dim(\widehat{A}_I) \geq \dim(A) = n$. Since $B[[x_1, \dots, x_c]]$ is integral of dimension n, the surjection $B[[x_1, \dots, x_c]] \to \widehat{A}_I$ is then necessarily an isomorphism.

We now argue as in the proof of [28, Corollary 2.7]. By a theorem of Scheiderer [28, Theorem 2.5], there exists an ideal J of A with radical I such that f is a sum of squares in A if it is a sum of squares in A/J. Since $I^m \subset J$ for some m, the proposition will be proven if we check that f is a sum of squares in A/I^m for all m. We will show the stronger fact that f is a sum of squares in $\widehat{A}_I = B[[x_1, \dots, x_c]]$. We have seen above that $f \in I^2 = \langle x_1, \dots, x_c\rangle^2$. Let κ be the residue field of B. Since the Hessian of f has rank $\geq c$, the image in $\kappa[x_1, \dots, x_c]$

of the quadratic term of f is a nondegenerate quadratic form. Applying Lemma 3.4 below concludes. $\square$

Lemma 3.4 *Let B be a local ring whose residue field κ is not of characteristic 2. Let $f \in B[[x_1, \ldots, x_n]]$ be positive semidefinite. If the lowest degree term of f is quadratic with nondegenerate image in $\kappa[x_1, \ldots, x_n]$, then f is a sum of squares.*

Proof Let $h \in B[x_1, \ldots, x_n]$ be the quadratic term of f. By [2, Chapter I, Proposition 3.4], we may assume after a suitable linear change of coordinates that $h = \sum_{i=1}^n \alpha_i x_i^2$ for some invertible elements $\alpha_i \in B$. Since f is positive semidefinite, the α_i are positive semidefinite, and Theorem 1.3 allows us to write $\alpha_i = \sum_j (\alpha_i^{(j)})^2$ for some $\alpha_i^{(j)} \in B$. After maybe permuting the $\alpha_i^{(j)}$, we may ensure that $\alpha_i^{(1)}$ is invertible in B. Choosing the $\alpha_i^{(1)} x_i$ as new variables, we may finally assume that $h - \sum_{i=1}^n x_i^2$ is a sum of squares in $B[x_1, \ldots, x_n]$. By Lemma 2.1 applied with $g = f - h$, there exist $a_1, \ldots, a_n$ in $B[[x_1, \ldots, x_n]]$ with

$$\sum_{i=1}^n x_i^2 + f - h = \sum_{i=1}^n (x_i + a_i)^2.$$

Combining these two facts shows that f is a sum of squares in $B[[x_1, \ldots, x_n]]$. $\square$

3.3 A nonreal bad point

As explained in Sect. 3.1, we are in need of an ideal whose square and symbolic square differ. Lemma 3.5 contains a simple example.

Lemma 3.5 *Let $C \subset \mathbb{A}^3_k$ be the image of the morphism $\nu : \mathbb{A}^1_k \to \mathbb{A}^3_k$ given by $\nu(t) = (t^3, t^4, t^5)$. Define $I_C := \langle u^3 - vw, v^2 - uw, w^2 - u^2 v \rangle \subset k[u, v, w]$ and consider the polynomial $f_1 := u^5 + uv^3 + w^3 - 3u^2 vw$. The following properties hold:*

(i) The zero locus of I_C is the geometrically integral curve $C \subset \mathbb{A}^3_k$.
(ii) One has $f_1 \in I_C$, $f_1 \notin I^2_{C,\langle u,v,w\rangle}$ and $vf_1 \in I_C^2$.

Proof. Since the morphism ν is finite, its image is a closed subvariety $C \subset \mathbb{A}^3_k$, which is geometrically integral because so is $\mathbb{A}^1_k$. That its ideal is exactly I_C is explained in [25, Example 3 p. 29]. This proves (i). As $f_1(t^3, t^4, t^5) = 0$, we see that $f_1 \in I_C$. To show that $f_1 \notin I^2_{C,\langle u,v,w\rangle}$, notice that in the development of an element of $I^2_{C,\langle u,v,w\rangle}$ as a power series in u, v and w, no term of degree ≤ 3 may appear (this argument may be found in [25, Example 3 p. 29]). The last assertion of (ii) follows from the identity

$$vf_1 = u(v^2 - uw)^2 + (w^2 - u^2 v)(wv - u^3). \quad \square \tag{3.2}$$

Now comes our first application of Lemma 3.1.

Theorem 3.6 *Consider the ideal $\mathfrak{m} := \langle x, y, z^2 + 1 \rangle \subset \mathbb{R}[x, y, z]$. The polynomial*

$$f := x^{10} + x^2 y^6 + (z^2 + 1)^3 - 3x^4 y^2 (z^2 + 1)$$

is positive semidefinite. It is a sum of squares in $\mathbb{R}[x, y, z]_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset \mathbb{R}[x, y, z]$ distinct from $\mathfrak{m}$, but it is not a sum of squares in $\mathbb{R}[x, y, z]_{\mathfrak{m}}$.

Proof That f is positive semidefinite stems from the inequality between the arithmetic and the geometric means of x^{10}, x^2y^6 and $(z^2+1)^3$.

Let $\psi : \mathbb{A}^3_{\mathbb{R}} \to \mathbb{A}^3_{\mathbb{R}}$ be the morphism defined by $\psi(x,y,z) = (x^2, y^2, z^2+1)$. Since the pull-back morphism $\psi^* : \mathbb{R}[u,v,w] \to \mathbb{R}[x,y,z]$ endows $\mathbb{R}[x,y,z]$ with a structure of free $\mathbb{R}[u,v,w]$-module, the morphism ψ is finite flat. Let C, I_C and f_1 be as in Lemma 3.5 applied with $k=\mathbb{R}$. Note that $f=\psi^*f_1$. Let $\Gamma := \psi^{-1}(C) \subset \mathbb{A}^3_{\mathbb{R}}$ be the curve defined by the ideal

$$I_\Gamma := \langle \psi^* I_C \rangle = \langle x^6 - y^2(z^2+1), y^4 - x^2(z^2+1), (z^2+1)^2 - x^4y^2 \rangle \subset \mathbb{R}[x,y,z].$$

Remark first that $f \in I_\Gamma$ by Lemma 3.5 (ii). The flatness of ψ and [22, Theorem 7.5 (ii)] imply that $f \notin I^2_{\Gamma,\mathfrak{m}}$, because $f_1 \notin I^2_{C,\langle u,v,w\rangle}$ as proven in Lemma 3.5 (ii). Since ψ is finite flat and since C is geometrically integral by Lemma 3.5 (i), the irreducible components of the curve Γ surject to C. For $t \in \mathbb{R}_{>1}$, the curve C is smooth at (t^3, t^4, t^5) and the morphism ψ is étale with only real points above (t^3,t^4,t^5). We deduce that all the irreducible components of Γ contain a smooth real point. It follows that $\Gamma(\mathbb{R})$ is Zariski-dense in Γ. In view of Lemma 1.1, the residue fields of the generic points of Γ are formally real. Moreover, as the curve Γ has no embedded point by flatness of ψ (see [22, Theorem 23.2]), it is reduced. Applying Lemma 3.1 with $I = I_{\Gamma,\mathfrak{m}}$, one shows that f is not a sum of squares in $\mathbb{R}[x,y,z]_{\mathfrak{m}}$.

It remains to check that, if $\mathfrak{p} \subset \mathbb{R}[x,y,z]$ is a prime ideal distinct from $\mathfrak{m}$, then f is a sum of squares in $\mathbb{R}[x,y,z]_{\mathfrak{p}}$. If $\mathfrak{p}$ is not maximal, this follows from Theorem 1.4. From now on, we assume that $\mathfrak{p}$ is maximal, and we let $p \in \mathbb{A}^3_{\mathbb{R}}$ be the closed point associated with $\mathfrak{p}$.

We claim that Γ is the Zariski closure of $\{(x_0,y_0,z_0) \in \mathbb{R}^3 \mid f(x_0,y_0,z_0) = 0\}$. We have already seen that f vanishes on Γ and that $\Gamma(\mathbb{R})$ is Zariski-dense in Γ. Conversely, if $(x_0,y_0,z_0) \in \mathbb{R}^3$ is such that $f(x_0,y_0,z_0)=0$, we deduce from the case of equality in the inequality between the arithmetic and the geometric means that $x_0^{10} = x_0^2y_0^6 = (z_0^2+1)^3$. These equations imply that $x_0 \neq 0$, so $x_0^8 = y_0^6$. One then easily verifies that (x_0,y_0,z_0) satisfies the defining equations of Γ, which proves the claim. By [3, Theorem 7.2.3], the Zariski closure of the image of $\mathrm{Sper}(\mathbb{R}[x,y,z]/\langle f\rangle) \to \mathrm{Spec}(\mathbb{R}[x,y,z]/\langle f\rangle)$ is also equal to Γ. If p does not belong to Γ, we deduce from Theorem 1.3 that f is a sum of squares in $\mathbb{R}[x,y,z]_{\mathfrak{p}}$.

Assume from now on that p belongs to Γ. In this case, we show that f is a sum of squares in $\mathbb{R}[x,y,z]_{\mathfrak{p}}$ by applying Proposition 3.3 with $X = \mathbb{A}^3_{\mathbb{R}}$, $Y=\Gamma$, $n=3$ and $c=2$. Let us verify its hypotheses. Let $q \in \mathbb{A}^3_{\mathbb{R}}$ be the point associated with $\mathfrak{m}$. Note that $p \neq q$ by hypothesis. The polynomials $x^8 - y^6$ and $x^{10} - (z^2+1)^3$ vanish on Γ and have independent differentials along $\Gamma \setminus \{q\}$. We deduce that Γ is smooth at p. Suppose for contradiction that the Hessian of f at p has rank ≤ 1. Then

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} \end{vmatrix} = 144x^5y^2z(4y^4 + x^2(z^2+1))$$

vanishes at p. As the polynomials x, y and $4y^4 + x^2(z^2+1)$ do not vanish on $\Gamma \setminus \{q\}$, we see that z vanishes at p. It follows that, at the point p, one has

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 90x^8 + 2y^6 - 36x^2y^2 & 12xy^5 - 24x^3y \\ 12xy^5 - 24x^3y & 30x^2y^4 - 6x^4 \end{vmatrix}, \tag{3.3}$$

and this quantity must vanish at p by the hypothesis on the Hessian. Since z vanishes at p, the equations of Γ show that $x^6 = y^2$ and $y^4 = x^2$ at the point p. Combining this with (3.3) shows that $\begin{pmatrix} 56x^2y^2 & -12x^3y \\ -12x^3y & 24x^4 \end{pmatrix} = 1200x^6y^2$ vanishes at p. As neither x nor y vanish on $\Gamma \setminus \{q\}$, this is a contradiction. We may thus apply Proposition 3.3 to complete the proof of the theorem. □

3.4 Sums of squares in the completion

In Sects. 3.4–3.5, we use Lemma 3.1 to prove Theorem 3.11. To do so, we construct an example of (A, I, f) as in Lemma 3.1, where A is local regular with maximal ideal $\mathfrak{m}$ and f is positive semidefinite, such that f is moreover a sum of squares in $\widehat{A}_{\mathfrak{m}}$. If one requires the residue field of A to be formally real, this is not easy to achieve. This is the goal of Proposition 3.7 in Sect. 3.4 and of Proposition 3.9 in Sect. 3.5. Let us first explain the principle of the argument of Proposition 3.7, where we ensure that f is a sum of squares in $\widehat{A}_{\mathfrak{m}}$.

Starting from the example of (A, I, f) with $f \in I^{(2)} \setminus I^2$ given by Lemma 3.5, we add to f a lot of squares of elements of I so as to improve the chances that it is a sum of squares in $\widehat{A}_{\mathfrak{m}}$. This works well only if the multiplicities of the squares of the generators of I are low compared to the multiplicity of f, and we can only arrange this after a change of variables of relatively high degree. Making sure that $\mathrm{Sper}(A/I)$ remains Zariski-dense in $\mathrm{Spec}(A/I)$ only complicates the change of variables that we need to use. The verification that the resulting element f is indeed a sum of squares in $\widehat{A}_{\mathfrak{m}}$ is computational since we do not know of a conceptual way to check it.

We recall that, in the whole of Sect. 3, we have fixed a field k of characteristic 0.

Proposition 3.7 *There exist $f_2 \in k[x, y, z]$ and an ideal $I_D \subset k[x, y, z]$ such that:*

(i) The ideal I_D defines a geometrically integral curve $D \subset \mathbb{A}^3_k$.
(ii) The point $(0, 0, 0)$ belongs to D. The curve $D \setminus \{(0, 0, 0)\}$ has a smooth k-point.
(iii) One has $f_2 \in I_D$ and $f_2 \notin I^2_{D,\langle x,y,z\rangle}$.
(iv) There exists $h \in k[x, y, z]$ such that $h \notin I_D$ and $hf_2 \in I^2_D$.
(v) The polynomial f_2 is a sum of squares in $k[[x, y, z]]$.

Proof Let $\phi : \mathbb{A}^3_k \to \mathbb{A}^3_k$ be defined by $\phi(x, y, z) = (x^2, y^8 - y^{10} + y^{11}, -z^2 + 2z^3)$. Since the pull-back morphism $\phi^* : k[u, v, w] \to k[x, y, z]$ endows $k[x, y, z]$ with a structure of free $k[u, v, w]$-module, the morphism ϕ is finite flat.

Let I_C, C and f_1 be as in Lemma 3.5. Let $D := \phi^{-1}(C) \subset \mathbb{A}^3_k$ be defined by the ideal $I_D := \langle \phi^* I_C \rangle \subset k[x, y, z]$. Let $\overline{k}$ be an algebraic closure of k. Since ϕ is flat, the curve D has no embedded point (see [22, Theorem 23.2]). To prove (i), it thus suffices to show that $D_{\overline{k}}$ is irreducible and generically reduced, i.e., that its total ring of fractions

$$F := \overline{k}(t)[x, y, z]/\langle x^2 - t^3, y^{11} - y^{10} + y^8 - t^4, 2z^3 - z^2 - t^5 \rangle$$

is a field. Remark that $F = \overline{k}(s)[y, z]/\langle y^{11} - y^{10} + y^8 - s^8, 2z^3 - z^2 - s^{10} \rangle$, where $s := x/t$. Since $2z^3 - z^2$ is not a nontrivial power in $\overline{k}(z)$, we see that $2z^3 - z^2 - s^{10}$ is irreducible in $\overline{k}(z)[s]$, hence in $\overline{k}(s)[z]$ by Gauss's lemma. The same reasoning shows that $y^{11} - y^{10} + y^8 - s^8$ is irreducible in $\overline{k}(s)[y]$. The two field extensions $K := \overline{k}(s)[z]/\langle 2z^3 - z^2 - s^{10} \rangle$ and $L := \overline{k}(s)[y]/\langle y^{11} - y^{10} + y^8 - s^8 \rangle$ of $\overline{k}(s)$ have coprime degree. Their tensor product $F = K \otimes_{\overline{k}(s)} L$ is thus a field. This proves (i).

One checks that $(0, 0, 0)$ and $(1, 1, 1)$ belong to $D(k)$. Since $\phi(1, 1, 1) = (1, 1, 1)$ is a smooth point of C, and since ϕ is étale at $(1, 1, 1)$, we see that $(1, 1, 1)$ is a smooth k-point of D. We have checked (ii).

Let $\hat{y} \in k[[y]]$ be the element such that $\hat{y}^8 = y^8 - y^{10} + y^{11}$ and $\hat{y} - y \in \langle y^2 \rangle$. Similarly, let $\hat{z} \in k[[z]]$ be such that $\hat{z}^2 = z^2 - 2z^3$ and $\hat{z} - z \in \langle z^2 \rangle$. For esthetic purposes, we also set $\hat{x} := x$. With this notation, one can write

$$I_D = \langle \hat{x}^6 + \hat{y}^8\hat{z}^2, \hat{y}^{16} + \hat{x}^2\hat{z}^2, \hat{z}^4 - \hat{x}^4\hat{y}^8 \rangle, \tag{3.4}$$

where the generators indeed belong to $k[x, y, z]$ since they only depend on $\hat{y}$ and $\hat{z}$ through $\hat{y}^8$ and $\hat{z}^2$. For the same reason, the element defined as

$$f_2 := -y^6\phi^* f_1 + 2(\hat{x}^6 + \hat{y}^8\hat{z}^2)^2 + y^4(\hat{y}^{16} + \hat{x}^2\hat{z}^2)^2 + (\hat{z}^4 - \hat{x}^4\hat{y}^8)^2, \tag{3.5}$$

belongs to $k[x, y, z]$ (we note that $\phi^* f_1 = \hat{x}^{10} + \hat{x}^2\hat{y}^{24} + \hat{z}^6 - 3\hat{x}^4\hat{y}^8\hat{z}^2$).

To see that $f_2 \in I_D$, combine Lemma 3.5 (ii) and (3.4). Assume for contradiction that $f_2 \in I^2_{D,\langle x,y,z\rangle}$. Then, in view of (3.4), one has $y^6\phi^* f_1 \in I^2_{D,\langle x,y,z\rangle}$. This is absurd, because the monomial $\hat{y}^6\hat{z}^6$ appears in the development of $y^6\phi^* f_1$ as a power series in $\hat{x}$, $\hat{y}$ and $\hat{z}$, but not in the development of any element of $I^2_{D,\langle x,y,z\rangle}$ as a power series in $\hat{x}$, $\hat{y}$ and $\hat{z}$ (as (3.4) shows). This proves (iii).

Choose $h := y^8 - y^{10} + y^{11}$. As $h(1, 1, 1) = 1 \neq 0$, we see that $h \notin I_D$. But $h\phi^* f_1 = \phi^*(vf_1) \in I_D^2$ by Lemma 3.5 (ii), so that $hf_2 \in I_D^2$ in view of (3.4). We have verified assertion (iv).

To prove assertion (v), we use a decomposition $f_2 = g + g' + g''$ in the ring $k[[x, y, z]] = k[[\hat{x}, \hat{y}, \hat{z}]]$. We choose

$$g := -\hat{y}^6\phi^* f_1 + (\hat{x}^6 + \hat{y}^8\hat{z}^2)^2 + \hat{y}^4(\hat{y}^{16} + \hat{x}^2\hat{z}^2)^2,$$

which is a sum of squares in $k[[\hat{x}, \hat{y}, \hat{z}]]$ in view of the identity:

$$g = (\hat{x}^2 - \hat{y}^6)^2(\hat{x}^8 + \hat{x}^6\hat{y}^6 + \hat{x}^4\hat{y}^{12} + \hat{x}^2\hat{y}^{18} + \hat{y}^{24} + 2\hat{x}^2\hat{y}^8\hat{z}^2) + \hat{z}^2\hat{y}^4(\hat{x}^4 + \hat{y}^2\hat{z}^2)(\hat{y}^{10} + \hat{z}^2).$$

We also set

$$g' := (y^4 - \hat{y}^4)(\hat{y}^{16} + \hat{x}^2\hat{z}^2)^2,$$

which is a sum of squares in $k[[\hat{x}, \hat{y}, \hat{z}]]$, because $y^4 - \hat{y}^4 - \hat{y}^6/4$ is a square in $k[[\hat{y}]]$ as its lowest degree term is $\hat{y}^6/4$. We finally define:

$$g'' := (\hat{y}^6 - y^6)\phi^* f_1 + (\hat{x}^6 + \hat{y}^8\hat{z}^2)^2 + (\hat{z}^4 - \hat{x}^4\hat{y}^8)^2. \tag{3.6}$$

To see that g'' is a sum of squares in $k[[\hat{x}, \hat{y}, \hat{z}]]$, we note that $\hat{y}^6 - y^6 = -\alpha\hat{y}^8$ for some $\alpha \in k[[\hat{y}]]$ whose constant term is equal to $3/4$. Pulling back equation (3.2) by the morphism ϕ and combining it with (3.6) yields the identity

$$g'' = \alpha\big(\hat{x}^2(\hat{y}^{16} + \hat{x}^2\hat{z}^2)^2 - (\hat{z}^4 - \hat{x}^4\hat{y}^8)(\hat{x}^6 + \hat{y}^8\hat{z}^2)\big) + (\hat{x}^6 + \hat{y}^8\hat{z}^2)^2 + (\hat{z}^4 - \hat{x}^4\hat{y}^8)^2,$$

which we rewrite as

$$g'' = \alpha\hat{x}^2(\hat{y}^{16} + \hat{x}^2\hat{z}^2)^2 + (\hat{x}^6 + \hat{y}^8\hat{z}^2 - \alpha/2(\hat{z}^4 - \hat{x}^4\hat{y}^8))^2 + (1 - \alpha^2/4)(\hat{z}^4 - \hat{x}^4\hat{y}^8)^2.$$

In the latter expression, all terms are sums of squares in $k[[\hat{x}, \hat{y}, \hat{z}]]$. Indeed, the power series α and $(1 - \alpha^2/4)$ are sums of squares in $k[[\hat{y}]]$ since their constant terms $3/4$ and $55/64$ are sums of squares in $\mathbb{Q}$, hence in k. □

Remark 3.8 To obtain a (complicated) closed formula for f_2, replace $\phi^* f_1$ by its value $\hat{x}^{10} + \hat{x}^2\hat{y}^{24} + \hat{z}^6 - 3\hat{x}^4\hat{y}^8\hat{z}^2$ in the formula (3.5), and use the change of variables $\hat{x} = x$, $\hat{y}^8 = y^8 - y^{10} + y^{11}$ and $\hat{z}^2 = z^2 - 2z^3$.

3.5 Bad points cannot be tested formally

In Proposition 3.9, we modify the polynomial constructed in Proposition 3.7 so as to make it positive semidefinite. We argue geometrically, on a well-chosen affine birational model of $\mathbb{A}^3_k$.

Proposition 3.9 *There exist $f_3 \in k[x, y, z]$ and an ideal $I_D \subset k[x, y, z]$ such that:*

(i) The ideal I_D defines a geometrically integral curve $D \subset \mathbb{A}^3_k$.
(ii) One has $(0, 0, 0) \in D(k)$. The curve $D \setminus \{(0, 0, 0)\}$ has a smooth k-point.
(iii) The polynomial f_3 is positive semidefinite and totally positive on $\mathbb{A}^3_k \setminus D$.
(iv) One has $f_3 \in I_D$ and $f_3 \notin I^2_{D,\langle x,y,z\rangle}$.
(v) The polynomial f_3 is a sum of squares in $k[[x, y, z]]$.
(vi) The polynomial f_3 is a sum of squares in $\mathcal{O}_{\mathbb{A}^3_k,p}$ for all $p \in \mathbb{A}^3_k \setminus \{(0, 0, 0)\}$.

Proof We may assume that $k = \mathbb{Q}$ since the general case follows by extending the scalars (use [22, Theorem 7.5 (ii)] to check that the second part of (iv) remains valid). Let f_2, I_D and D be as in Proposition 3.7 applied with $k = \mathbb{Q}$. Define $o := (0, 0, 0) \in D(\mathbb{Q})$. Assertions (i) and (ii) are exactly Proposition 3.7 (i) and (ii). We fix a smooth $\mathbb{Q}$-point q of $D \setminus \{o\}$.

Let $\overline{D}$ be the closure of D in $\mathbb{P}^3_{\mathbb{Q}}$. Resolving the singularities of $\overline{D} \setminus \{o\}$ as in [21, Chapter 8, Proposition 1.26] shows the existence of a composition of blow-ups at closed points $\widetilde{\mathbb{P}}^3 \to \mathbb{P}^3_{\mathbb{Q}}$ that is an isomorphism above $\mathbb{A}^3_{\mathbb{Q}}$ and such that o is the only singular point of the strict transform $\widetilde{D} \subset \widetilde{\mathbb{P}}^3$ of $\overline{D}$. Choose a very ample line bundle $\mathcal{L}$ on $\widetilde{\mathbb{P}}^3$ and a basis (σ_i) of $H^0(\widetilde{\mathbb{P}}^3, \mathcal{L})$, and define U to be the complement in $\widetilde{\mathbb{P}}^3$ of the ample divisor $\{\sum_i \sigma_i^2 = 0\}$. Then $U \subset \widetilde{\mathbb{P}}^3$ is an affine open subset such that $U(\mathbb{R}) = \widetilde{\mathbb{P}}^3(\mathbb{R})$. Define $Z := \widetilde{D} \cap U \subset U$ and let $I_Z \subset \mathcal{O}(U)$ be the ideal of Z.

Notice that $o \in Z(\mathbb{Q})$. View f_2 as a rational function on U that is well-defined at $o \in U(\mathbb{Q})$. Hence, there exists $a \in \mathcal{O}(U)$ nonzero at o such that $af_2 \in \mathcal{O}(U)$. Let $b_1, \dots, b_m \in \mathcal{O}(U)$ be generators of I_Z, and define:

$$g := a^2 f_2 + \lambda^2 \cdot \sum_{i=1}^{m} b_i^2 \in \mathcal{O}(U), \tag{3.7}$$

where $\lambda \in \mathbb{Q}$ is to be chosen later.

We claim that, for all $p \in U(\mathbb{R})$, and for all $\lambda \in \mathbb{Q}$ big enough, there exists a neighbourhood Ω_p of p in $U(\mathbb{R})$ such that g is nonnegative on Ω_p . We distinguish three cases. If $p \notin Z(\mathbb{R})$, then g is nonnegative at p for $\lambda \gg 0$ since one of the b_i does not vanish at p. If $p = o$, then f_2 is nonnegative in a neighbourhood of $p \in U(\mathbb{R})$ by Proposition 3.7 (v) and Lemma 2.6, so that any $\lambda \geq 0$ works. If $p \in Z(\mathbb{R})$ is distinct from o, then it is a smooth point of $Z_{\mathbb{R}}$. Consequently, after maybe permuting the b_i, we may assume that there exists $b' \in \mathcal{O}(U)$ such that (b_1, b_2, b') forms a regular system of parameters in $\widehat{\mathcal{O}}_{U_{\mathbb{R}},p} \simeq \mathbb{R}[[b_1, b_2, b']]$ and such that the ideal $J := I_Z \cdot \widehat{\mathcal{O}}_{U_{\mathbb{R}},p} \subset \widehat{\mathcal{O}}_{U_{\mathbb{R}},p}$ is generated by b_1 and b_2. Proposition 3.7 (iv) implies that f_2, hence also g, vanish at the generic point of the spectrum of the ring $\widehat{\mathcal{O}}_{U_{\mathbb{R}},p}/J^2$, hence vanish in $\widehat{\mathcal{O}}_{U_{\mathbb{R}},p}/J^2$ by [30, Appendix 6, Lemma 5] (or by [18, (2.1)]). As a consequence, there exist $\alpha, \beta, \gamma \in \mathbb{R}[[b_1, b_2, b']]$ such that $g = \alpha b_1^2 + \beta b_1 b_2 + \gamma b_2^2$ in $\mathbb{R}[[b_1, b_2, b']]$. If $\lambda \gg 0$, the constant terms of both α and $\gamma - \beta^2/(4\alpha)$ are positive, so that there exist $\delta, \varepsilon \in \mathbb{R}[[b_1, b_2, b']]$ such that $\delta^2 = \alpha$ and $\varepsilon^2 = \gamma - \beta^2/(4\alpha)$. We may then write $g = (\delta b_1 + \beta b_2/(2\delta))^2 + (\varepsilon b_2)^2$ in $\mathbb{R}[[b_1, b_2, b']]$. Lemma 2.6 thus shows that g is nonnegative in a neighbourhood Ω_p of p in $U(\mathbb{R})$. The claim is proved.

Since $U(\mathbb{R}) = \widetilde{\mathbb{P}}^3(\mathbb{R})$ is compact, it is covered by finitely many of the Ω_p. Consequently, for $\lambda \gg 0$, the function g is nonnegative on $U(\mathbb{R})$. We fix such a λ. In view of [3, Theorem 7.2.3], the element $g \in \mathcal{O}(U_{\mathbb{R}})$ is positive semidefinite. As the only field ordering of $\mathbb{Q}$ extends to $\mathbb{R}$, we deduce that $g \in \mathcal{O}(U)$ is positive semidefinite.

View (3.7) as an identity in $\mathcal{O}_{U,o} = \mathbb{Q}[x, y, z]_{\langle x,y,z\rangle}$. Choose $a' \in \mathbb{Q}[x, y, z]$ that does not vanish at o such that $a'a$ and the $a'b_i$ all belong to $\mathbb{Q}[x, y, z]$. Let $b'_1, \dots, b'_{m'} \in \mathbb{Q}[x, y, z]$ be generators of I_D. Define

$$f_3 := (a')^2 g + \sum_{i=1}^{m'} (b'_i)^2 \in \mathbb{Q}[x, y, z]. \tag{3.8}$$

Since $g \in \mathcal{O}(U)$ is positive semidefinite, we see that $(a')^2 g$ is positive semidefinite as an element of $\mathbb{Q}(x, y, z)$, hence as an element of $\mathbb{Q}[x, y, z]$ by Lemma 1.1. Assertion (iii) follows at once from (3.8).

Assertions (iv) and (v) are consequences of Proposition 3.7 (iii) and (v) and of the formulas (3.7) and (3.8) since a and a' do not vanish at o.

If $p \in \mathbb{A}^3_{\mathbb{Q}}$ is not a closed point, then f_3 is a sum of squares in $\mathcal{O}_{\mathbb{A}^3_{\mathbb{Q}},p}$ by Theorem 1.4 and (iii). Let us check that f_3 is a sum of squares in $\mathcal{O}_{\mathbb{A}^3_{\mathbb{Q}},q}$. To do so, we apply Proposition 3.3 with $X = \mathbb{A}^3_{\mathbb{R}}$, $Y = D$, $n = 3$ and $c = 2$. Let us verify its hypotheses. As q is a smooth $\mathbb{Q}$-point of D, the function field of D is formally real by Lemma 1.1. Since f vanishes on D by (iv) and is totally positive on $\mathbb{A}^3_{\mathbb{Q}} \setminus D$ by (iii), we see that D is the Zariski closure of $\mathrm{Sper}(\mathbb{Q}[x, y, z]/\langle f\rangle)$. As $(a')^2 g$ is positive semidefinite and vanishes at q, its differential at q vanishes and its Hessian at q is positive semidefinite (see Lemma 1.2). Hence, by (3.8) and smoothness of D at q, the Hessian of f_3 at q is positive semidefinite of rank ≥ 2.

By the above, there are only finitely many closed points $p_1, \dots, p_r$ in $\mathbb{A}^3_{\mathbb{Q}} \setminus \{o\}$, all distinct from q, such that f_3 is not a sum of squares in $\mathcal{O}_{\mathbb{A}^3_{\mathbb{Q}},p}$. By Lemma 3.10 below, there exists a birational morphism $\pi : \mathbb{A}^3_{\mathbb{Q}} \to \mathbb{A}^3_{\mathbb{Q}}$ such that o and q are in the open subset over which π is an isomorphism, and such that none of the p_i are in the image of π. After a change of coordinates, we may assume that $\pi(o) = (o)$. After replacing f_3, q and D with $\pi^* f_3, \pi^{-1}(q)$ and the strict transform of D by π, properties (i), (ii), (iii), (iv) and (v) are still satisfied, and (vi) now holds. □

Lemma 3.10 *Fix $n \geq 2$. Let $p_1, \dots, p_r, q_1, \dots, q_s \in \mathbb{A}^n_k$ be distinct closed points. Then there exists a birational morphism $\pi : \mathbb{A}^n_k \to \mathbb{A}^n_k$ such that the p_i are not in the image of π and the q_j are in the open subset above which π is an isomorphism.*

Proof Arguing by induction on r, we may assume that $r = 1$. After a general linear change of coordinates, we may assume that the first coordinate of p_1 is distinct from the first coordinates of each of the q_i, and that the n-th coordinate of p_1 is nonzero. Let P be the minimal polynomial of the first coordinate of p_1. Then one may define π by setting $\pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, P(x_1)x_n)$. □

We may now give our second application of Lemma 3.1.

Theorem 3.11 *There exists a positive semidefinite polynomial $f \in \mathbb{R}[x, y, z]$ that is a sum of squares in $\mathbb{R}[[x, y, z]]$ and in $\mathbb{R}[x, y, z]_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset \mathbb{R}[x, y, z]$ distinct from $\langle x, y, z\rangle$, but that is not a sum of squares in $\mathbb{R}[x, y, z]_{\langle x,y,z\rangle}$.*

Proof Let f_3, I_D and D be as in Proposition 3.9 applied with $k = \mathbb{R}$. Set $f := f_3$. In view of Proposition 3.9 (iii), (v) and (vi), we only need to show that f is not a sum of squares in $\mathbb{R}[x, y, z]_{\langle x,y,z\rangle}$.

Proposition 3.9 (i) and (ii) and Lemma 1.1 imply that the function field of D is formally real. In view of Proposition 3.9 (i) and (iv), one may apply Lemma 3.1 with $I = I_D$ to show that f is not a sum of squares in $\mathbb{R}[x, y, z]_{\langle x,y,z\rangle}$. □

3.6 Bad points on varieties

We extend the example of Proposition 3.9 first to higher dimensions in Proposition 3.12, then to arbitrary varieties in Theorem 3.13.

Proposition 3.12 *For all $n \geq c \geq 3$, there exist $f_4 \in k[x_1, \dots, x_n]$ and an ideal $I_Z \subset k[x_1, \dots, x_n]$ such that, setting $W := \{(0, \dots, 0)\} \times \mathbb{A}^{n-c}_k \subset \mathbb{A}^n_k$, the following assertions hold:*

- *(i) The variety $Z \subset \mathbb{A}^n_k$ defined by I_Z is geometrically integral of dimension $n - c + 1$ and contains W. The variety $Z \setminus W$ has a smooth k-point.*
- *(ii) Let η be the generic point of W. One has $f_4 \in I_Z$ and $f_4 \notin I^2_{Z,\eta}$.*
- *(iii) The polynomial f_4 is positive semidefinite and totally positive on $\mathbb{A}^n_k \setminus Z$.*
- *(iv) The polynomial f_4 is a sum of squares in $\widehat{\mathcal{O}}_{\mathbb{A}^n_k,\eta}$.*
- *(v) The polynomial f_4 is a sum of squares in $\mathcal{O}_{\mathbb{A}^n_k,p}$ for all $p \in \mathbb{A}^n_k \setminus W$.*

Proof Let f_3, I_D and D be as in Proposition 3.9. We consider the subvariety $Z := \{(0, \dots, 0)\} \times D \times \mathbb{A}^{n-c}_k$ of $\mathbb{A}^{c-3}_k \times \mathbb{A}^3_k \times \mathbb{A}^{n-c}_k = \mathbb{A}^n_k$, and we let I_Z be the ideal of Z. View f_3 as a function on $\mathbb{A}^{c-3}_k \times \mathbb{A}^3_k \times \mathbb{A}^{n-c}_k = \mathbb{A}^n_k$ by pull-back from the second factor, and define $f_4 := f_3 + \sum_{j=1}^{c-3} x_j^2$. Assertions (i), (ii), (iii) and (iv) are consequences of Proposition 3.9. Assertion (v) follows from Proposition 3.9 (vi) if $p \notin \mathbb{A}^{c-3}_k \times \{(0, \dots, 0)\} \times \mathbb{A}^{n-c}_k$ and from (iii) and Theorem 1.3 if $p \notin Z$. □

Now comes the main theorem of this section.

Theorem 3.13 *Let X be an affine variety over k and let $x \in X$ be a regular point. Define $A := \mathcal{O}_{X,x}$, with maximal ideal $\mathfrak{m}$. Assume that* $\dim(A) \geq 3$ *and that* $\mathrm{Frac}(A)$ *is formally real. Then there exists $f \in \mathcal{O}(X)$ such that:*

- *(i) The regular function f is a sum of squares in $\widehat{A}_{\mathfrak{m}}$.*
- *(ii) For all prime ideals $\mathfrak{p} \neq \mathfrak{m}$ of A, the function f is a sum of squares in $A_{\mathfrak{p}}$.*
- *(iii) But f is not a sum of squares in A.*

Proof At any point of the proof, we may replace X by an affine open neighbourhood $V \subset X$ of x. To see it, suppose that $f \in \mathcal{O}(V)$ satisfies (i), (ii) and (iii). Choose $f' \in \mathcal{O}(X)$ that does not vanish at x with the property that $f'f \in \mathcal{O}(V)$ lifts to an element $f'' \in \mathcal{O}(X)$. Then $f'f'' \in \mathcal{O}(X)$ also satisfies (i), (ii) and (iii) since $(f'f'')|_V = (f'|_V)^2 f$. As a consequence, we may assume X to be smooth and irreducible. Replacing k with its algebraic closure in $k(X)$, we may assume that X is geometrically irreducible. We set $n := \dim(X)$ and $c := \dim(A)$.

Let f_4, Z and W be as in Proposition 3.12 and let $q \in (Z \setminus W)(k)$ be a smooth k-point (see Proposition 3.12 (i)). Let $\overline{X}$ be a smooth projective compactification of X, let $Y \subset \overline{X}$ be the closed integral subvariety whose generic point is x, and let $\overline{Z}$ and $\overline{W}$ be the closures of Z and W in $\mathbb{P}^n_k$. Choose homogeneous coordinates $[y_1 : \dots : y_{n+1}]$ of $\mathbb{P}^n_k$ with $\overline{W} = \{y_1 = \dots = y_c = 0\}$ and $q = \{y_2 = \dots = y_{n+1} = 0\}$.

By the Artin-Lang homomorphism theorem [3, Theorem 4.1.2] applied over the real closure of k associated with the restriction of an ordering of $\mathrm{Frac}(A)$, one may choose a closed point $p \in X \setminus (Y \cap X)$ whose residue field is formally real.

Let $\mathcal{L}$ be a very ample line bundle on $\overline{X}$. Choose $e \gg 0$, and let $\sigma_1, \ldots, \sigma_{n+1}$ be sections in $H^0(\overline{X}, \mathcal{L}^{\otimes e})$ such that $\sigma_1, \ldots, \sigma_c$ vanish on Y, such that $\sigma_2, \ldots, \sigma_{n+1}$ vanish on p, and that are general among the sections satisfying these properties.

Lemma 3.14 *The following holds:*

(a) The formula $t \mapsto [\sigma_1(t) : \cdots : \sigma_{n+1}(t)]$ defines a morphism $\sigma : \overline{X} \to \mathbb{P}^n_k$.
(b) The morphism σ is finite flat, and étale at $p \in \overline{X}$.
(c) One has $\sigma(Y) = \overline{W}$ and $\sigma(p) = q$. The point p is a smooth point of $\sigma^{-1}(\overline{Z})$.
(d) The subvariety $\sigma^{-1}(\overline{Z}) \subset \overline{X}$ is geometrically integral.

Proof (a) Let $\mathcal{I}_Y$, $\mathcal{I}_{\{p\}}$ and $\mathcal{I}_{Y\cup\{p\}}$ be the ideal sheaves of Y, of $\{p\}$ and of $Y \cup \{p\}$ in $\overline{X}$. Since $e \gg 0$, the sheaves $\mathcal{I}_Y \otimes \mathcal{L}^{\otimes e}$, $\mathcal{I}_{\{p\}} \otimes \mathcal{L}^{\otimes e}$ and $\mathcal{I}_{Y\cup\{p\}} \otimes \mathcal{L}^{\otimes e}$ are all globally generated. It follows that $\{\sigma_1 = 0\}$ does not contain p. It then also follows, by induction on $1 \leq i \leq n+1$, that σ_i does not vanish identically on any irreducible component of $\{\sigma_1 = \cdots = \sigma_{i-1} = 0\}$, and hence that $\{\sigma_1 = \cdots = \sigma_i = 0\}$ has dimension $n-i$. When $i = n+1$, this means that the σ_i have no common zero.

(b) Each fiber of σ has the property that one of the σ_i does not vanish at all on it. Since $\mathcal{L}$ is ample and $\overline{X}$ is proper, this shows that no fiber of σ may be positive-dimensional. The morphism σ is thus quasi-finite, hence finite since it is proper. That σ is flat now follows from [22, Theorem 23.1]. As the sheaves $\mathcal{I}_{\{p\}} \otimes \mathcal{L}^{\otimes e}$ and $\mathcal{I}_{Y\cup\{p\}} \otimes \mathcal{L}^{\otimes e}$ are globally generated and the σ_i are general, the differentials of $\sigma_2, \ldots, \sigma_{n+1}$ at p are linearly independent. The fiber $\{\sigma_2 = \cdots = \sigma_{n+1} = 0\}$ of σ through p is thus smooth of dimension 0 at p. This completes the verification that σ is étale at p.

(c) The inclusion $\sigma(Y) \subset \overline{W}$ holds by our choice of $\sigma_1, \ldots, \sigma_c$. Since σ is finite by (b), a dimension argument shows that $\sigma(Y) = \overline{W}$. Our choice of $\sigma_2, \ldots, \sigma_{n+1}$ implies that $\sigma(p) = q$. Since q is a smooth point of $\overline{Z}$, we deduce from (b) that p is a smooth point of $\sigma^{-1}(\overline{Z})$.

(d) Assertion (d) is a consequence of Bertini's irreducibility theorem. In what follows, we explain how to reduce it to the classical statement [19, Théorème 6.3 4)].
The subvariety $\sigma^{-1}(\overline{Z})$ of $\overline{X}$ has no embedded point by [22, Theorem 23.2] which applies by flatness of σ, and has p as a smooth closed point by (c). To prove (d), it thus suffices to verify that $\sigma^{-1}(\overline{Z})$ is geometrically irreducible. Define $\Omega := \{\sigma_1 \neq 0\} \subset \overline{X}$. By finiteness of σ, none of the irreducible components of $\sigma^{-1}(\overline{Z})$ lie over the hyperplane $\{y_1 = 0\}$, so we only need to show that $\sigma^{-1}(\overline{Z}) \cap \Omega$ is geometrically irreducible.
Consider the open subset $\Theta := \{y_1 \neq 0\} \subset \overline{Z}$. For $2 \leq i \leq n+1$, define $z_i := y_i/y_1 \in \mathcal{O}(\Theta)$ and $g_i := \sigma_i/\sigma_1 \in \mathcal{O}(\Omega)$. The variety $\sigma^{-1}(\overline{Z}) \cap \Omega$ may be naturally identified with the zero locus in $\Omega \times \Theta$ of the n equations $(z_i - g_i)_{2\leq i\leq n+1}$. Define $\Sigma_i := \{z_2 - g_2 = \cdots = z_i - g_i = 0\} \subset \Omega \times \Theta$, so that $\sigma^{-1}(\overline{Z}) \cap \Omega = \Sigma_{n+1}$. As $\mathcal{I}_{\{p\}} \otimes \mathcal{L}^{\otimes e}$ and $\mathcal{I}_{Y\cup\{p\}} \otimes \mathcal{L}^{\otimes e}$ are globally generated and the σ_i are general, the differentials of the g_i at p are general. It follows that Σ_i is smooth of dimension $2n - c - i + 2$ at p and that the differential at p of the first projection $\pi_i : \Sigma_i \to \Omega$ has maximal rank.
We will prove by induction on $1 \leq i \leq n+1$ that Σ_i is geometrically irreducible. Assertion (d) will follow by taking $i = n+1$. In view of Proposition 3.12 (i), both X and $\overline{Z}$ are geometrically irreducible, hence so is $\Sigma_1 = \Omega \times \Theta$. This shows that the base case of the induction is valid.

As for the induction step, assume that Σ_{i-1} is geometrically irreducible. Let $(\tau_j^{(i)})_{1\leq j\leq m_i}$ be a basis of $H^0(\overline{X}, \mathcal{I}_{Y\cup\{p\}}\otimes\mathcal{L}^{\otimes e})$ if $2\leq i\leq c$ (resp. a basis of $H^0(\overline{X}, \mathcal{I}_{\{p\}}\otimes\mathcal{L}^{\otimes e})$ if $c+1\leq i\leq n+1$). Set $h_j^{(i)} := \tau_j^{(i)}/\sigma_1 \in \mathcal{O}(\Omega)$. Consider the morphism $\rho_i : \Sigma_{i-1}\to\mathbb{A}_k^{m_i+1}$ given by $t\mapsto (h_1^{(i)}(t),\dots,h_{m_i}^{(i)}(t), z_i(t))$. Since σ_i was chosen general, the subvariety $\Sigma_i\subset\Sigma_{i-1}$ identifies with the inverse image by ρ_i of a general affine hyperplane of $\mathbb{A}_k^{m_i+1}$. The facts verified above that Σ_{i-1} is smooth of dimension $2n-c-i+3$ at p and that the differential of π_{i-1} at p has maximal rank imply that the image of π_{i-1} has dimension $\min(n, 2n-c-i+3)$. In particular, this image cannot be included in $Y\cup\{p\}$. Since $e\gg 0$, the linear system generated by the $\tau_j^{(i)}$ induces an embedding of $\overline{X}\setminus(Y\cup\{p\})$. It follows that the transcendence degree of the subfield of $k(\Sigma_{i-1})$ generated by the $h_j^{(i)}$ is equal to the dimension $\min(n, 2n-c-i+3)$ of the image of π_{i-1}, hence is ≥ 2. We deduce that the image of ρ_i has dimension ≥ 2. Bertini's irreducibility theorem as stated in [19, Théorème 6.3 4)] shows that Σ_i is geometrically irreducible. This concludes the induction and the proof of the lemma. □

We resume the proof of Theorem 3.13. Let $U\subset\mathbb{P}_k^n$ be an open affine subset containing q and the generic point of $\overline{W}$, and such that f_4 is regular on U. Set $V := \sigma^{-1}(U)\cap X\subset\overline{X}$. It is an open affine subset (note that σ is affine by Lemma 3.14 (b)) containing q and the generic point x of Y by Lemma 3.14 (c).

We now define $f := \sigma^*(f_4|_U)\in\mathcal{O}(V)$ and check one by one the claims of Theorem 3.13. Assertions (i) and (ii) follow from Proposition 3.12 (iv) and (v). To prove assertion (iii), we consider the ideal $I\subset A$ of functions vanishing on the subscheme $\sigma^{-1}(\overline{Z})\times_{\overline{X}}\mathrm{Spec}(A)$ of $\mathrm{Spec}(A)$, and we apply Lemma 3.1. Let us check its hypotheses. That I is radical stems from Lemma 3.14 (d). Since p is a smooth point of $\sigma^{-1}(\overline{Z})$ with formally real residue field by Lemma 3.14 (c), and since $\sigma^{-1}(\overline{Z})$ is integral by Lemma 3.14 (d), we deduce from Lemma 1.1 that the function field of $\sigma^{-1}(\overline{Z})$ is formally real, hence that $\mathrm{Sper}(A/I)$ is Zariski-dense in $\mathrm{Spec}(A/I)$. That $f\in I$ follows from the first statement of Proposition 3.12 (ii). Finally, since σ is flat by Lemma 3.14 (b), that $f\notin I^2$ may be deduced from the second statement of Proposition 3.12 (ii) by applying [22, Theorem 7.5 (ii)]. Lemma 3.1 now applies and shows that f is not a sum of squares in A. □

3.7 An additional example

It is not straightforward to extract a concrete polynomial from the proof of Theorem 3.11. Giving an example in ≥ 4 variables is much easier, as the next theorem shows.

We use a variation on Motzkin's famous polynomial [24, p. 217]: we have only modified its coefficients to be elements of $\mathbb{R}[w]$ instead of real numbers.

Theorem 3.15 *The polynomial* $f = x^6 + w^2y^2z^4 + w^2y^4z^2 + (1-w)x^2y^2z^2$ *is positive semidefinite and a sum of squares in* $\mathbb{R}[[w,x,y,z]]$*, but it is not a sum of squares in* $\mathbb{R}(w)[[x,y,z]]$*, hence not in* $\mathbb{R}[w,x,y,z]_{\langle w,x,y,z\rangle}$ *either.*

Proof The inequality between the arithmetic and geometric means of x_0^6, $w_0^2y_0^2z_0^4$ and $w_0^2y_0^4z_0^2$ implies that $f(w_0,x_0,y_0,z_0)\geq(3w_0^{4/3}-w_0+1)x_0^2y_0^2z_0^2\geq 0$ for all $(w_0,x_0,y_0,z_0)\in\mathbb{R}^4$. This shows that f is positive semidefinite. The polynomial f is a sum of squares in $\mathbb{R}[[w,x,y,z]]$ because $1-w$ is a square in this ring.

Assume for contradiction that f is a sum of squares in $\mathbb{R}(w)[[x,y,z]]$. Then, for all but countably many $w_0\in\mathbb{R}$ the polynomial $f(w_0,x,y,z)\in\mathbb{R}[x,y,z]$ is a sum of squares in $\mathbb{R}[[x, y, z]]$. Fix such a w_0 with $w_0 > 1$ and define the polynomial $g(y, z) := f(w_0, 1, y, z) \in \mathbb{R}[y, z]$. One can then write $g(y, z) = \sum_i h_i^2$ for some $h_i \in \mathbb{R}[y, z]$. Setting $y = 0$, one shows that no monomial of the form z^a can appear in the h_i. By symmetry, no monomial of the form y^a can appear in the h_i. The identity $g(y, z) = \sum_i h_i^2$ now implies that the coefficient of y^2z^2 in g is nonnegative, which contradicts our choice of $w_0 > 1$.

That f is not a sum of squares in $\mathbb{R}[w, x, y, z]_{\langle w,x,y,z\rangle}$ follows, since $\mathbb{R}(w)[[x, y, z]]$ is the completion of the localization of $\mathbb{R}[w, x, y, z]_{\langle w,x,y,z\rangle}$ at the ideal $\langle x, y, z\rangle$. □

4 Regular local rings without bad points

In this last section, we construct examples of regular local rings in which all positive semidefinite elements are sums of squares. The regular local rings A that we consider have the peculiar feature that their function field may be ordered in a unique way. The idea of the construction is to start with a regular local ring B and with an ordering ξ of $\mathrm{Frac}(B)$, and to choose A to be a maximal sub-B-algebra of the Henselization B^{h} of B to which ξ lifts.

Theorem 4.1 *For all $n \geq 0$, there exists a regular local $\mathbb{R}$-algebra A of dimension n such that* $\mathrm{Sper}(A)$ *consists of exactly one point, which is a field ordering of* $\mathrm{Frac}(A)$.

Proof If $n = 0$, take $A := \mathbb{R}$. If $n \geq 1$, we split the proof in seven steps.

Step 1 Construction of the local ring A.

Let $y \in \mathbb{P}^n_{\mathbb{R}}$ be a closed point with complex residue field, define $B := \mathcal{O}_{\mathbb{P}^n_{\mathbb{R}},y}$, let $\mathfrak{m}$ be the maximal ideal of B and set $L := \mathbb{R}(x_1, \ldots, x_n) = \mathrm{Frac}(B)$. We recall the definition of the Henselization B^{h} of B (see [15, Définition 18.6.5]). Let $(B_i)_{i\in I}$ be a set of representatives of all isomorphism classes of local essentially étale B-algebras $u_i : B \to B_i$ such that u_i induces an isomorphism of residue fields. Say that $i \leq i'$ if there exists a (necessarily unique) morphism of B-algebras $B_i \to B_{i'}$. The set I is partially ordered and filtered. One defines $B^{\mathrm{h}} := \varinjlim_{i\in I} B_i$.

Let $\alpha_1, \ldots, \alpha_n$ be n elements of $\mathbb{R}[[t]]$ that are algebraically independent over $\mathbb{R}$ (see [23, Lemma 1]). They give rise to a morphism $\alpha : \mathrm{Spec}(\mathbb{R}[[t]]) \to \mathbb{A}^n_{\mathbb{R}}$. Since the α_i are algebraically independent, the morphism α induces an inclusion $\alpha^* : L \hookrightarrow \mathbb{R}((t))$. The field ordering of $\mathbb{R}((t))$ for which t is a positive infinitesimal restricts, by the inclusion α^*, to an ordering ξ of L.

Define $L_i := \mathrm{Frac}(B_i)$. Consider all the subsets $J \subset I$ such that:

(i) For all $i \in J$, the ordering ξ may be extended to an ordering of L_i.
(ii) For all $i, i' \in J$, there exists $i'' \in J$ with $i'' \geq i$ and $i'' \geq i'$.

Since an increasing union of such subsets again has these two properties, we may use Zorn's lemma to choose one that is maximal with respect to the inclusion. Call it J. It is partially ordered and filtered, and we consider the B-algebra $A := \varinjlim_{i\in J} B_i$.

The arguments used in [15, Théorème 18.6.6, Corollaire 18.6.10] to show that B^{h} is a flat local regular B-algebra with maximal ideal $\mathfrak{m}B^{\mathrm{h}}$ and residue field $\mathbb{C}$ show, *mutatis mutandis*, that A is a flat local regular B-algebra with maximal ideal $\mathfrak{m}A$ and residue field $\mathbb{C}$. Its dimension is n by [14, Proposition 6.1.1].

Step 2 Construction of an ordering ζ of $K := \mathrm{Frac}(A)$.

Consider the set $Z_i \subset \mathrm{Sper}(L_i)$ of orderings whose image in $\mathrm{Sper}(L)$ is ξ. Since $\mathrm{Sper}(L)$ is Hausdorff [20, VIII, Theorem 6.3], its point ξ is closed. It follows from [20, Corollary p. 272] that $Z_i \subset \mathrm{Sper}(L_i)$ is closed, hence compact by [20, VIII, Theorem 6.3]. Since the Z_i are nonempty for $i \in J$ by property (i) of Step 1, the subset $Z := \varprojlim_{i\in J} Z_i$ of $\mathrm{Sper}(K) = \varprojlim_{i\in J} \mathrm{Sper}(L_i)$ is nonempty by Tychonoff's theorem. This shows that the field K is formally real. We choose a point $\zeta \in Z$.

Step 3 In the remainder of the proof, we suppose that $\mathrm{Sper}(A)$ contains a point χ distinct from ζ, and we use this hypothesis to contradict the maximality of J.

In Step 3, we show that one may assume that χ is an ordering of K.

The point $\chi \in \mathrm{Sper}(A)$ corresponds to an ordering of $\kappa := \mathrm{Frac}(A/\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$. Set $c := \dim(A_{\mathfrak{p}})$ and let $(t_1, \dots, t_c)$ be a regular system of parameters in $A_{\mathfrak{p}}$. By Cohen's structure theorem [22, Theorem 29.7], there exists an isomorphism $\widehat{A}_{\mathfrak{p}} \simeq \kappa[[t_1, \dots, t_c]]$, which induces inclusions $K \subset \kappa((t_1, \dots, t_c)) \subset \kappa((t_1)) \dots ((t_c))$. Any ordering of a field k extends in two ways to an ordering of $k((t))$, one for which t is a positive infinitesimal and one for which t is a negative infinitesimal. If $c \geq 1$, it follows that K admits at least two orderings, one for which t_1 is positive and one for which t_1 is negative. Replacing χ by one of these, we may assume that $c = 0$, i.e., that ζ and χ are two distinct orderings of K.

Step 4 Study of the valuations associated with the orderings ζ and χ.

Let $f \in K$ be such that $f \succ_\zeta 0$ but $f \prec_\chi 0$. There exists $j \in J$ such that $f \in L_j$, and we fix such an element j. Since B_j is a local essentially étale B-algebra, there exist a projective variety X over $\mathbb{R}$, a closed point $x \in X$, a morphism $\pi : X \to \mathbb{P}^n_{\mathbb{R}}$ étale at x such that $\pi(x) = y$, and an isomorphism of B-algebras $B_j \simeq \mathcal{O}_{X,x}$. In particular, $L_j \simeq \mathbb{R}(X)$. By resolution of singularities, we may assume that X is smooth over $\mathbb{R}$. After multiplying f by a square, we may assume that $f \in B_j$. Let $D \subset X$ be the effective Cartier divisor obtained by taking the Zariski closure in X of the subscheme $\{f = 0\} \subset \mathrm{Spec}(B_j)$.

Let $V_\prec := \{g \in L_j \mid -r \prec g \prec r \text{ for some } r \in \mathbb{R}\}$ be the valuation ring associated with an ordering $\prec$ of L_j (see [3, Proposition 10.1.13]). Its maximal ideal is $\mathfrak{m}_\prec := \{g \in L_j \mid -r \prec g \prec r \text{ for all } r \in \mathbb{R}_{>0}\}$ and its residue field is isomorphic to $\mathbb{R}$. We let $v_\prec$ be the corresponding valuation of L_j and $c_\prec \in X(\mathbb{R})$ be its center. Since ζ restricts to ξ on L, the restriction of v_ζ to L is induced by the t-adic valuation on $\mathbb{R}((t))$ and the inclusion $\alpha^* : L \hookrightarrow \mathbb{R}((t))$. As L_j is a finite extension of L, we deduce that v_ζ is a discrete valuation. If $c_\zeta \in D$, replace X with its blow-up at c_ζ, and D with its strict transform in the blow-up. This has the effect of decreasing the image by the valuation v_ζ of a local equation of D at c_ζ. As v_ζ is discrete, repeating this procedure finitely many times ensures that $c_\zeta \notin D$.

Step 5 Construction of a subset $J' \subset I$.

Let S be the spectrum of the semilocal ring of X at the points x, c_ζ and c_χ. We note that x is distinct from either c_ζ or c_χ since its residue field is $\mathbb{C}$ (but c_ζ and c_χ might coincide). As $\mathcal{O}_X(-D)$ is invertible and as any locally free module of constant rank over the spectrum of a semilocal ring is free, the ideal sheaf $\mathcal{O}_X(-D)|_S$ is principal, generated by some $g \in H^0(S, \mathcal{O}_X(-D)|_S)$. Since $c_\zeta \notin D$, one has $g(c_\zeta) \neq 0$ and we may assume, after maybe replacing g with $-g$, that $g(c_\zeta) > 0$. In particular, $g \succ_\zeta 0$. If $c_\zeta = c_\chi$ or if $g \succ_\chi 0$, define $h = 1$. If $c_\zeta \neq c_\chi$ and $g \prec_\chi 0$, let $h \in \mathcal{O}(S)^*$ be an invertible element such that $h(x) = h(c_\zeta) = 1$ and $h(c_\chi) = -1$, so that $h \succ_\zeta 0$ and $h \prec_\chi 0$. Then the element

$e := fh/g \in L_j$ has the property that $e \succ_\zeta 0$ and $e \prec_\chi 0$. Moreover, $e \in (B_j)^*$ because both f and g generate the invertible sheaf $\mathcal{O}_X(-D)$ at the point x.

Consider the ring A' obtained by localizing $A[z]/\langle z^2 - e\rangle$ at one of its maximal ideals. We define $J' \subset I$ to be the set of $i \in I$ such that there exists a morphism of B-algebras $B_i \to A'$.

Step 6 The subset $J' \subset I$ satisfies the properties (i) and (ii) of Step 1.

Since $e \prec_\chi 0$, the element $e \in K$ is not a square, and it follows that A' is integral with fraction field $K' := K[z]/\langle z^2 - e\rangle$. Since $e \succ_\zeta 0$, the element e has a square root in the real closure of K associated with ζ. This shows that ζ may be extended to an ordering ζ' of K'. If $i \in J'$, the restriction of ζ' to L_i is an ordering of L_i that extends ξ. The shows (i).

Choose $i, i' \in J'$. The two morphism $B_i \to A'$ and $B_{i'} \to A'$ induce a morphism $B_i \otimes_B B_{i'} \to A'$. The localization of $B_i \otimes_B B_{i'}$ at its maximal ideal induced by the maximal ideal of A' is a local essentially étale B-algebra with residue field $\mathbb{C}$ that admits a morphism to A'. It is thus of the form $B_{i''}$ for some $i'' \in J'$, and the element $i'' \in J'$ satisfies $i'' \geq i$ and $i'' \geq i'$. We have verified the property (ii).

Step 7 The subset $J' \subset I$ contradicts the maximality of J.

It is clear that $J \subset J'$ since for all $i \in J$, there exist morphisms of B-algebras $B_i \to A \to A'$.

It remains to show that $J' \neq J$. Consider the ring B'_j obtained by localizing $B_j[z]/\langle z^2 - e\rangle$ at its maximal ideal induced by the maximal ideal of A'. The ring B'_j is a local B-algebra with residue field $\mathbb{C}$ that is essentially étale because $e \in (B_j)^*$. It is therefore of the form $B_{j'}$ for some $j' \in I$. Since there exists a morphism $B'_j \to A'$ by construction, we see that $j' \in J'$. However j' cannot belong to J. Indeed, if it were the case, there would exist a morphism of B-algebras $B'_j \to A$. This is impossible since e is a square in B'_j but $e \prec_\chi 0$. □

That the positive semidefinite elements in the regular local rings constructed in Theorem 4.1 are sums of squares is a straightforward application of Scheiderer's results on sums of squares in local rings.

Theorem 4.2 *For all $n \geq 0$, there exists a regular local $\mathbb{R}$-algebra A of dimension n with the following properties:*

(i) All positive semidefinite elements of A are sums of squares in A.
(ii) The field $\mathrm{Frac}(A)$ *is formally real.*

Proof Let A be the $\mathbb{R}$-algebra constructed in Theorem 4.1. It satisfies (ii). To verify (i), choose a nonzero positive semidefinite element $f \in A$. Since the only point of $\mathrm{Sper}(A)$ is supported on the ideal (0) of A, the space $\mathrm{Sper}(A/\langle f^2\rangle)$ is empty. It follows from the real Nullstellensatz that -1 is a sum of squares in $A/\langle f^2\rangle$ (see [3, Theorem 4.3.7]). As a consequence, $f = ((f+1)/2)^2 - ((f-1)/2)^2$ is a sum of squares in $A/\langle f^2\rangle$, hence a sum of squares in A by [28, Corollary 2.3 (b)]. □

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