ON THE SUBVARIETIES WITH NONSINGULAR REAL LOCI OF A REAL ALGEBRAIC VARIETY

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ABSTRACT. Let X be a smooth projective real algebraic variety. We give new positive and negative results on the problem of approximating a submanifold of the real locus of X by real loci of subvarieties of X, as well as on the problem of determining the subgroups of the Chow groups of X generated by subvarieties with nonsingular real loci, or with empty real loci.

INTRODUCTION

In this article, we study the subvarieties with nonsingular real loci of a smooth projective real algebraic variety X. We consider their classes in the Chow groups of X and whether their real loci can approximate a fixed $\mathcal{C}^\infty$ submanifold of $X(\mathbb{R})$.

Let c and d denote the codimension and the dimension of these subvarieties. The guiding principle of our results is that, for each of the three problems that we will consider in §§0.1-0.3, subvarieties with nonsingular real loci are abundant when $d < c$ (see Theorems 0.1, 0.4 and 0.6), but may be scarce for $d \geq c$ (see Theorems 0.2, 0.5 and 0.7). The geometric rationale behind this principle, in the spirit of Whitney's theorem in differential geometry [Whi36], is that a d-dimensional variety mapped generically to X is expected not to self-intersect, hence to have nonsingular image in X, only if $d < c$.

0.1. **Chow groups.** It is an old question, going back to Borel and Haefliger [BH61, §5.17], to decide when the Chow group $\mathrm{CH}_d(X)$ of a smooth projective variety X of dimension $c+d$ over a field is generated by classes of smooth subvarieties of X. This is not true in general (a first counterexample appeared in [HRT74, Theorem 1], for $c = 2$ and $d = 7$). The main positive result, due to Hironaka [Hir68, Theorem p. 50], gives an affirmative answer if $d < c$ and $d \leq 3$ (his arguments now work over any infinite perfect field, thanks to [CP09]). One may wonder whether Hironaka's theorem holds as soon as $d < c$.

In real algebraic geometry, it is natural to consider, more generally, subvarieties that are smooth along their real loci. Our first theorem is a variant of Hironaka's result in this setting, valid for all values of (c, d) such that $d < c$.

Theorem 0.1 (Theorem 2.3)**.** *Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$. If $d < c$, then the group $\mathrm{CH}_d(X)$ is generated by classes of closed subvarieties of X that are smooth along their real loci.*

Our proof is based on the smoothing technique developed by Hironaka in [Hir68]. We need to refine it for two reasons: to control real loci, and to deal with the singularities that inevitably appear in the course of our proof if $d > 3$. To do so, we rely on the theory of linkage, as developed by Peskine and Szpiro [PS74] and Huneke and Ulrich [HU85]. Our argument works over an arbitrary real closed field.

Our second theorem shows that Theorem 0.1 is optimal, for infinitely many values of c. We let $\alpha(m)$ denote the number of ones in the dyadic expansion of m.

Theorem 0.2 (Theorem 4.14)**.** *If $d \geq c$ are such that $\alpha(c+1) \geq 3$, there exists an abelian variety X of dimension $c+d$ over $\mathbb{R}$ such that $\mathrm{CH}_d(X)$ is not generated by classes of closed subvarieties of X that are smooth along their real loci.*

Theorem 0.2 is entirely new. The hypothesis that $\alpha(c+1) \geq 3$ cannot be weakened to $\alpha(c+1) \geq 2$. Indeed, Kleiman has showed that the Chow group of codimension 2 cycles on a smooth projective fourfold or fivefold over an infinite field is generated by classes of smooth subvarieties (see [Kle69, Theorem 5.8], where the hypothesis that the base field is algebraically closed may be discarded as a theory of Chow groups and Chern classes is now available in the required generality [Ful98]).

Let us briefly explain the principle of the proof of Theorem 0.2 in the key case where $c = d$. Assume to simplify that $\beta \in \mathrm{CH}_d(X)$ is the class of closed subvariety $Y \subset X$ which is smooth along $Y(\mathbb{R})$. Let $g : W \to X$ be a morphism obtained by resolving the singularities of Y. The double locus of g, which is well-defined as a 0-cycle on W, has degree divisible by 4. Indeed, double points come two by two, and each such pair has a distinct complex conjugate. On the other hand, a double point formula due to Fulton [Ful98, §9.3] computes the degree of this double locus in terms of the Chern classes of X and W and of the self-intersection of Y in X. Divisibility results for Chern numbers due to Rees and Thomas [RT77, Theorem 3] now give restrictions on β, which sometimes lead to a contradiction.

This strategy applies as well over $\mathbb{C}$, and yields new examples of smooth projective complex varieties whose Chow groups are not generated by smooth subvarieties.

Theorem 0.3 (Theorem 4.17)**.** *If $d \geq c$ are such that $\alpha(c+1) \geq 3$, there exists a smooth projective variety X of dimension $c+d$ over $\mathbb{C}$ such that $\mathrm{CH}_d(X)$ is not generated by classes of smooth closed subvarieties of X.*

This complements the counterexamples of [HRT74, Theorem 1], [Deb95, Théorème 6] and [BD22, Theorem 1.2] to the question of Borel and Haefliger. Theorem 0.3 is closely related to [RT80, Proposition 1], where the easier problem of showing that a cycle is not rationally equivalent to the class of a smooth subvariety (as opposed to a linear combination of classes of smooth subvarieties) is considered.

0.2. **The kernel of the Borel–Haefliger map.** If X is a smooth projective variety of dimension $c+d$ over $\mathbb{R}$, a related problem is to determine the subgroup of $\mathrm{CH}_d(X)$ generated by classes of subvarieties with no real points. This subgroup is included in the kernel $\mathrm{CH}_d(X)_{\mathbb{R}-\mathrm{hom}}$ of the Borel–Haefliger cycle class map $\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}), \mathbb{Z}/2)$ (see [BH61] or [BW20a, §1.6.2]), which associates with the class of an integral closed subvariety $Y \subset X$ the homology class of its real locus. One may wonder when these two subgroups coincide. This question was known to have a positive answer for $c = 1$ (Bröcker [Brö80], see also [Sch95, §4]), for $d = 0$ (Colliot-Thélène and Ischebeck [CTI81, Proposition 3.2 (ii)]), and for $d = 1$ and $c = 2$ (Kucharz [Kuc04, Theorem 1.2]).

Combining our improvements of Hironaka's smoothing technique and a theorem of Ischebeck and Schülting according to which $\mathrm{CH}_d(X)_{\mathbb{R}-\mathrm{hom}}$ is generated by classes of integral closed subvarieties of X whose real locus is not Zariski-dense [IS88, Main Theorem 4.3], we obtain an affirmative answer for all values of (c, d) such that $d < c$.

Theorem 0.4 (Theorem 2.4)**.** *Let X be a smooth projective variety of dimension $c+d$ over $\mathbb{R}$. If $d < c$, then $\mathrm{CH}_d(X)_{\mathbb{R}-\mathrm{hom}}$ is generated by classes of closed integral subvarieties of X with empty real loci.*

Theorem 0.4 fails in general over non-archimedean real closed fields (see Remark 2.5). Kucharz has shown in [Kuc04, Theorem 1.1] that the hypothesis $d < c$ of Theorem 0.4 cannot be improved, for all even values of $c \geq 2$. We extend this result to all the values of c not of the form $2^k - 1$.

Theorem 0.5 (Theorem 4.16)**.** *If $d \geq c$ are such that $\alpha(c+1) \geq 2$, there exists an abelian variety X of dimension $c+d$ over $\mathbb{R}$ such that $\mathrm{CH}_d(X)_{\mathbb{R}-\mathrm{hom}}$ is not generated by classes of closed subvarieties of X with empty real loci.*

The proof of Theorem 0.5 follows the same path as that of Theorem 0.2, using additionally a new result on congruences of Chern numbers (Theorem 3.6). The hypothesis that $\alpha(c+1) \geq 2$ in Theorem 0.5 cannot be removed in general, in view of Bröcker's above-mentioned theorem [Brö80] when $c = 1$.

0.3. **Algebraic approximation.** Let X be a smooth projective variety of dimension $c+d$ over $\mathbb{R}$, and fix a closed d-dimensional $\mathcal{C}^\infty$ submanifold $j : M \hookrightarrow X(\mathbb{R})$. We now focus on the classical question whether M can be approximated by real loci of algebraic subvarieties of X (see [BCR98, Definition 12.4.10] or [AK92, §2.8]):

Property (A)*.* For all neighbourhoods $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$ of the inclusion, there exist $\phi \in \mathcal{U}$ and a closed subvariety $Y \subset X$ which is smooth along $Y(\mathbb{R})$ such that $\phi(M) = Y(\mathbb{R})$.

One must take into account the topological obstructions to the validity of (A). The finest ones are based on cobordism theory and originate from [BT80b] or [AK81]. If T is a topological space, recall that two continuous maps $f_1 : N_1 \to T$ and $f_2 : N_2 \to T$, where the N_i are d-dimensional compact $\mathcal{C}^\infty$ manifolds, are said to be *cobordant* if there exist a compact $\mathcal{C}^\infty$ manifold with boundary C, a diffeomorphism $\partial C \simeq N_1 \cup N_2$, and a continuous map $f : C \to T$ such that $f|_{N_i} = f_i$ for $i \in \{1, 2\}$. The group (for the disjoint union) of cobordism equivalence classes of such maps is the d-th *unoriented cobordism group* $MO_d(T)$ of T.

Let $MO_d^{\mathrm{alg}}(X(\mathbb{R})) \subset MO_d(X(\mathbb{R}))$ be the subgroup generated by cobordism classes of continuous maps of the form $g(\mathbb{R}) : W(\mathbb{R}) \to X(\mathbb{R})$, where $g : W \to X$ is a morphism of smooth projective varieties over $\mathbb{R}$ and W has dimension d. The following property is a necessary condition for the validity of (A).

Property (B)*.* One has $[j : M \hookrightarrow X(\mathbb{R})] \in MO_d^{\mathrm{alg}}(X(\mathbb{R}))$.

We show that it is the only obstruction to the validity of (A) for low values of d.

Theorem 0.6 (Theorem 2.7)**.** *Properties (A) and (B) are equivalent if $d < c$.*

Theorem 0.6 was already known when $d = 1$, thanks to Bochnak and Kucharz [BK03, Theorem 1.1] for $c = 2$, and to Wittenberg and the author [BW20b, Theorem 6.8] for any $c \geq 2$ (improving earlier results by Akbulut and King [AK88]). Theorem 0.6 is new for $d \geq 2$.

Our proof is based on a relative Nash–Tognoli theorem (see [BT80b, Proposition 4.1] or [AK81, Proposition 0.2]), which solves the approximation problem up to unwanted singular points. To remove these singular points, we use the refinements of Hironaka's smoothing method already mentioned in §§0.1-0.2. Hironaka's smoothing technique, as developed in [Hir68], had already been applied in the context of real algebraic approximation in the proof of [BW20b, Theorem 6.8].

We also prove that Theorem 0.6 is sharp: it may fail as soon as $d \geq c$, for infinitely many values of c. Recall that $\alpha(m)$ is the number of ones in the dyadic expansion of $m \geq 0$.

Theorem 0.7 (Theorem 4.19)**.** *If $d \geq c$ are such that $\alpha(c+1) = 2$, there exist X and M such that (A) fails but (B) holds.*

To the best of our knowledge, Theorem 0.7 features the first examples demonstrating that properties (A) and (B) are not equivalent in general. The hypothesis that $\alpha(c+1) = 2$ cannot be entirely dispensed with, as (A) and (B) are equivalent when $c = 1$ (see Proposition 5.5). The values of c for which Theorem 0.7 applies are $c \in \{2, 4, 5, 8, 9, 11, 16, \dots\}$. We have not been able to disprove the equivalence of (A) and (B) for other values of c, for instance for $c = 3$.

The proof of Theorem 0.7 uses techniques similar to that of Theorems 0.2 and 0.5. Although both properties (A) and (B) only involve real loci, the proof of Theorem 0.7 makes use in an essential way of global topological properties of sets of complex points, through their classes in the complex cobordism ring MU_*.

In [KvH09, p. 269], Kucharz and van Hamel ask whether property (A) always holds when $X = \mathbb{P}^n_{\mathbb{R}}$. The obstructions used in the proof of Theorem 0.7 show that this question would have a negative answer if one replaced $\mathbb{P}^n_{\mathbb{R}}$ with other very similar varieties, such as some products of projective spaces (property (B) always holds for these varieties by [BT80b, Remark 3 p. 103]). This demonstrates the very particular role played by projective spaces in the question of Kucharz and van Hamel.

Theorem 0.8 (Theorem 4.23)**.** *For all $k \geq 1$, property (A) fails in general for $c = d = 2^k$ and $X = \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^{2^{k+1}-1}_{\mathbb{R}}$.*

0.4. **Structure of the article.** We study linkage to expand the scope of Hironaka's smoothing technique in §1, and use it in §2 to prove Theorems 0.1, 0.4 and 0.6. Generalities about complex cobordism and an application to the divisibility of the top Segre class may be found in §3. This result and a double point formula are combined in §4 to prove Theorems 0.2, 0.3, 0.5, 0.7 and 0.8. Finally, variants of properties (A) and (B) and their interactions are considered in §5.

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0.6. **Notation and conventions.** A variety over a field k is a separated scheme of finite type over k. Smooth varieties over k are understood to be equidimensional. If $f : X \to Y$ is a morphism of varieties over k and k' is a field extension of k, we let $f(k') : X(k') \to Y(k')$ be the map induced at the level of k'-points. We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers.

All $\mathcal{C}^\infty$ manifolds are assumed to be Hausdorff and second countable. We endow the set $\mathcal{C}^\infty(M, N)$ of $\mathcal{C}^\infty$ maps between two $\mathcal{C}^\infty$ manifolds with the weak $\mathcal{C}^\infty$ topology [Hir94, p. 36].

For $m \geq 0$, we let $\alpha(m)$ be the number of ones in the dyadic expansion of m.

1. Linkage

In the whole of §1, we fix an infinite field k, a smooth projective variety V over k, a very ample line bundle $\mathcal{O}_V(1)$ on V, and a (possibly empty) Cohen–Macaulay closed subscheme $W \subset V$ of pure codimension r in V.

We study the subvarieties of V that are linked to W by complete intersections defined by sections of multiples of $\mathcal{O}_V(1)$ (§1.1), and their behaviour in families (§§1.2-1.3), focusing in particular on their images by a morphism and on their real loci when k is a real closed field (§§1.4-1.5).

If $\underline{l} = (l_1, \ldots, l_r)$ is an r-tuple of integers and if $\mathcal{F}$ is a coherent sheaf on V, we set $\mathcal{F}(\underline{l}) := \bigoplus_{i=1}^r \mathcal{F}(l_i)$. In particular, $H^0(V, \mathcal{F}(\underline{l})) = \bigoplus_{i=1}^r H^0(V, \mathcal{F}(l_i))$. A statement depending on an r-tuple of integers $\underline{l} = (l_1, \ldots, l_r)$ is said to hold for $\underline{l} \gg 0$ if it holds for $l_r \gg \cdots \gg l_2 \gg l_1 \gg 0$, i.e., if l_1 is big enough, if l_2 is big enough (depending on l_1), and so forth.

1.1. **Linked subvarieties.** Let $\mathcal{I}_W \subset \mathcal{O}_V$ be the ideal sheaf of W in V. Choose an r-tuple of integers $\underline{l} = (l_1, \ldots, l_r)$ and a section $\underline{F} \in H^0(V, \mathcal{I}_W(\underline{l}))$ such that $\underline{F} = (F_1, \ldots, F_r)$ is a regular sequence (such $\underline{F}$ always exists if $\mathcal{I}_W(\underline{l})$ is generated by its global sections, for instance for $\underline{l} \gg 0$).

Let $Z := \{F_1 = \cdots = F_r = 0\} \subset V$ be the complete intersection it defines, and let $\mathcal{I}_Z = \langle \underline{F} \rangle \subset \mathcal{O}_V$ be its ideal sheaf. Let $W' \subset V$ be the subvariety with ideal sheaf $\mathcal{I}_{W'} := (\mathcal{I}_Z : \mathcal{I}_W) \subset \mathcal{O}_V$, where a local section $s \in \mathcal{O}_V$ belongs to $(\mathcal{I}_Z : \mathcal{I}_W)$ if multiplication by s induces a morphism $\mathcal{I}_W \xrightarrow{s} \mathcal{I}_Z$.

One has $Z = W \cup W'$ set-theoretically. It is a theorem of Peskine and Szpiro [PS74, Proposition 1.3] that $W' \subset V$ is also Cohen–Macaulay of pure codimension r, and that $\mathcal{I}_W = (\mathcal{I}_Z : \mathcal{I}_{W'}) \subset \mathcal{O}_V$. In view of the symmetry of the relation between the subschemes W and W' of V, they are said to be *linked* by the regular sequence $\underline{F}$. We write $W \sim W'$, or $W \sim_{\underline{l}} W'$ if we want to emphasize that the regular sequence is a section of $\mathcal{O}_V(\underline{l})$. We also say that $W \sim W'$ is the *link* defined by $\underline{F}$.

Remarks 1.1. (i) In the whole of §1, we could have only considered links with respect to complete intersections of multidegree $(l, \ldots, l)$, with $l \gg 0$ when needed. The reason why we allow multidegrees $\underline{l} = (l_1, \ldots, l_r)$, requiring $\underline{l} \gg 0$ when needed, is to be able to apply directly the proof of [Hir68, Lemma 5.1.1] in the proof of Proposition 1.6 below.

(ii) In §§1.1–1.3, we could allow V to be any Gorenstein projective variety (see especially [PS74, Proposition 1.3]).

Lemma 1.2. *Let $x \in V$, and let $g_1, \ldots, g_r \in \mathcal{I}_{W,x} \subset \mathcal{O}_{V,x}$ be a regular sequence. Then, for an r-tuple of integers $\underline{l} \gg 0$, there exists a regular sequence $\underline{F} \in H^0(V, \mathcal{I}_W(\underline{l}))$ such that the ideals $\langle \underline{F} \rangle$ and $\langle g_1, \ldots, g_r \rangle$ of $\mathcal{O}_{V,x}$ coincide.*

Proof. Let $Y \subset V$ and $Y_i \subset V$ be the schematic closures of $\{g_1 = \cdots = g_r = 0\}$ and $\{g_i = 0\}$ in V. Let $\mathcal{I}_W, \mathcal{I}_Y, \mathcal{I}_{Y_i} \subset \mathcal{O}_V$ be the ideal sheaves of W, Y and Y_i in V and define $\mathcal{I} := \mathcal{I}_W \cap \mathcal{I}_Y$ and $\mathcal{I}_i := \mathcal{I}_W \cap \mathcal{I}_{Y_i}$. The subscheme of V defined by $\mathcal{I}$ has support $Y \cup W$, hence has pure codimension r in V, and coincides with Y in a neighbourhood of x.

Choose $\underline{l} \gg 0$ so that the sheaves $\mathcal{I}(\underline{l})$ and $\mathcal{I}_i(l_i)$ are generated by their global sections, and choose a general element $\underline{F} \in H^0(V, \mathcal{I}(\underline{l}))$. Since $\mathcal{I}_i(l_i)$ is globally generated, there exist $G_i \in H^0(V, \mathcal{I}_i(l_i)) \subset H^0(V, \mathcal{I}(l_i))$ with $\langle G_i \rangle = \langle g_i \rangle \subset \mathcal{O}_{V,x}$, hence with $\langle G_1, \ldots, G_r \rangle = \langle g_1, \ldots, g_r \rangle \subset \mathcal{O}_{V,x}$. Since $\underline{F}$ has been chosen general, we deduce that $\langle \underline{F} \rangle = \langle g_1, \dots, g_r \rangle \subset \mathcal{O}_{V,x}$. Since $\mathcal{I}(\underline{l})$ is globally generated, we also see that $\underline{F}$ forms a regular sequence. The lemma is proven. □

Lemma 1.3. *Let $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \cdots \sim_{\underline{l}_j} W_j$ be links of subschemes of V. Assume that W is a local complete intersection at $x \in W$. For all r-tuples of integers $\underline{l}_{2j+1} \gg \cdots \gg \underline{l}_{j+1} \gg 0$, there exists a chain $W_j \sim_{\underline{l}_{j+1}} W_{j+1} \sim_{\underline{l}_{j+2}} \cdots \sim_{\underline{l}_{2j+1}} W_{2j+1}$ of links of subschemes of V such that $x \notin W_{2j+1}$.*

Proof. For $1 \leq i \leq j$, let $\underline{F}_i$ be the regular sequence yielding the link $W_{i-1} \sim_{\underline{l}_i} W_i$. Thanks to Lemma 1.2, one may choose inductively, for $j+1 \leq i \leq 2j$, an r-tuple $\underline{l}_i \gg 0$, a regular sequence $\underline{F}_i \in H^0(V, \mathcal{I}_{W_{i-1}}(\underline{l}_i))$ such that the ideals $\langle \underline{F}_i \rangle$ and $\langle \underline{F}_{2j+1-i} \rangle$ of $\mathcal{O}_{V,x}$ coincide. This gives rise to a link $W_{i-1} \sim_{\underline{l}_i} W_i$ with the property that W_i and W_{2j-i} coincide in a neighourhood of x, by the symmetry of the link construction.

The subschemes W_{2j} and $W_0 = W$ then coincide in a neighbourhood of x, hence W_{2j} is a local complete intersection at x, defined by a regular sequence $g_1, \dots, g_r \in \mathcal{I}_{W_{2j},x}$. A final application of Lemma 1.2 provides us with an r-tuple $\underline{l}_{2j+1} \gg 0$, and with a link $W_{2j} \sim_{\underline{l}_{2j+1}} W_{2j+1}$ associated with a regular sequence $\underline{F}_{2j+1}$ such that $\langle \underline{F}_{2j+1} \rangle = \langle g_1, \dots, g_r \rangle \subset \mathcal{O}_{V,x}$. It follows that $x \notin W_{2j+1}$. □

1.2. **Linkage in families.** Let B be a smooth variety over k. Let $\mathfrak{W} \subset V \times B$ be a closed subscheme of pure codimension r with ideal sheaf $\mathcal{I}_{\mathfrak{W}} \subset \mathcal{O}_{V \times B}$, such that the second projection $f : \mathfrak{W} \to B$ is flat with Cohen–Macaulay fibers. Let $\underline{l}$ be an r-tuple such that, letting $p : V \times B \to B$ denote the second projection, the adjunction morphism $p^*p_*(\mathcal{I}_{\mathfrak{W}}(\underline{l})) \to \mathcal{I}_{\mathfrak{W}}(\underline{l})$ is surjective and $R^ip_*(\mathcal{I}_{\mathfrak{W}}(\underline{l})) = 0$ for $i > 0$. Note that, under these conditions, the push-forward sheaf $E := p_*(\mathcal{I}_{\mathfrak{W}}(\underline{l}))$ is a vector bundle such that the natural morphism $E|_b \to H^0(V, \mathcal{I}_{\mathfrak{W}_b}(\underline{l}))$ is an isomorphism for all $b \in B$ [Har77, III, Theorem 12.11].

View $E \to B$ as a geometric vector bundle over B. A point of E over $b \in B$ corresponds to a section $\underline{F} \in H^0(V, \mathcal{I}_{\mathfrak{W}_b}(\underline{l}))$. Let $B' \subset E$ be the open subset of those points such that $\underline{F}$ forms a regular sequence, and hence defines a complete intersection in V. Let $\mathfrak{Z} \subset V \times B'$ be the universal family of these complete intersections, and let $\mathfrak{W}_{B'} \subset V \times B'$ be the base change of $\mathfrak{W}$, with ideal sheaves $\mathcal{I}_{\mathfrak{Z}}, \mathcal{I}_{\mathfrak{W}_{B'}} \subset \mathcal{O}_{V \times B'}$. We consider the subscheme $\mathfrak{W}' \subset V \times B'$ with ideal sheaf $\mathcal{I}_{\mathfrak{W}'} := (\mathcal{I}_{\mathfrak{Z}} : \mathcal{I}_{\mathfrak{W}_{B'}})$ and we let $f' : \mathfrak{W}' \to B'$ denote the second projection.

By Proposition 1.4, this extends the construction of §1.1 in the relative setting.

Proposition 1.4. *The morphism $f' : \mathfrak{W}' \to B'$ is flat with Cohen–Macaulay fibers. For $b \in B'$, one has $\mathcal{I}_{\mathfrak{W}'_b} = (\mathcal{I}_{\mathfrak{Z}_b} : \mathcal{I}_{\mathfrak{W}_b})$.*

Proof. The scheme $\mathfrak{W}$ is Cohen–Macaulay by [EGA42, Corollaire 6.3.5 (ii)], hence so is $\mathfrak{W}'$ by [PS74, Proposition 1.3]. Since f' is equidimensional with regular base and Cohen–Macaulay total space, it is flat by [EGA42, Proposition 6.1.5].

Choose a regular system of parameters $x_1, \dots, x_N$ of the regular local ring $\mathcal{O}_{B',b}$. To show the equality of ideals $\mathcal{I}_{\mathfrak{W}'_b} = (\mathcal{I}_{\mathfrak{Z}_b} : \mathcal{I}_{\mathfrak{W}_b})$ at a point $v \in V_b$, one may apply N times sucessively [HU85, Lemma 2.12] in the local ring $\mathcal{O}_{V_b,v}$ (this is essentially what is done in [HU85, Proposition 2.13]). That the fibers $\mathfrak{W}'_b$ of f' are Cohen–Macaulay now follows from [PS74, Proposition 1.3]. □

1.3. **Moduli of links.** We now iterate the construction of §1.2, thus adapting to our global setting a local construction due to Huneke and Ulrich [HU87, HU88].

Recall that $W \subset V$ is a Cohen–Macaulay closed subscheme of pure codimension r. We set $L_0(W) := \mathrm{Spec}(k)$, $\mathfrak{W}_0 := W$ and $f_0 : \mathfrak{W}_0 \to L_0(W)$ to be the structural morphism. We inductively construct $f_i : \mathfrak{W}_i \to L_i(W)$ for $i \geq 1$, by choosing an r-tuple $\underline{l}_i \gg 0$, by applying the construction of §1.2 to $f_{i-1} : \mathfrak{W}_{i-1} \to L_{i-1}(W)$, and by setting $(f_i : \mathfrak{W}_i \to L_i(W)) = (f'_{i-1} : \mathfrak{W}'_{i-1} \to L_{i-1}(W)')$. The varieties $L_i(W)$ are irreducible, smooth over k and k-rational, and the morphisms f_i are flat with Cohen–Macaulay fibers, as an induction based on Proposition 1.4 shows.

We call $f_j : \mathfrak{W}_j \to L_j(W)$ the j-th *moduli of links* of W (with respect to the degrees $\underline{l}_1, \dots, \underline{l}_j$). Its points parametrize sequences $(\underline{F}_i)_{1 \leq i \leq j}$ of regular sequences that give rise to chains of linked subvarieties $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \cdots \sim_{\underline{l}_j} W_j$ of V.

Remark 1.5. The construction of the j-th moduli of links $f_j : \mathfrak{W}_j \to L_j(W)$ goes through with no modifications if k is finite, but beware that $L_j(W)(k)$ might be empty in this case.

1.4. **Images by a morphism.** In §§1.4-1.5, we fix a smooth morphism $\pi : V \to X$ of smooth projective varieties over k and we let d and n be the dimensions of W and X. In Propositions 1.6, 1.7, 1.9 and 1.14, we study the images by π of subvarieties of X linked to W, under the assumption that $n > 2d$. Proposition 1.6 is due to Hironaka [Hir68]. Proposition 1.7 is a simple Bertini theorem. Proposition 1.9, which is the main result of §1.4, is more delicate since one must deal with singularities of varieties linked to W. Proposition 1.14, which is the main result of §1.5, is specific to the case where k is a real closed field.

Proposition 1.6 (Hironaka)**.** *Assume that $n > 2d$, that $d \leq 3$, and that W is smooth. Then, for $j \geq 4$ and r-tuples of integers $\underline{l}_j \gg \cdots \gg \underline{l}_1 \gg 0$, there exists a chain of linked smooth subvarieties $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \cdots \sim_{\underline{l}_j} W_j$ of V such that $\pi|_{W_j} : W_j \to X$ is a closed embedding.*

Proof. Choose general links $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \cdots \sim_{\underline{l}_j} W_j$. Since $d \leq 3$ and W is smooth, the W_i are smooth by [Hir68, Corollary 3.9.1].

That $\pi|_{W_j} : W_j \to X$ is a closed embedding may be checked over an algebraic closure of k, where it follows from the proof of [Hir68, Lemma 5.1.1]. More precisely, define $A(W_i) \subset W_i$ to be the closed subset of those $x \in W_i$ such that $\pi|_{W_i} : W_i \to X$ is not a closed embedding above a neighbourhood of $\pi(x)$. One has $\dim(A(W_0)) \leq \dim(W) \leq 3$. Moreover, $\dim(A(W_{i+1})) \leq \dim(A(W_i))$, and the inequality is strict if $A(W_i) \neq \varnothing$, by the proof of [Hir68, Lemma 5.1.1]. It follows that $A(W_i) = \varnothing$ for $i \geq 4$, hence that $\pi|_{W_j} : W_j \to X$ is a closed embedding. □

Proposition 1.7. *Assume that $n > 2d$, and let $\underline{l} = (l_1, \dots, l_r)$ be an r-tuple of integers such that the linear systems $H^0(V, \mathcal{I}_W(l_i))$ embed $V \setminus W$ in projective spaces. Then, for $\underline{F} \in H^0(V, \mathcal{I}_W(\underline{l}))$ general, the link $W \sim_{\underline{l}} W'$ associated with $\underline{F}$ satisfies:*

(i) The variety $S := W' \setminus (W \cap W')$ is smooth.
(ii) The morphism $\pi|_S : S \to X$ is an embedding.
(iii) The subsets $\pi(S)$ and $\pi(W)$ of X are disjoint.

Proof. Apply Lemma 1.8 below with $Y = V \setminus W$, $f = \pi|_{V \setminus W}$, and $F = \pi(W)$. □

Lemma 1.8. *Let $f : Y \to X$ be a smooth morphism of smooth varieties over k, let $F \subset X$ be a closed subset, and let $V_1, \dots, V_r$ be linear systems on Y inducing embeddings of Y in projective spaces. If one has* $\dim(X) > 2(\dim(Y) - r)$ *and*

$\dim(X) > \dim(F)+\dim(Y)-r$, then, for general $\sigma_i \in V_i$, the variety $S := \{\sigma_i = 0\}$ is smooth, the morphism $f|_S : S \to X$ is an embedding, and $f(S) \cap F = \varnothing$.

Proof. The smoothness of S follows from the Bertini theorem. For general σ_i, the complete intersection $S \cap f^{-1}(F)$ in $f^{-1}(F)$ is empty, hence $f(S) \cap F = \varnothing$.

Let H be the Hilbert scheme parametrizing zero-dimensional subschemes $Z \subset Y$ of length 2 in the fibers of f. Let $B \subset H \times V_1 \times \cdots \times V_r$ be the closed subset parametrizing tuples $([Z], \sigma_1, \ldots, \sigma_r)$ such that the σ_i vanish on Z. Computing

$$\dim(B) = \dim(H) + \sum_i \dim(V_i) - 2r = 2\dim(Y) - \dim(X) + \sum_i \dim(V_i) - 2r,$$

shows that $\dim(B) < \sum_i \dim(V_i)$. To ensure that $f|_S$ is an embedding, it suffices to choose $(\sigma_1, \ldots, \sigma_r)$ outside of the image of the projection $B \to V_1 \times \cdots \times V_r$. □

Proposition 1.9. *Assume that $n > 2d$ and that W is a local complete intersection. For $j \gg 0$ and for r-tuples of integers $\underline{l}_j \gg \cdots \gg \underline{l}_1 \gg 0$, there exists a chain of linked subvarieties $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \cdots \sim_{\underline{l}_j} W_j$ of V with the property that $\pi|_{W_j} : W_j \to X$ is geometrically injective.*

Proof. Let $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \cdots \sim_{\underline{l}_j} W_j$ be a general chain of links. Define $C(W_i) \subset W_i$ to be the constructible subset of those $x \in W_i$ such that $(\pi|_{W_i})^{-1}(\pi(x))$ has more than one geometric point. Of course, one has $\dim(C(W_0)) \leq \dim(W) = d$. Proposition 1.7 implies that $C(W_{i+1}) \subset C(W_i)$ for all $i \geq 0$. We also claim that $\dim(C(W_{2i+1})) < \dim(C(W_i))$ if $2i + 1 \leq j$ and $C(W_i) \neq \varnothing$. These facts imply that $C(W_j) = \varnothing$ if $j \geq 2^{d+1} - 1$, which concludes.

It remains to prove the claim. Assume that $2i + 1 \leq j$ and that $C(W_i) \neq \varnothing$. Choose finitely many points $x_1, \ldots, x_N \in C(W_i)$, including at least one in each irreducible component of the Zariski closure of $C(W_i)$. By Lemma 1.3, there exists a chain of links $W_i \sim_{\underline{l}_i} W^{(s)}_{i+1} \sim_{\underline{l}_{i+1}} \cdots \sim_{\underline{l}_{2i+1}} W^{(s)}_{2i+1}$ such that $x_s \notin W^{(s)}_{2i+1}$, for all $1 \leq s \leq N$. Since $W_i \sim_{\underline{l}_i} W_{i+1} \sim_{\underline{l}_{i+1}} \cdots \sim_{\underline{l}_{2i+1}} W_{2i+1}$ corresponds to a general point of the $(i + 1)$-th moduli of links of W_i in the sense of §1.3, we deduce that $x_s \notin W_{2i+1}$, hence that $x_s \notin C(W_{2i+1})$ for $1 \leq s \leq N$. The chain of inclusions $C(W_{2i+1}) \subset C(W_{2i}) \subset \cdots \subset C(W_i)$ implies that $\dim(C(W_{2i+1})) < \dim(C(W_i))$, which proves the claim. □

Remark 1.10. The proof of Proposition 1.9 requires to use a huge number of links (exponential in d). We do not know if this is necessary, or a quirk of the proof.

1.5. **Real loci.** In §1.5, we keep the notation of §1.4 and assume moreover that $k = R$ is a real closed field, for instance the field $\mathbb{R}$ of real numbers.

Lemma 1.11. *Fix r-tuples of integers $\underline{l}_1 = (l_{1,1}, \ldots, l_{1,r})$ and $\underline{l}_2 = (l_{2,1}, \ldots, l_{2,r})$ with $l_{2,i} - l_{1,i}$ nonnegative and even for $1 \leq i \leq r$. Let $W \sim_{\underline{l}_1} W_1$ be a link. Then there exists a link $W_1 \sim_{\underline{l}_2} W_2$ such that $W_2 = W$ in a neighbourhood of $V(R)$.*

Proof. Let $(u_1, \ldots, u_N)$ be a basis of $H^0(V, \mathcal{O}_V(1))$. The section $u := \sum_{m=1}^N u_m^2$ does not vanish on $V(R)$. Let $v_1, \ldots, v_r \in H^0(V, \mathcal{O}_V(2))$ be general small deformations of u. They are general elements of $H^0(V, \mathcal{O}_V(2))$ that do not vanish on $V(R)$.

Let $\underline{F} = (F_1, \ldots, F_r) \in H^0(V, \mathcal{I}_W(\underline{l}_1))$ be a regular sequence defining $W \sim_{\underline{l}_1} W_1$. There exist integers $a_i \geq 0$ such that $\underline{G} := (v_1^{a_1} F_1, \ldots, v_r^{a_r} F_r) \in H^0(V, \mathcal{I}_{W_1}(\underline{l}_2))$. Since the v_i are general and since $\underline{F}$ is a regular sequence, we see that $\underline{G}$ is also regular sequence. Let $W_1 \sim_{\underline{l}_2} W_2$ be the link it defines.

The ideal sheaves $\langle \underline{F} \rangle$ and $\langle \underline{G} \rangle$ of $\mathcal{O}_V$ coincide in a neighbourhood of $V(R)$ since the v_i do not vanish on $V(R)$. It thus follows from the symmetry of linkage (see §1.1) that $W_2 = W$ in a neighbourhood of $V(R)$. □

Lemma 1.12. *Set $W_0 := W$. Suppose that W_0 is smooth along $W_0(R)$ and that $n > 2d$. Fix r-tuples of even integers $\underline{l}_2 \gg \underline{l}_1 \gg 0$. If $R = \mathbb{R}$, fix a neighbourhood $\mathcal{U} \subset \mathcal{C}^\infty(W_0(\mathbb{R}), V(\mathbb{R}))$ of the inclusion. Then there exist links $W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} W_2$ with the following properties.*

(i) The variety W_2 is smooth along $W_2(R)$.
(ii) Let $D(W_i) \subset W_i$ be the subset of those $x \in W_i$ such that $\pi|_{W_i}$ is not immersive at x. Set $d_i := \sup_{x \in D(W_i)(R)} \dim_x D(W_i)$. If $D(W_0)(R) \neq \varnothing$, then $d_2 < d_0$.
(iii) If $R = \mathbb{R}$, there exists a diffeomorphism $\phi : W_0(\mathbb{R}) \xrightarrow{\sim} W_2(\mathbb{R})$ such that, letting $\iota : W_2(\mathbb{R}) \to V(\mathbb{R})$ denote the inclusion, one has $\iota \circ \phi \in \mathcal{U}$.

Proof. Choose finitely many points $x_1, \dots, x_N \in W_0(R)$, including at least one in each irreducible component of $D(W_0)$ that has real points. By Lemma 1.3, a link $W_0 \sim_{\underline{l}_1} W_1$ corresponding to a general point of the first moduli of links of W_0 (in the sense of §1.3) has the property that $x_s \notin W_1$ for $1 \leq s \leq N$. Lemma 1.11 shows the existence of a link $W_1 \sim_{\underline{l}_2} \widetilde{W}$ such that $\widetilde{W} = W_0$ in a neighbourhood of $V(R)$. We deduce that $\widetilde{W}$ is smooth along $\widetilde{W}(R) = W_0(R)$.

Let $\widetilde{f} : \widetilde{\mathfrak{W}} \to L_1(W_1)$ be the first moduli of links of W_1 with respect to the degree $\underline{l}_2$ (as in §1.3) and let $b \in L_1(W_1)(R)$ be the point associated with $W_1 \sim_{\underline{l}_2} \widetilde{W}$. Proposition 1.4 shows that $\widetilde{\mathfrak{W}}_b = \widetilde{W}$ and that $\widetilde{f}$ is flat, hence smooth in a neighbourhood of $\widetilde{\mathfrak{W}}_b(R)$. As $\widetilde{f}$ is proper, the map $\widetilde{f}(R)$ is closed by [DK81, Theorem 9.6]. We deduce that there exists a Euclidean neighourhood Ω of b in $L_1(W_1)(R)$ such that the morphism $\widetilde{f}$ is smooth along $\widetilde{f}(R)^{-1}(\Omega)$. Choose such an Ω small enough. Since $L_1(W_1)$ is smooth and irreducible (see §1.3), the subset $\Omega \subset L_1(W_1)$ is Zariski-dense (apply [BCR98, Proposition 2.8.14]). Consequently, one may choose $a \in \Omega$ general. Let $W_1 \sim_{\underline{l}_2} W_2 := \widetilde{\mathfrak{W}}_a$ be the associated link. Assertion (i) holds by our choice of Ω.

Let $D \subset \widetilde{\mathfrak{W}}$ be the closed subset of those $x \in \widetilde{\mathfrak{W}}$ such that the morphism $\pi|_{\tilde{f}^{-1}(\tilde{f}(x))} : \widetilde{\mathfrak{W}}_{\tilde{f}(x)} \to X$ is not immersive at x. Set $E := D \cap (W_1 \times L_1(W_1)) \subset \widetilde{\mathfrak{W}}$. By Proposition 1.7 (ii), there is a proper closed subset $F \subset L_1(W_1)$ such that $D \subset \widetilde{f}^{-1}(F) \cup (W_1 \times L_1(W_1))$ as subsets of $V \times L_1(W_1)$. Since a has been chosen general, it lies outside of F. We deduce the inclusion $D(W_2) \subset E_a$ of subsets of $\widetilde{\mathfrak{W}}_a$. The function $x \mapsto \dim_x(E_{\tilde{f}(x)})$ is upper semicontinuous for the Zariski topology on E by [EGA43, Théorème 13.1.3], hence upper semicontinuous for the Euclidean topology on $E(R)$. Since $\widetilde{f}|_E : E \to L_1(W_1)$ is proper, the map $\widetilde{f}|_E(R)$ is closed by [DK81, Theorem 9.6]. If Ω has been chosen small enough, we deduce at once the inequality:

$$\sup_{x \in E_b(R)} \dim_x E_b \geq \sup_{x \in E_a(R)} \dim_x E_a. \tag{1.13}$$

As $\widetilde{\mathfrak{W}}_b = W_0$ in a neighbourhood of $V(\mathbb{R})$, one has $E_b \subset D(W_0)$ in a neighbourhood of $V(\mathbb{R})$. Since none of the x_s belong to W_1, we see that no irreducible components of $D(W_0)$ that has real points is included in E_b. If $D(W_0)(R) \neq \varnothing$, it follows that the left-hand side of (1.13) is $< d_0$. On the other hand, the right-hand side of (1.13) is $\geq d_2$ since $D(W_2) \subset E_a$. The inequality $d_2 < d_0$ follows, proving (ii).

If $R=\mathbb{R}$, one may assume Ω to be connected. Ehresmann's theorem applied to the proper submersion $\widetilde{f}(\mathbb{R})|_{\widetilde{f}(\mathbb{R})^{-1}(\Omega)}:\widetilde{f}(\mathbb{R})^{-1}(\Omega)\to\Omega$ yields a diffeomorphism $\psi:\Omega\times\widetilde{\mathfrak{W}}_b(\mathbb{R})\xrightarrow{\sim}\widetilde{f}(\mathbb{R})^{-1}(\Omega)$ compatible with the projections to Ω. If Ω has been chosen small enough, the composition

$$W_0(\mathbb{R})=\widetilde{\mathfrak{W}}_b(\mathbb{R})\xrightarrow{\psi_a}\widetilde{\mathfrak{W}}_a(\mathbb{R})=W_2(\mathbb{R})\xrightarrow{\iota}V(\mathbb{R})$$

belongs to $\mathcal{U}$ for all $a\in\Omega$. Assertion (iii) is proven. □

Proposition 1.14. *Suppose that W is smooth along $W(R)$ and that $n>2d$. Let $j\gg 0$ be even and let $\underline{l}_j\gg\cdots\gg\underline{l}_1\gg 0$ be r-tuples of even integers. Define $f_j:\mathfrak{W}_j\to L_j(W)$ as in §1.3. Then there exists a nonempty subset $\Omega\subset L_j(W)(R)$ which is open for the Euclidean topology such that the following holds for all $b\in\Omega$.*

(i) The variety $\mathfrak{W}_{j,b}$ is smooth along $\mathfrak{W}_{j,b}(R)$.
(ii) The morphism $\pi|_{\mathfrak{W}_{j,b}}$ is immersive along $\mathfrak{W}_{j,b}(R)$.

If moreover $R=\mathbb{R}$ and $\mathcal{U}\subset\mathcal{C}^\infty(W(\mathbb{R}),V(\mathbb{R}))$ is a neighbourhood of the inclusion, one may ensure that the following holds.

(iii) There exists a diffeomorphism $\phi_b:W(\mathbb{R})\xrightarrow{\sim}\mathfrak{W}_{j,b}(\mathbb{R})$ such that, letting $\iota_b:\mathfrak{W}_{j,b}(\mathbb{R})\to V(\mathbb{R})$ denote the inclusion, one has $\iota_b\circ\phi_b\in\mathcal{U}$.

Proof. Choose $j\geq 2d+2$ even. Applying $j/2$ times Lemma 1.12 shows the existence of a chain of linked subvarieties $W=W_0\sim_{\underline{l}_1}W_1\sim_{\underline{l}_2}\cdots\sim_{\underline{l}_j}W_j$ of V such that W_j is smooth along $W_j(R)$ and such that $\pi|_{W_j}$ is immersive along $W_j(R)$. Moreover, if $R=\mathbb{R}$, Lemma 1.12 ensures the existence of a diffeomorphism $\phi:W(\mathbb{R})\xrightarrow{\sim}W_j(\mathbb{R})$ such that, letting $\iota:W_j(\mathbb{R})\to V(\mathbb{R})$ denote the inclusion, one has $\iota\circ\phi\in\mathcal{U}$.

Let $a\in L_j(W)(R)$ be the point associated with $W=W_0\sim_{\underline{l}_1}W_1\sim_{\underline{l}_2}\cdots\sim_{\underline{l}_j}W_j$. Proposition 1.4 shows that $\mathfrak{W}_{j,a}=W_j$ and that f_j is flat, hence smooth in a neighbourhood of $\mathfrak{W}_{j,a}(R)$. As f_j is proper, the map $f_j(R)$ is closed by [DK81, Theorem 9.6]. We deduce the existence of a neighourhood Ω of a in $L_j(W)(R)$ such that f_j is smooth along $f_j(R)^{-1}(\Omega)$. Assertion (i) follows. So does assertion (ii) after maybe shrinking Ω. If $R=\mathbb{R}$, that assertion (iii) holds after shrinking Ω further follows from Ehresmann's theorem applied to the proper submersion $f_j(\mathbb{R})|_{f_j(\mathbb{R})^{-1}(\Omega)}:f_j(\mathbb{R})^{-1}(\Omega)\to\Omega$. □

2. Smoothing by linkage

Let us apply linkage theory as developed in §1 to prove Theorems 0.1, 0.4 and 0.6.

2.1. **Main statement.** Here is the technical result from which the main theorems of §2 will follows.

Proposition 2.1. *Let $g:W\to X$ be a morphism of smooth projective varieties over a real closed field R. Let d and n be the dimensions W and X and assume that $n>2d$. If $R=\mathbb{R}$, fix a neighbourhood $\mathcal{V}\subset\mathcal{C}^\infty(W(\mathbb{R}),X(\mathbb{R}))$ of $g(\mathbb{R})$. Then there exists a closed subvariety $i:Y\hookrightarrow X$ of dimension d with the following properties.*

(i) The variety Y is smooth along $Y(R)$.
(ii) If $d\leq 3$, then Y is smooth.
(iii) The class $g_*[W]-[Y]\in\mathrm{CH}_d(X)$ is a linear combination of classes of smooth closed subvarieties of X with empty real loci.
(iv) If $R=\mathbb{R}$, there is a diffeomorphism $\psi:W(\mathbb{R})\xrightarrow{\sim}Y(\mathbb{R})$ such that $i(\mathbb{R})\circ\psi\in\mathcal{V}$.

Proof. Define $V := X \times W$, let $\pi : V \to X$ be the first projection and let $\mathcal{O}_V(1)$ be a very ample line bundle on V. Let r be the codimension of the closed embedding $(g, \mathrm{Id}) : W \hookrightarrow V$. If $R = \mathbb{R}$, define $\mathcal{U} := \{\xi \in \mathcal{C}^\infty(W(\mathbb{R}), V(\mathbb{R})) \mid \pi(\mathbb{R}) \circ \xi \in \mathcal{V}\}$, which is a neighbourhood of the inclusion $W(\mathbb{R}) \hookrightarrow V(\mathbb{R})$ by [Mic80, Theorem 11.4].

Choose an even integer $j \gg 0$ and r-tuples of even integers $\underline{l}_j \gg \cdots \gg \underline{l}_1 \gg 0$. Let $f_j : \mathfrak{W}_j \to L_j(W)$ be the j-th moduli of links of W as in §1.3, and choose $\Omega \subset L_j(W)(R)$ be as in Proposition 1.14. Since $L_j(W)$ is smooth and irreducible (see §1.3), the subset $\Omega \subset L_j(W)$ is Zariski-dense (apply [BCR98, Proposition 2.8.14]). Consequently, one may choose a general point $b \in \Omega$. Define $Y := \pi(\mathfrak{W}_{j,b})$ with inclusion $i : Y \hookrightarrow X$. By Proposition 1.9, the morphism $\pi|_{\mathfrak{W}_{j,b}} : \mathfrak{W}_{j,b} \to Y$ is geometrically injective. By Proposition 1.14 (i)-(ii), in a neighbourhood of $\mathfrak{W}_{j,b}(R)$, the variety $\mathfrak{W}_{j,b}$ is smooth and the morphism $\pi|_{\mathfrak{W}_{j,b}}$ is immersive. These facts show that $\theta := \pi(R)|_{\mathfrak{W}_{j,b}(R)} : \mathfrak{W}_{j,b}(R) \to Y(R)$ is bijective and that Y is smooth along the image of θ. This proves (i). The argument also shows that θ is a diffeomorphism. Consequently, if $R = \mathbb{R}$, one may take $\psi := \theta \circ \phi_b$, where ϕ_b is as in Proposition 1.14 (iii), which proves (iv). If $d \leq 3$, Proposition 1.6 shows that $\mathfrak{W}_{j,b}$ is smooth and that $\pi|_{\mathfrak{W}_{j,b}}$ is a closed immersion, proving (ii).

It remains to prove (iii). The chain of links relating W and $\mathfrak{W}_{j,b}$ shows that the difference $[W] - [\mathfrak{W}_{j,b}] \in \mathrm{CH}_d(V)$ is a multiple of $(2\lambda)^r \in \mathrm{CH}^r(V) = \mathrm{CH}_d(V)$, where $\lambda := c_1(\mathcal{O}_V(1)) \in \mathrm{CH}^1(V)$. Consequently, $\pi_*[W] - \pi_*[\mathfrak{W}_{j,b}] = g_*[W] - [Y]$ is a multiple of $\pi_*((2\lambda)^r) \in \mathrm{CH}_d(X)$. Let $(u_1, \ldots, u_N)$ be a basis of $H^0(V, \mathcal{O}_V(1))$. Proposition 1.7 applied with $W = \varnothing$, with $l_i = 2$, and with the F_i chosen to be general small deformations of $\sum_{m=1}^N u_m^2$, shows that $\pi_*((2\lambda)^r)$ is the class of a smooth closed subvariety of X with empty real locus, which concludes. □

Remark 2.2. Over a general real closed field, one could replace Proposition 2.1 (iv) by the assertion that there exists a Nash diffeomorphism $\psi : W(R) \xrightarrow{\sim} Y(R)$ such that $g(R) : W(R) \to X(R)$ and $i(R) \circ \psi : Y(R) \to X(R)$ are Nash homotopic. The proof is identical, replacing the use of Ehresmann's theorem in the proofs of Lemma 1.12 (iii) and Proposition 1.14 by its Nash analogue proven by Coste and Shiota [CS92, Theorem 2.4 (iii)'].

2.2. **Low-dimensional cycles.** We first give applications to the Chow groups of smooth projective varieties over real closed fields.

Theorem 2.3. *Let X be a smooth projective variety of dimension n over a real closed field R. For $n > 2d$, the Chow group $\mathrm{CH}_d(X)$ is generated by classes of closed integral subvarieties of X which are smooth along their real loci.*

Proof. Let $Z \subset X$ be a closed integral subvariety of X, let $W \to Z$ be a resolution of singularities, and let $g : W \to X$ be the induced morphism. Proposition 2.1 furnishes a closed subvariety $Y \subset X$ which is smooth along $Y(R)$ and such that $g_*[W] - [Y] \in \mathrm{CH}_d(X)$ is a linear combination of classes of smooth closed subvarieties of X. This proves the theorem. □

Theorem 2.4. *Let X be a smooth projective variety of dimension n over $\mathbb{R}$. For $n > 2d$, the group $\mathrm{Ker}\big[\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}), \mathbb{Z}/2)\big]$ is generated by classes of closed integral subvarieties of X with empty real loci.*

Proof. Ischebeck and Schülting [IS88, Main Theorem 4.3] have shown that the group $\mathrm{Ker}\big[\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}), \mathbb{Z}/2)\big]$ is generated by classes of closed integral subvarieties of $Z \subset X$ with the property that $Z(\mathbb{R})$ is not Zariski-dense in Z. Let $W \to Z$ be a resolution of singularities of such a subvariety, and let $g : W \to X$ be the induced morphism. Since the real locus of a smooth irreducible variety over $\mathbb{R}$ is empty or Zariski-dense, one has $W(\mathbb{R}) = \varnothing$.

By Proposition 2.1, there is a closed subvariety $Y \subset X$ with $Y(\mathbb{R}) \simeq W(\mathbb{R}) = \varnothing$ and such that $g_*[W] - [Y] \in \mathrm{CH}_d(X)$ is a linear combination of classes of closed subvarieties of X with empty real loci. This concludes the proof. □

Remark 2.5. The proof of Theorem 2.4 does not extend to general real closed fields because of its use of [IS88, Main Theorem 4.3]. As a matter of fact, the statement of Theorem 2.4 does not hold over the real closed field $R := \cup_n \mathbb{R}((t^{1/n}))$, for any $d > 0$, as we show in the next proposition.

Proposition 2.6. *For all $c, d \geq 1$, there exist a smooth projective variety $X_{c,d}$ of dimension $c + d$ over $R := \cup_n \mathbb{R}((t^{1/n}))$ and a class $\beta_{c,d} \in \mathrm{CH}_d(X_{c,d})$ such that:*

(i) *One has $\mathrm{cl}_R(\beta_{c,d}) = 0 \in H_d(X_{c,d}(R), \mathbb{Z}/2)$.*
(ii) *For all identities $\beta_{c,d} = \sum_{i \in I} n_i[Z_i] \in \mathrm{CH}_d(X_{c,d})$ with $n_i \in \mathbb{Z}$ and $Z_i \subset X$ integral, there exists $i \in I$ such that n_i is odd and $Z_i(\mathbb{R})$ is Zariski-dense in Z_i.*

Proof. Such $X_{1,1}$ and $\beta_{1,1}$ have been constructed in [BW20b, Propositions 9.17 and 9.19 (ii)] (they form a counterexample to Bröcker's EPT theorem over the field R). Let $x \in \mathbb{P}^{c-1}(R)$ and $y \in \mathbb{P}^{d-1}(R)$ be general points. One may then define $X_{c,d} := X_{1,1} \times \mathbb{P}^{c-1}_R \times \mathbb{P}^{d-1}_R$ and $\beta_{c,d} := pr_1^* \beta_{1,1} \cdot pr_2^*[x]$. The required property of $(X_{c,d}, \beta_{c,d})$ follows from that of $(X_{1,1}, \beta_{1,1})$, and from the equation $\beta_{1,1} = (pr_1)_*(\beta_{c,d} \cdot pr_3^*[y])$. □

2.3. **Approximation of submanifolds.** We now give an application to the existence of algebraic approximations for submanifolds of the real locus of a smooth projective variety X over $\mathbb{R}$. We refer to §0.3 for the definition of the algebraic cobordism group $MO_d^{\mathrm{alg}}(X(\mathbb{R}))$.

Theorem 2.7. *Let X be a smooth projective variety of dimension n over $\mathbb{R}$, and let $j : M \hookrightarrow X(\mathbb{R})$ be a closed $\mathcal{C}^\infty$ submanifold of dimension d. If $n > 2d$, the following properties are equivalent.*

(i) *One has $[j] \in MO_d^{\mathrm{alg}}(X(\mathbb{R}))$.*
(ii) *For all neighbourhoods $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$ of j, there exist a closed d-dimensional subvariety $i : Y \hookrightarrow X$ smooth along $Y(\mathbb{R})$ and a diffeomorphism $\phi : M \xrightarrow{\sim} Y(\mathbb{R})$ such that $i(\mathbb{R}) \circ \phi \in \mathcal{U}$.*

If moreover $d \leq 3$, one may choose Y to be smooth in assertion (ii).

Proof. Assume that (i) holds and let $\mathcal{U}$ be as in (ii). A relative variant of the Nash–Tognoli theorem (see [BT80b, Proposition 4.1] or [AK81, Proposition 0.2]) shows the existence of a morphism $g : W \to X$ of smooth projective varieties over $\mathbb{R}$ and of a diffeomorphism $\chi : M \xrightarrow{\sim} W(\mathbb{R})$ such that $g(\mathbb{R}) \circ \chi \in \mathcal{U}$. Assertion (ii) and the last statement of Theorem 2.7 now follow by applying Proposition 2.1 to $\mathcal{V} := \{\xi \circ \chi^{-1}, \xi \in \mathcal{U}\} \subset \mathcal{C}^\infty(W(\mathbb{R}), X(\mathbb{R}))$ and by defining $\phi := \psi \circ \chi$.

Suppose conversely that (ii) holds. Applying it to a small enough neighbourhood $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$ of j shows that j is homotopic, hence cobordant, to a $\mathcal{C}^\infty$ map of the form $i(\mathbb{R})$, by [Wal16, Proposition 4.4.4]. To get (i), take $W \to Y$ to be a resolution of singularities which is an isomorphism over $Y(\mathbb{R})$, define $g : W \to X$ be the induced morphism, and note that j is cobordant to $g(\mathbb{R})$. □

3. Complex cobordism and Chern numbers

After a short review of cobordism theory (in §3.1) and of its relation with characteristic classes (in §3.2), we study the top Segre class in §3.3, our goal being Theorem 3.6.

3.1. **The cobordism rings.** Two compact $\mathcal{C}^\infty$ manifolds M_1 and M_2 of dimension n are said to be *cobordant* if there exists a compact $\mathcal{C}^\infty$ manifold with boundary C and a diffeomorphism $\partial C \simeq M_1 \cup M_2$. Let MO_n be the set of cobordism classes of such manifolds, and define $MO_* := \bigoplus_{n\geq 0} MO_n$.

Let M be a $\mathcal{C}^\infty$ manifold. A *stably almost complex structure* on M is a complex structure J on the real vector bundle $T_M \oplus \mathbb{R}^k$ for some $k \geq 0$, modulo the equivalence relation generated by $(T_M \oplus \mathbb{R}^k, J) \simeq (T_M \oplus \mathbb{R}^{k+2} = T_M \oplus \mathbb{R}^k \oplus \mathbb{C}, (J, i))$. Two n-dimensional stably almost complex compact $\mathcal{C}^\infty$ manifolds M_1 and M_2 are said to be *complex cobordant* if there exists a stably almost complex compact $\mathcal{C}^\infty$ manifold with boundary C and a diffeomorphism $\partial C \simeq M_1 \cup M_2$ compatible with the stably almost complex structures. Let MU_n be the set of complex cobordism classes of such manifolds, and define $MU_* := \bigoplus_{n\geq 0} MU_n$.

Disjoint union and cartesian product endow MO_* and MU_* with graded ring structures: they are the *unoriented cobordism ring* and the *complex cobordism ring*. Thom [Tho54, Théorème IV.12] and Milnor [Mil60] (see also Quillen [Qui71, Theorem 6.5]) have computed that $MO_* \simeq \mathbb{Z}/2[x_d]_{d\neq 2^k-1}$ and $MU_* \simeq \mathbb{Z}[t_{2d}]_{d\geq 1}$, where $x_d \in MO_d$ and $t_{2d} \in MU_{2d}$. A comprehensive treatment of these results may be found in [Koc96]. As was noted by Milnor [Mil65, Lemma 1], a striking consequence of Thom's computation is that any element of MO_* may be represented by the real locus of a smooth projective variety over $\mathbb{R}$ (a disjoint union of products of projective spaces and Milnor hypersurfaces).

Let $\phi : MU_* \to MO_*$ be the graded ring homomorphism forgetting the stably almost complex structures. Milnor [Mil65, Theorem 1] has shown that the image of ϕ consists exactly of the squares in MO_*, hence that there exists a surjective ring homomorphism $\psi : MU_* \to MO_*$ such that $\phi(x) = \psi(x)^2$ for all $x \in MU_*$.

Lemma 3.1. *The ideal* $\ker(\psi) \subset MU_*$ *is generated by* 2 *and by* $(\ker(\psi)_{2^{k+1}-2})_{k\geq 1}$.

Proof. We use the isomorphisms $MO_* \simeq \mathbb{Z}/2[x_d]_{d\neq 2^k-1}$ and $MU_* \simeq \mathbb{Z}[t_{2d}]_{d\geq 1}$. Since $\psi : MU_* \to MO_*$ is surjective, $\psi(t_{2d}) - x_d \in MO_d$ is decomposable for $d \neq 2^k - 1$, and $\psi(t_{2d})$ is of course decomposable for $d = 2^k - 1$. We deduce from the surjectivity of ψ the existence of $t'_{2d} \in MU_{2d}$ such that $t_{2d} - t'_{2d}$ is decomposable (so that $MU_* = \mathbb{Z}[t'_{2d}]_{d\geq 1}$) and such that $\psi(t'_{2d}) = x_d$ if $d \neq 2^k - 1$ and $\psi(t'_{2d}) = 0$ otherwise. It is now clear that $\ker(\psi)$ is generated by 2 and by the $(t'_{2^{k+1}-2})_{k\geq 1}$. □

3.2. **Stiefel–Whitney and Chern numbers.** Let M be a compact $\mathcal{C}^\infty$ manifold of dimension n, and let $w_r(M) \in H^r(M, \mathbb{Z}/2)$ be the r-th Stiefel–Whitney class of its tangent bundle. For a sequence of nonnegative integers $I = (i_1, i_2, \dots)$ with $|I| := \sum_r r i_r = n$, we define $w_I(M) := \langle \prod_r w_r(M)^{i_r}, [M]\rangle \in \mathbb{Z}/2$, where $[M]$ is the fundamental class of M. The $w_I(M)$ are the *Stiefel–Whitney numbers* of M. Thom has shown that they only depend on the cobordism class of M, and that they determine this cobordism class [Tho54, Théorèmes IV.3 and IV.11].

Similarly, if M is a stably almost complex compact $\mathcal{C}^\infty$ manifold of dimension n, we let $c_r(M) \in H^{2r}(M, \mathbb{Z})$ denote the r-th Chern class of its stable tangent bundle, and we define the *Chern numbers* $c_I(M) := \langle \prod_r c_r(M)^{i_r}, [M]\rangle \in \mathbb{Z}$ of M for

$|I| = n/2$. These numbers only depend on the complex cobordism class of M, and determine it (see [Qui71, Theorem 6.5]).

Lemma 3.2. *For all $y \in MU_{2d}$ and all $I = (i_1, i_2, \dots)$ with $|I| = d$, the reduction modulo 2 of $c_I(y)$ is equal to $w_I(\psi(y))$.*

Proof. Represent y by a stably almost complex compact $\mathcal{C}^\infty$ manifold M and $\psi(y)$ by a compact $\mathcal{C}^\infty$ manifold N. For $I' = (0, i_1, 0, i_2, \dots)$, one has

$$c_I(M) = w_{I'}(M) = w_{I'}(N \times N) = w_I(N) \in \mathbb{Z}/2,$$

where the first equality holds by [MS74, Problem 14-B], the second since M is cobordant to $N \times N$, and the third is [Mil65, Lemma 2]. □

The relation $(\sum_r c_r(M))(\sum_r s_r(M)) = 1$ defines the *Segre classes* or *normal Chern classes* $s_r(M) \in H^{2r}(M, \mathbb{Z})$ of M. Pairing the top Segre class with $[M]$ yields a morphism $s_d : MU_{2d} \to \mathbb{Z}$ which is a linear combination of Chern numbers. This characteristic number is multiplicative in the following sense.

Lemma 3.3. *For $y \in MU_{2d}$ and $y' \in MU_{2d'}$, one has $s_{d+d'}(yy') = s_d(y)s_{d'}(y')$.*

Proof. Represent y and y' by stably almost complex compact $\mathcal{C}^\infty$ manifolds and apply the Whitney sum formula [MS74, (14.7)]. □

3.3. **Divisibility of the top Segre class.** In [RT77], Rees and Thomas study divisibility properties of some Chern numbers. In Theorem 3.4, we recall one of their results, which we complement in Theorem 3.6. Recall from §0.6 that we let $\alpha(m)$ be the number of ones in the dyadic expansion of m.

Theorem 3.4 (Rees–Thomas)**.** *For $d \geq 0$ and $e \geq 1$, the function $s_d : MU_{2d} \to \mathbb{Z}$ is divisible by 2^e if and only if $\alpha(d + e - 1) > 2(e - 1)$.*

Proof. Rees and Thomas [RT77, Theorem 3] show that $s_d : MU_{2d} \to \mathbb{Z}$ is divisible by 2^e if and only if $\alpha(d+f) > 2f$ for all $0 \leq f \leq e-1$, and it is easily verified that $\alpha(d + f) > 2f$ implies that $\alpha(d + f - 1) > 2(f - 1)$ for all $f \geq 1$. □

We point out for later use an easy corollary of Lemma 3.3 and Theorem 3.4.

Corollary 3.5. *For $d \geq 1$, the function $s_d : MU_{2d} \to \mathbb{Z}$ takes even values, and takes values divisible by 4 on decomposable elements.*

Here is the main result of §3.

Theorem 3.6. *Let $d \geq 0$ and $e \geq 1$ be such that $\alpha(d+e-1) > 2(e-1)$. Then the function $\frac{s_d}{2^e} : MU_{2d} \to \mathbb{Z}$ coincides modulo 2 with an integral linear combination of Chern numbers if and only if $\alpha(d + e) \geq 2e$.*

Proof. Assume first that $\alpha(d + e) < 2e$, and let $d + e = 2^{a_1} + \cdots + 2^{a_f}$ be the dyadic expansion of $d+e$, with $f \leq 2e-1$. Define $d_1 := 2^{a_1} + \cdots + 2^{a_{f-1}} - (e-1)$ and $d_2 := 2^{a_f} - 1$. One has $\alpha(d_1 + e - 1) = f - 1 \leq 2e - 2$. It thus follows from Theorem 3.4 that there exists $y_1 \in MU_{2d_1}$ such that $s_{d_1}(y_1)$ is not divisible by 2^e. Theorem 3.4 also shows the existence of $y_2 \in MU_{2d_2}$ such that $s_{d_2}(y_2)$ is not divisible by 4. We deduce from Lemma 3.3 that $s_d(y_1y_2)/2^e \in \mathbb{Z}$ is odd.

Since the map $\psi : MU_* \to MO_*$ is surjective and $MO_{2^{a_f}-1}$ contains no indecomposable element (see §3.1), there exists a decomposable element $z_2 \in MU_{2d_2}$ with $\psi(z_2) = \psi(y_2)$. By Corollary 3.5, one may replace y_2 with $y_2 - z_2$ and thus assume that $\psi(y_2) = 0$. But then $\psi(y_1y_2) = \psi(y_1)\psi(y_2) = 0$, which shows, in view of Lemma 3.2, that all the Chern numbers of y_1y_2 are even. Consequently, $s_d(y_1y_2)/2^e \pmod 2$ cannot be a linear combination with $\mathbb{Z}/2$ coefficients of Chern numbers of y_1y_2. We have thus proven the direct implication of the theorem.

Assume now that $\alpha(d+e) \geq 2e$. Let k be such that $1 \leq 2^k - 1 \leq d$. We claim that $MU_{2^{k+1}-2} \cdot MU_{2d-2^{k+1}+2}$ is included in the kernel of the morphism $\chi : MU_{2d} \to \mathbb{Z}/2$ obtained by reducing $\frac{s_d}{2^e} : MU_{2d} \to \mathbb{Z}$ modulo 2. To see it, choose $y \in MU_{2^{k+1}-2}$ and $z \in MU_{2d-2^{k+1}+2}$. We now compute

$$\alpha(d - 2^k + 1 + (e-1)) = \alpha(d - 2^k + e) \geq \alpha(d+e) - 1 > 2(e-1),$$

and Theorem 3.4 shows that $s_{d-2^k+1}(z)$ is divisible by 2^e. Since $s_{2^k-1}(y)$ is even by Corollary 3.5, Lemma 3.3 shows that $s_d(yz)$ is divisible by 2^{e+1}, as wanted.

We deduce from Lemma 3.1 that the kernel of $\psi : MU_{2d} \to MO_d$ is included in the kernel of χ, hence that $\chi = \mu \circ \psi$ for some morphism $\mu : MO_d \to \mathbb{Z}/2$. Since a class in MO_d is determined by its Stiefel–Whitney numbers (see §3.2), the morphism μ is a linear combination of Stiefel–Whitney numbers. Lemma 3.2 now implies that χ is the reduction modulo 2 of an integral linear combination of Chern numbers, which concludes the proof. ☐

Example 3.7. The first interesting case of Theorem 3.6 is $d = 2$ and $e = 1$. Since $\alpha(d+e) = \alpha(3) = 2 = 2e$, it predicts the existence of an integral linear combination of Chern numbers which coincides modulo 2 with $\frac{s_2}{2} : MU_4 \to \mathbb{Z}$.

We claim that this linear combination may be chosen to be c_2. Indeed, MU_4 is generated by classes of projective complex surfaces (see [Ada74, II, Corollary 10.8]), and for such a surface S, our claim follows from Noether's formula

$$s_2(S) = (c_1^2 - c_2)(S) = 12\chi(S, \mathcal{O}_S) - 2c_2(S).$$

4. The double point class

We use the results of §3 in combination with a double point formula. We give applications to Chow groups in §4.3, proving Theorems 0.2, 0.3 and 0.5. We also construct new examples of submanifolds of real loci of smooth projective varieties over $\mathbb{R}$ without algebraic approximations in §4.5, proving Theorems 0.7 and 0.8.

4.1. A consequence of Fulton's double point formula.

Formulas for the rational equivalence class of the double point locus of a morphism go back to Todd [Tod40, (7.01)] and Laksov [Lak78, Theorem 26] under strong assumptions on the morphism. The following proposition is an application of a refined double point formula of Fulton [Ful98, Theorem 9.3], which is valid for an arbitrary morphism.

Proposition 4.1. *Let $g : W \to X$ be a morphism of smooth projective varieties over $\mathbb{R}$. Let d be the dimension of W and assume that X has dimension $2d$. Let $N_{W/X} := [g^*T_X] - [T_W]$ be the virtual normal bundle of g.*

(i) If g is an embedding, then

$$\deg((g_*[W])^2) = \deg(c_d(N_{W/X})). \tag{4.2}$$

(ii) If g is an embedding in a neighbourhood of $g(\mathbb{C})^{-1}(X(\mathbb{R}))$, then

$$\deg((g_*[W])^2) \equiv \deg(c_d(N_{W/X})) \pmod 4. \tag{4.3}$$

Proof. Let $D(g) \subset W$ be the closed subset consisting of those $x \in W$ such that g is not an embedding above a neighbourhood of $g(x)$. Let $\mathbb{D}(g) \in \mathrm{CH}_0(D(g))$ be the double point class of g defined in [Ful98, §9.3]. By [Ful98, Theorem 9.3], one has

$$\mathbb{D}(g) = g^* g_*[W] - c_d(N_{W/X}) \in \mathrm{CH}_0(W).$$

Since $g_* g^* g_*[W] = (g_*[W])^2$ by the projection formula, we deduce that

$$\deg(g_*\mathbb{D}(g)) = \deg((g_*[W])^2) - \deg(c_d(N_{W/X})) \in \mathbb{Z}. \tag{4.4}$$

In case (i), one has $D(g) = \varnothing$, hence $\mathbb{D}(g) = 0$, and (4.4) implies (4.2).

Define $\overline{D}(g) := g(D(g))$. By [Ful98, Example 9.3.14], there exists a 0-cycle $\overline{\mathbb{D}}(g) \in \mathrm{CH}_0(\overline{D}(g))$ such that $g_*\mathbb{D}(g) = 2\overline{\mathbb{D}}(g) \in \mathrm{CH}_0(\overline{D}(g))$. The hypothesis of (ii) implies that $\overline{D}(g) \subset X$ has no real points, hence that $\deg(g_*\mathbb{D}(g)) = 2\deg(\overline{\mathbb{D}}(g))$ is divisible by 4. The desired congruence (4.3) now follows from (4.4). □

Remark 4.5. Proposition 4.1 (i) is of course much easier than Proposition 4.1 (ii). It follows for instance from [Ful98, Corollary 6.3].

4.2. **Weil restrictions of scalars and quotients of abelian varieties.** We gather here the geometric constructions that will be used in §4.3.

We let (A, λ) denote a very general principally polarized abelian variety of dimension $d \geq 1$ over $\mathbb{C}$. Let $e_1, \dots, e_d, f_1, \dots, f_d \in H^1(A(\mathbb{C}), \mathbb{Z})$ be a basis such that the principal polarization $\lambda \in H^2(A(\mathbb{C}), \mathbb{Z}) = \bigwedge^2 H^1(A(\mathbb{C}), \mathbb{Z})$ of A is equal to $\sum_i e_i \wedge f_i$. Denote by $(A', \lambda' = \sum_i e'_i \wedge f'_i)$ another copy of (A, λ), which we identify with the dual of A by means of the principal polarization. Let $\mu \in H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$ be the class of the Poincaré bundle. As in [BL04, Lemma 14.1.10] (whose notation is different from ours), one computes that $\mu = \sum_i (e_i \wedge f'_i + e'_i \wedge f_i)$.

Lemma 4.6. *The subring* $\mathrm{Hdg}^*(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q}) \subset H^*(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q})$ *of Hodge classes is generated by* λ, λ' *and* μ.

Proof. Let $V := H_1(A(\mathbb{C}), \mathbb{Q})$. In [Mur84], two subgroups $\mathrm{Hod}(A) \subset L(A)$ of the symplectic group $\mathrm{Sp}(V, \lambda)$ are considered. As A is very general, the subgroup $\mathrm{Hod}(A)$ is equal to $\mathrm{Sp}(V, \lambda)$ (see [BL04, Proposition 17.4.2]). It follows that $\mathrm{Hod}(A) = L(A) = \mathrm{Sp}(V, \lambda)$ and that $\mathrm{End}(A)_{\mathbb{Q}} = \mathbb{Q}$ (see *e.g.* [BL04, §17.6 (3)]). This implies that A has no factors of type III (in the sense of [Mur84, p. 199]). We can now apply [Mur84, Theorem 3.1] to show that $\mathrm{Hdg}^*(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q})$ is generated as a ring by $\mathrm{Hdg}^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q})$.

The Künneth formula induces an isomorphism of Hodge structures:

$$H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q}) \simeq H^2(A(\mathbb{C}), \mathbb{Q}) \oplus H^2(A'(\mathbb{C}), \mathbb{Q}) \oplus \mathrm{End}(H^1(A(\mathbb{C}), \mathbb{Q})).$$

The $\mathbb{Q}$-vector spaces $\mathrm{Hdg}^2(A(\mathbb{C}), \mathbb{Q})$ and $\mathrm{Hdg}^2(A'(\mathbb{C}), \mathbb{Q})$ are respectively generated by λ and λ', by Mattuck's theorem [BL04, Theorem 17.4.1]. As $\mathrm{End}(A)_{\mathbb{Q}} = \mathbb{Q}$, the $\mathbb{Q}$-vector space of Hodge classes in $\mathrm{End}(H^1(A(\mathbb{C}), \mathbb{Q}))$ is one-dimensional, generated by μ. This concludes the proof. □

In the next lemmas, we let $\mathbb{Z}/2$ act on $A \times A'$ by exchanging the two factors. We define $D := \mathrm{Hdg}^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$ and $E := D^{\mathbb{Z}/2}$, and we consider the subgroup $F \subset H^1(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$ generated by $e_2, \dots, e_d$, $f_1, \dots, f_d$, $e'_2, \dots, e'_d$, $f'_1, \dots, f'_d$, $2e_1, e_1 + e'_1, 2e'_1$. We also define $\varepsilon_{k,l,m} := \frac{\lambda^k}{k!} \cdot \frac{(\lambda')^l}{l!} \cdot \frac{\mu^m}{m!}$, and we set $\zeta_{k,l,m} := \varepsilon_{k,l,m} + \varepsilon_{l,k,m}$ if $k > l$ and $\zeta_{k,k,m} := \varepsilon_{k,k,m}$.

Lemma 4.7. *(i) The classes $\varepsilon_{k,l,m}$, where (k,l,m) ranges over the triples of nonnegative integers such that $k+l+m=d$, form a $\mathbb{Z}$-basis of D.*
(ii) The classes $\zeta_{k,l,m}$, where (k,l,m) ranges over the triples of nonnegative integers such that $k\geq l$ and $k+l+m=d$, form a $\mathbb{Z}$-basis of E.
(iii) The subgroups $E\cap\bigwedge^{2d}F$ and $2E$ of $H^{2d}(A(\mathbb{C})\times A'(\mathbb{C}),\mathbb{Z})$ are equal.

Proof. By Lemma 4.6, the integral classes listed in (i) span $\mathrm{Hdg}^{2d}(A(\mathbb{C})\times A'(\mathbb{C}),\mathbb{Q})$ as a $\mathbb{Q}$-vector space. To prove (i), it remains to show that these classes are linearly independent, and that they span a primitive sublattice of $\mathrm{Hdg}^{2d}(A(\mathbb{C})\times A'(\mathbb{C}),\mathbb{Z})$. Both assertions follow from the fact that, when one decomposes them in a basis of $H^{2d}(A(\mathbb{C})\times A'(\mathbb{C}),\mathbb{Z})=\bigwedge^{2d}(\mathbb{Z}e_1\oplus\cdots\oplus\mathbb{Z}f'_d)$ consisting of wedges of elements of $\{e_1,\ldots,f'_d\}$, the coefficient of the basis element which equals

$$(\bigwedge_{i=1}^{k} e_i\wedge f_i)\wedge(\bigwedge_{i=k+1}^{k+l} e'_i\wedge f'_i)\wedge(\bigwedge_{i=k+l+1}^{d} e_i\wedge f'_i) \tag{4.8}$$

up to a sign is nonzero only in the decomposition of $\varepsilon_{k,l,m}$.

Assertion (ii) follows from (i) and from the fact that the $\mathbb{Z}/2$-action exchanges λ and λ' and preserves μ.

We now prove (iii). That $2E\subset E\cap\bigwedge^{2d}F$ is obvious, and we check the reverse inclusion. Let Q be the quotient of $H^{2d}(A(\mathbb{C})\times A'(\mathbb{C}),\mathbb{Z})=\bigwedge^{2d}(\mathbb{Z}e_1\oplus\cdots\oplus\mathbb{Z}f'_d)$ by the subgroup generated by elements of the form $(e_1-e'_1)\wedge g_2\wedge\cdots\wedge g_{2d}$ with $g_2,\ldots,g_{2d}\in\{e_1,\ldots,f'_d\}$ on the one hand, and by elements of the form $g_1\wedge\cdots\wedge g_{2d}$ with $g_1,\ldots,g_{2d}\in\{e_1,\ldots,f'_d\}\setminus\{e_1,e'_1\}$ on the other hand. Consider the $\mathbb{Z}/2$-basis of $Q\otimes_{\mathbb{Z}}\mathbb{Z}/2$ consisting of the images of the elements of the form $e_1\wedge g_2\wedge\cdots\wedge g_{2d}$ with $g_2,\ldots,g_{2d}\in\{e_1,\ldots,f'_d\}\setminus\{e_1,e'_1\}$. Decompose in this basis the images in $Q\otimes_{\mathbb{Z}}\mathbb{Z}/2$ of the classes $\zeta_{k,l,m}$ considered in (ii). For $k\geq l$, the basis element (4.8) appears with nonzero coefficient only in the decomposition of the image of $\zeta_{k,l,m}$, with one exception: when $(k,l,m)=(1,0,d-1)$, the basis element (4.8) appears with nonzero coefficient only in the decomposition of the images of $\zeta_{1,0,d-1}$ and of $\zeta_{0,0,d}$. This shows that the classes $\zeta_{k,l,m}$ are $\mathbb{Z}/2$-linearly independent in $Q\otimes_{\mathbb{Z}}\mathbb{Z}/2$. As the image of $\bigwedge^{2d}F$ in $Q\otimes_{\mathbb{Z}}\mathbb{Z}/2$ is zero, we deduce that all the coefficients appearing in the decomposition of an element of $E\cap\bigwedge^{2d}F$ in the $\mathbb{Z}$-basis of E described in (ii) must be even. This shows that $E\cap\bigwedge^{2d}F\subset 2E$, as wanted. ☐

Lemma 4.9. *If $\delta,\delta'\in E$, then $\deg(\delta\cdot\delta')$ is even.*

Proof. It suffices to prove the lemma when δ and δ' belong to the $\mathbb{Z}$-basis of E described in Lemma 4.7 (ii). We may thus assume that $\delta=\zeta_{k,l,m}$ and $\delta'=\zeta_{k',l',m'}$. If $k>l$, then

$$\deg(\delta\cdot\delta')=\deg((\varepsilon_{k,l,m}+\varepsilon_{l,k,m})\cdot\delta')=2\deg(\varepsilon_{k,l,m}\cdot\delta')$$

by $\mathbb{Z}/2$-invariance of δ', and this number is even. The same argument applies if $k'>l'$. Assume now that $k=l$ and $k'=l'$. Then one has

$$\delta\cdot\delta'=\zeta_{k,k,m}\cdot\zeta_{k',k',m'}=\binom{k+k'}{k}^2\binom{m+m'}{m}\varepsilon_{k+k',k+k',m+m'}. \tag{4.10}$$

As a consequence of [Bea83, Lemme 1 p. 247] and of the projection formula applied to the morphism $A \times A' \to A'$, one computes that

$$\begin{aligned}\deg(\varepsilon_{k+k',k+k',m+m'}) &= (-1)^{d-k-k'} \deg_{A'} \big(\frac{(\lambda')^{k+k'}}{(k+k')\,!} \cdot \frac{(\lambda')^{d-k-k'}}{(d-k-k')\,!}\big) \\ &= (-1)^{d-k-k'} \binom{d}{k+k'}.\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11) yields $\deg(\delta \cdot \delta') = (-1)^{d-k-k'} \binom{k+k'}{k}^2 \binom{m+m'}{m} \binom{d}{k+k'}$.

Assume for contradiction that this number is odd. Let us say that two integers $n, n' \geq 0$ are *dyadically disjoint* if their dyadic expansions do not share any nonzero digit. The formula for the 2-adic valuation of the factorial appearing in [Rob00, p. 241] shows that $\binom{n+n'}{n}$ is odd if and only if n and n' are dyadically disjoint. It follows that k and k' are dyadically disjoint, and that so are $m = d - 2k$ and $m' = d - 2k'$, and $k + k'$ and $d - k - k'$. As $d - 2k$ and $d - 2k'$ are dyadically disjoint, the integer d is even. Write $d = 2e$. Then $e - k$ and $e - k'$ are dyadically disjoint. As k and k', as well as $e - k$ and $e - k'$ and also $k + k'$ and $(e - k) + (e - k')$ are dyadically disjoint, the four integers k, k', $e - k$ and $e - k'$ are pairwise dyadically disjoint. Since $k + (e - k) = k' + (e - k')$, this is impossible unless these four numbers all vanish. This contradicts the assumption that $d \geq 1$. □

Let $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and consider the G-module $\mathbb{Z}(j) := (\sqrt{-1})^j \mathbb{Z} \subset \mathbb{C}$. If X is a variety over $\mathbb{R}$, letting G act both on $X(\mathbb{C})$ and on $\mathbb{Z}(j)$ endows $H^k(X(\mathbb{C}), \mathbb{Z}(j))$ with an action of G.

Proposition 4.12. *For all $d \geq 1$, there exists an abelian variety X of dimension $2d$ over $\mathbb{R}$ and a class $\beta \in \mathrm{CH}^d(X)$ with the following properties.*

(i) *If $\gamma, \gamma' \in H^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$ are Hodge and G-invariant, then $\deg(\gamma \cdot \gamma')$ is even.*
(ii) $\deg(\beta^2) \equiv 2 \pmod 4$.
(iii) $\mathrm{cl}_{\mathbb{R}}(\beta) = 0 \in H_d(X(\mathbb{R}), \mathbb{Z}/2)$.

Proof. Let (A, λ) be a very general principally polarized abelian variety of dimension d which is defined over $\mathbb{R}$ (by Baire's theorem, one may choose a very general real point of the moduli space M of d-dimensional principally polarized abelian varieties with level 3 structure, since M is a smooth variety). Write $x \mapsto \bar{x}$ for the action of complex conjugation on $A(\mathbb{C})$.

Let (A', λ') be another copy of (A, λ). The real abelian variety $A \times A'$ has set of complex points $A(\mathbb{C}) \times A'(\mathbb{C})$ with an action of complex conjugation given by $(x, y) \mapsto (\overline{x}, \overline{y})$. Define $X := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(A)$ to be the Weil restriction of scalars of $A_{\mathbb{C}}$. It is the abelian variety over $\mathbb{R}$ whose set of complex points is $X(\mathbb{C}) = A(\mathbb{C}) \times A'(\mathbb{C})$, with an action of complex conjugation given by $(x, y) \mapsto (\overline{y}, \overline{x})$. The subvariety $A_{\mathbb{C}} \times \{0\} \cup \{0\} \times A'_{\mathbb{C}}$ of $A_{\mathbb{C}} \times A'_{\mathbb{C}}$ descends to a subvariety $Z \subset X$, and we define $\beta := [Z] \in \mathrm{CH}^d(X)$. Since the normalization of Z has no real points, one has $\mathrm{cl}_{\mathbb{R}}(\beta) = 0$. Moreover, $\deg(\beta^2) = 2$. Assertions (ii) and (iii) are proven.

Let $\mu \in H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}(1))$ be the class of the Poincaré bundle. As the cycle class map $\mathrm{CH}^1(A_{\mathbb{C}} \times A'_{\mathbb{C}}) \to H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}(1))$ is G-equivariant, the classes λ, λ' and μ all belong to $H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}(1))^G$. It then follows from Lemma 4.7 (i) that the group $\mathrm{Hdg}^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}(d))$ consists exclusively of G-invariant classes.

As the action of complex conjugation on $X(\mathbb{C})$ is the composition of the action of complex conjugation on $A(\mathbb{C}) \times A'(\mathbb{C})$ and of the exchange of the two factors, we deduce that group of G-invariant Hodge classes in $H^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$ is exactly the group E described in Lemma 4.7 (ii). Assertion (i) now follows from Lemma 4.9. □

The next proposition is a variant of Proposition 4.12 which works over the complex numbers, but which is slightly more complicated.

Proposition 4.13. *For all $d \geq 1$, there exists a $2d$-dimensional smooth projective variety X over $\mathbb{C}$ and a class $\beta \in \mathrm{CH}^d(X)$ with the following properties.*

(i) If $\gamma, \gamma' \in H^{2d}(X(\mathbb{C}), \mathbb{Z})$ are Hodge, then $\deg(\gamma \cdot \gamma')$ is even.
(ii) One has $\deg(\beta^2) \equiv 2 \pmod 4$.
(iii) All higher Chern classes of X are torsion, i.e., $c(X) = 1 \in \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. Let (A, λ) be a very general principally polarized abelian variety of dimension d over $\mathbb{C}$. Let $e_1, \dots, e_d, f_1, \dots, f_d \in H^1(A(\mathbb{C}), \mathbb{Z})$ be a basis such that $\lambda = \sum_i e_i \wedge f_i$. Let $\tau \in A(\mathbb{C})[2] \simeq H_1(A(\mathbb{C}), \mathbb{Z}/2)$ be the 2-torsion point associated with the morphism $H^1(A(\mathbb{C}), \mathbb{Z}) \to \mathbb{Z}/2$ sending e_1 to 1 and $e_2, \dots, e_d, f_1, \dots, f_d$ to 0. Denote by $(A', \lambda' = \sum_i e'_i \wedge f'_i, \tau')$ another copy of $(A, \lambda = \sum_i e_i \wedge f_i, \tau)$.

Let $\mathbb{Z}/4$ act on $A \times A'$ via $(x, x') \mapsto (x' + \tau, x)$. Let $p : A \times A' \to X$ (resp. $q : A \times A' \to B$) be the quotient of $A \times A'$ by $\mathbb{Z}/4$ (resp. by the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/4$). Since $\mathbb{Z}/2$ acts on $A \times A'$ via $(x, x') \mapsto (x + \tau, x' + \tau')$, we see that B is an abelian variety, and that $q^*(H^1(B(\mathbb{C}), \mathbb{Z})) \subset H^1(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$ is the subgroup generated by $e_2, \dots, e_d, f_1, \dots, f_d, e'_2, \dots, e'_d, f'_1, \dots, f'_d, 2e_1, e_1 + e'_1, 2e'_1$.

Assertion (iii) follows at once from the fact that $c(A \times A') = 1 \in \mathrm{CH}^*(A \times A')$ since $p : A \times A' \to X$ is finite étale.

Consider $Z := p(A \times \{0\}) \subset A \times A'$, and define $\beta := [Z] \in \mathrm{CH}^d(X)$. As $p^*\beta = [A \times \{0\}] + [A \times \{\tau'\}] + [\{0\} \times A'] + [\{\tau\} \times A']$, one computes that $\deg(p^*\beta^2) = 8$. Since $\deg(p) = 4$, one has $\deg(\beta^2) = 2$. This proves assertion (ii).

Let $\gamma, \gamma' \in \mathrm{Hdg}^{2d}(X(\mathbb{C}), \mathbb{Z})$ be Hodge classes. As translations act trivially on the cohomology of $A \times A'$, the automorphism $(x, x') \mapsto (x' + \tau, x)$ acts in cohomology as $(x, x') \mapsto (x', x)$. It follows that $\delta := p^*\gamma$ and $\delta' := p^*\gamma'$ are invariant under the involution of $\mathrm{Hdg}^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$ exchanging the two factors. In other words, the classes δ and δ' belong to the group denoted by E in Lemma 4.7. Since moreover δ and δ' belong to $\mathrm{Im}(q^*)$, Lemma 4.7 (iii) shows that δ and δ' are divisible by 2 in E. Lemma 4.9 now implies that $\deg(\delta \cdot \delta') \equiv 0 \pmod 8$. Since $\deg(p) = 4$, we deduce that $\deg(\gamma \cdot \gamma') \equiv 0 \pmod 2$, which proves (i). □

4.3. **High-dimensional cycles.** Here are applications to Chow groups.

Theorem 4.14. *Let $d \geq c$ be such that $\alpha(c+1) \geq 3$. Then there exists an abelian variety X of dimension $c + d$ over $\mathbb{R}$ such that $\mathrm{CH}_d(X)$ is not generated by classes of closed subvarieties of X that are smooth along their real loci.*

Proof. Suppose first that $d = c$, and let X and β be as in Proposition 4.12. Assume for contradiction that $\beta = \sum_i n_i[Y_i] \in \mathrm{CH}_d(X)$ where $n_i \in \mathbb{Z}$ and the Y_i are integral closed subvarieties of X that are smooth along their real loci. One computes:

$$\deg(\beta^2) = \sum_i n_i^2 \deg([Y_i]^2) + 2 \sum_{i<j} n_i n_j \deg([Y_i] \cdot [Y_j]). \tag{4.15}$$

The existence of Krasnov's cycle class map $\mathrm{cl} : \mathrm{CH}_d(X) \to H^{2d}_G(X(\mathbb{C}), \mathbb{Z}(d))$ to G-equivariant Betti cohomology refining the usual complex cycle class map

$\mathrm{cl}_{\mathbb{C}} : \mathrm{CH}_d(X) \to H^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$ to Betti cohomology [Kra94, Theorem 0.6], the fact that the image of $\mathrm{cl}_{\mathbb{C}}$ consists of Hodge classes, and assertion (i) of Proposition 4.12, combine to show that $2\sum_{i<j} n_i n_j \deg([Y_i]\cdot[Y_j])$ is divisible by 4. Let $g_i : W_i \to Y_i$ be a resolution of singularities which is an isomorphism above $Y_i(\mathbb{R})$. Proposition 4.1 (ii) and the fact that all higher Chern classes of the abelian variety X vanish show that $\deg([Y_i]^2) \equiv \deg(s_d(W_i)) \pmod 4$. Since $\alpha(d+1) \geq 3$ by hypothesis, Theorem 3.4 implies that $\deg(s_d(W_i)) = s_d([W_i(\mathbb{C})]) \equiv 0 \pmod 4$. These congruences and assertion (ii) of Proposition 4.12 contradict (4.15).

To deal with the general case, apply the $d = c$ case to get a smooth projective variety X' of dimension $2c$ over $\mathbb{R}$ and a class $\beta' \in \mathrm{CH}_c(X')$ that is not a linear combination of classes of subvarieties of X' that are smooth along their real loci. Define $X := X' \times A$ where A is any abelian variety of dimension $d - c$ over $\mathbb{R}$, and $\beta := pr_1^*\beta' \in \mathrm{CH}_d(X)$. That β is not a linear combination of classes of subvarieties of X that are smooth along their real loci follows from the corresponding property of β' and from the Bertini theorem. □

Theorem 4.16. *If $d \geq c$ are such that $\alpha(c+1) \geq 2$, there exists an abelian variety X of dimension $c+d$ over $\mathbb{R}$ such that $\mathrm{Ker}\big[\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}), \mathbb{Z}/2)\big]$ is not generated by classes of closed subvarieties of X with empty real loci.*

Proof. The proof is almost identical to the proof of Theorem 4.14, replacing *that are smooth along their real loci* by *with empty real loci* everywhere. Only the argument used in the $d = c$ case to show that $s_d([W_i(\mathbb{C})]) \equiv 0 \pmod 4$ needs to be modified, as follows. Since $W_i(\mathbb{R}) = \varnothing$, one has $\psi([W_i(\mathbb{C})]) = 0 \in MO_d$ by [Con79, Theorem 22.4]. We deduce from Lemma 3.2 that all the Chern numbers of $[W_i(\mathbb{C})] \in MU_{2d}$ are even. Theorem 3.6 and the hypothesis that $\alpha(d+1) \geq 2$ imply that $s_d([W_i(\mathbb{C})]) \equiv 0 \pmod 4$, as wanted. □

Theorem 4.17. *If $d \geq c$ are such that $\alpha(c+1) \geq 3$, there exists a smooth projective variety X of dimension $c+d$ over $\mathbb{C}$ such that $\mathrm{CH}_d(X)$ is not generated by classes of smooth closed subvarieties of X.*

Proof. The proof is similar to that of Theorem 4.14. The argument at the end of the proof of Theorem 4.14 shows that we may assume that $d = c$.

Let X and β be as in Proposition 4.13. Assume that $\beta = \sum_i n_i[Y_i] \in \mathrm{CH}_d(X)$ where $n_i \in \mathbb{Z}$ and the Y_i are smooth closed subvarieties of X, and consider equation (4.15). Since the Betti cohomology classes of the Y_i are Hodge, assertion (i) of Proposition 4.13 shows that $2\sum_{i<j} n_i n_j \deg([Y_i]\cdot[Y_j])$ is divisible by 4. Assertion (iii) of Proposition 4.13 and [Ful98, Corollary 6.3] together imply that $\deg([Y_i]^2) = \deg(s_d(W_i))$. Since $\alpha(d+1) \geq 3$, Theorem 3.4 implies that $\deg(s_d(W_i))$ is divisible by 4. Assertion (ii) of Proposition 4.13 now contradicts (4.15). □

4.4. **Hypersurfaces in abelian varieties.** We give here a geometric construction based on a Noether–Lefschetz argument, on which the proof of Theorem 4.19 relies.

Proposition 4.18. *For all $d, e \geq 1$, there exists a $2d$-dimensional smooth projective variety X over $\mathbb{R}$ with the following properties.*

(i) The total Chern class $c(X) \in \mathrm{CH}^*(X)$ of X satisfies $c(X) \equiv 1 \pmod{2^{e+1}}$.

(ii) The subgroup of Hodge classes $\mathrm{Hdg}^{2d}(X(\mathbb{C}), \mathbb{Z}) \subset H^{2d}(X(\mathbb{C}), \mathbb{Z})$ is generated by a class $\eta \in \mathrm{Hdg}^{2d}(X(\mathbb{C}), \mathbb{Z})$ with $\deg(\eta^2) \equiv 0 \pmod{2^{e+1}}$.

(iii) One has $X(\mathbb{R}) \neq \varnothing$.

Proof. Arguing as in the proof of Proposition 4.12, choose a very general principally polarized abelian variety A of dimension $2d+1$ over $\mathbb{R}$. The principal polarization of A is represented by an ample line bundle $\mathcal{L}$ on A which is defined over $\mathbb{R}$ (see [SS89, Theorem 4.1]). The group $\mathrm{Hdg}^{2d}(A(\mathbb{C}),\mathbb{Z})$ of degree $2d$ Hodge classes on $A_{\mathbb{C}}$ is generated by $\frac{1}{d!}c_1(\mathcal{L})^d$ by Mattuck's theorem (see [BL04, Theorem 17.4.1]) since $\frac{1}{d!}c_1(\mathcal{L})^d$ is a primitive integral cohomology class.

Let $l \gg 0$ be such that $2^{e+1} \mid l$ and $\mathcal{L}^{\otimes l}$ is very ample. Choose a Lefschetz pencil of sections $\mathcal{L}^{\otimes l}$, and let $X \subset A$ be a very general member of this pencil with $X(\mathbb{R}) \neq \varnothing$.

The restriction morphism $H^{2d}(A(\mathbb{C}),\mathbb{Z}) \to H^{2d}(X(\mathbb{C}),\mathbb{Z})$ is injective with torsion free cokernel Λ, by the weak Lefschetz theorem [AF59, Theorem 2]. Let $\Xi \subset \Lambda_{\mathbb{C}}$ be the subspace of Hodge classes. Since X was chosen very general, any class in Ξ remains Hodge when transported horizontally using the Gauss–Manin connection of the Lefschetz pencil. It follows that Ξ is stabilized by the monodromy of the Lefschetz pencil on $\Lambda_{\mathbb{C}}$. Since this action is irreducible as a consequence of the hard Lefschetz theorem (see [Lam81, Theorem 7.3.2]) and since the Hodge structure on $\Lambda_{\mathbb{C}}$ is not trivial (as the restriction map $H^{2d}(A,\mathcal{O}_A) \to H^{2d}(X,\mathcal{O}_X)$ is not surjective), we deduce that $\Xi = 0$. This shows that all Hodge classes in $H^{2d}(X(\mathbb{C}),\mathbb{Z})$ are in the image of the injective restriction map $H^{2d}(A(\mathbb{C}),\mathbb{Z}) \to H^{2d}(X(\mathbb{C}),\mathbb{Z})$, hence that $\mathrm{Hdg}^{2d}(X(\mathbb{C}),\mathbb{Z})$ is generated by the restriction η of $\frac{1}{d!}c_1(\mathcal{L})^d$. We now compute $\deg(\eta^2) = \deg(\frac{1}{(d!)^2}c_1(\mathcal{L})^{2d} \cdot c_1(\mathcal{L}^{\otimes l})) = \frac{l(2d+1)!}{(d!)^2}$, which proves (ii).

The normal exact sequence $0 \to T_X \to (T_A)|_X \to \mathcal{L}^{\otimes l}|_X \to 0$ finally shows that $c(X) = c(A)|_X \cdot c(\mathcal{L}^{\otimes l}|_X)^{-1} = (1 + lc_1(\mathcal{L})|_X)^{-1} \equiv 1 \pmod{2^{e+1}}$, proving (i). □

4.5. **Submanifolds with no algebraic approximations.** Now come the promised applications to algebraic approximation. We refer to §0.3 for the definition of the cobordism group $MO_d(X(\mathbb{R}))$ of the real locus of a smooth projective variety X over $\mathbb{R}$, and of its subgroup $MO_d^{\mathrm{alg}}(X(\mathbb{R}))$ of algebraic cobordism classes. Recall that MO_d denotes the cobordism ring, which is the cobordism group of the point (see §3.1).

Theorem 4.19. *Assume that $c,d,e \geq 1$ are such that $d \geq c$ and $\alpha(c+e) = 2e$. Then there exist a smooth projective variety X of dimension $c+d$ over $\mathbb{R}$ and a d-dimensional closed $\mathcal{C}^\infty$ submanifold $j : M \hookrightarrow X(\mathbb{R})$ such that the following properties hold for all d-dimensional closed subvarieties $i : Y \hookrightarrow X$.*

(i) One has $[j] \in MO_d^{\mathrm{alg}}(X(\mathbb{R}))$.
(ii) If Y is smooth, then $[M] \neq [Y(\mathbb{R})] \in MO_d$.
(iii) If $e = 1$ and Y is smooth along $Y(\mathbb{R})$, then $[M] \neq [Y(\mathbb{R})] \in MO_d$.

We first prove Theorem 4.19 in the particular case where $c = d$.

Lemma 4.20. *If $c = d$, Theorem 4.19 holds with any X as in Proposition 4.18.*

Proof. Recall from §3.2 that if $I = (i_1, i_2, \dots)$ is a sequence of nonnegative integers, we set $|I| := \sum_r r i_r$. By Theorem 3.6, there exists a degree 1 homogeneous polynomial $P \in \mathbb{Z}[x_I]_{|I|=d}$ such that $s_d(y) \equiv 2^e P(c_I(y)) \pmod{2^{e+1}}$ for all $y \in MU_{2d}$.

Theorem 3.4 shows the existence of $y_0 \in MU_{2d}$ such that $s_d(y_0) \not\equiv 0 \pmod{2^{e+1}}$, hence such that $P(c_I(y_0)) \not\equiv 0 \pmod 2$. Letting M be a compact $\mathcal{C}^\infty$ manifold representing the $\psi(y_0) \in MO_d$, Lemma 3.2 shows that $P(w_I(M)) \neq 0 \in \mathbb{Z}/2$.

A theorem of Whitney [Whi44, Theorem 5] asserts that any d-dimensional compact $\mathcal{C}^\infty$ manifold may be embedded in $\mathbb{R}^{2d}$. It follows that one may choose an embedding $j : M \hookrightarrow X(\mathbb{R})$ of M in a small ball of $X(\mathbb{R})$. Since this small ball is contractible, j is homotopic (hence cobordant) to a constant map $M \to X(\mathbb{R})$. As M is cobordant to the real locus of a smooth projective variety over $\mathbb{R}$ (see §3.1), one deduces that $[j] \in MO_d^{\mathrm{alg}}(X(\mathbb{R}))$, proving (i).

Let $W \to Y$ be a desingularization of Y which is an isomorphism above $Y(\mathbb{R})$ and let $g : W \to X$ be the induced morphism. Proposition 4.1 shows, under the assumptions of either (ii) or (iii), that

$$\deg([Y]^2) = \deg((g_*[W])^2) \equiv \deg(c_d(N_{W/X})) \pmod{2^{e+1}}. \tag{4.21}$$

Since the Betti cohomology class of Y is a Hodge class, assertion (ii) of Proposition 4.18 shows that $\deg([Y]^2) \equiv 0 \pmod{2^{e+1}}$. The total Chern class of $N_{W/X}$ is $c(N_{W/X}) = c(g^*T_X) \cdot c(T_W)^{-1} = g^*c(X) \cdot s(W)$, where $s(W) \in \mathrm{CH}^*(W)$ denotes the total Segre class of W. We deduce from assertion (iii) of Proposition 4.18 that $c_d(N_{W/X}) \equiv s_d(W) \pmod{2^{e+1}}$. Together with (4.21), these facts shows that $s_d(W) \equiv 0 \pmod{2^{e+1}}$.

Consequently, we have $s_d(W(\mathbb{C})) \equiv 0 \pmod{2^{e+1}}$. By our choice of P, we deduce that $P(c_I(W(\mathbb{C}))) \equiv 0 \pmod 2$. Conner and Floyd [Con79, Theorem 22.4] have proven that $W(\mathbb{C})$ and $W(\mathbb{R}) \times W(\mathbb{R})$ are cobordant, and it follows that $\psi([W(\mathbb{C})]) = [W(\mathbb{R})] \in MO_d$. Lemma 3.2 now shows that $P(w_I(W(\mathbb{R}))) = 0 \in \mathbb{Z}/2$, hence that $P(w_I(Y(\mathbb{R}))) = P(w_I(W(\mathbb{R}))) \neq P(w_I(M))$. Since Stiefel–Whitney numbers are cobordism invariants (see §3.2), we have $[M] \neq [Y(\mathbb{R})] \in MO_d$. □

The proof of Theorem 4.19 in general easily reduces to the above lemma.

Proof of Theorem 4.19. Lemma 4.20 produces a smooth projective variety X' of dimension $2c$ over $\mathbb{R}$ and a c-dimensional closed $\mathcal{C}^\infty$ submanifold $j' : M' \hookrightarrow X'(\mathbb{R})$ satisfying properties (i)-(iii) of Theorem 4.19 (with d replaced by c). Define $X := X' \times \mathbb{P}^{d-c}_{\mathbb{R}}$ and $M := M' \times \mathbb{P}^{d-c}(\mathbb{R})$, and consider the embedding $j := (j', \mathrm{Id}) : M \hookrightarrow X(\mathbb{R})$.

Our choice of X' and M', shows the existence of a morphism $g' : W' \to X'$ of smooth projective varieties over $\mathbb{R}$ such that j' is cobordant to $g'(\mathbb{R})$. Defining $W := W' \times \mathbb{P}^{d-c}_{\mathbb{R}}$ and $g := (g', \mathrm{Id})$, we see that j is cobordant to $g(\mathbb{R})$, hence that $[j] \in MO_d^{\mathrm{alg}}(X(\mathbb{R}))$, which proves (i).

Suppose now that $i : Y \hookrightarrow X$ is as in the statement of Theorem 4.19 and satisfies the hypothesis of either (ii) or (iii). Let $x \in \mathbb{P}^{d-c}(\mathbb{R})$ be a general point, and define $Y' := Y \cap (X' \times \{x\})$ and $i' : Y' \hookrightarrow X' \times \{x\} \simeq X'$ to be the natural inclusion. Bertini's theorem ensures that Y' is smooth in case (ii) and that Y' is smooth along $Y'(\mathbb{R})$ in case (iii). If M were cobordant to $Y(\mathbb{R})$, Sard's theorem would imply that M' is cobordant to $Y'(\mathbb{R})$. This contradicts our choice of M' and proves (ii) and (iii). □

Remarks 4.22. (i) Assertions (i) and (iii) of Theorem 4.19 prove Theorem 0.7.

(ii) It is striking that the obstructions to M being approximable by real loci of algebraic subvarieties of X provided by Theorem 4.19 (ii)-(iii) involve cobordism theory, although $[j] \in MO_d^{\mathrm{alg}}(X(\mathbb{R}))$ by Theorem 4.19 (i). Loosely speaking, the map j is cobordant to an algebraic map, but not to an algebraic embedding.

(iii) Complex cobordism and Theorem 3.6 are not needed to prove Theorem 4.19 for $c = 2$. One may use Noether's formula instead, as in Example 3.7.

(iv) In the setting of Theorem 4.19, one could moreover arrange that the inclusion $j : M \hookrightarrow X(\mathbb{R})$ be approximable in the $\mathcal{C}^\infty$ topology by the inclusion of the set of smooth real points of an algebraic subvariety $Z \subset X$ with compact set of smooth real points (which necessarily also has some singular real points if $e = 1$, in view of Theorem 4.19 (iii)). To prove it when $c = d$, one can use linkage and general position arguments as in Sections 1 and 2. We do not give a detailed proof here. The general case reduces to the case $c = d$ as in the proof of Theorem 4.19.

The proof of Theorem 4.23 is a variant of the proof of Lemma 4.20. Since $c(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^{2^{k+1}-1}_{\mathbb{R}}) \not\equiv 1 \pmod 4$, the argument is slightly more complicated.

The relation $(\sum_r w_r(M))(\sum_r \overline{w}_r(M)) = 1$ defines the *normal Stiefel–Whitney classes* $\overline{w}_r(M) \in H^r(M, \mathbb{Z}/2)$ of a compact $\mathcal{C}^\infty$ manifold M. We also recall that if $i : Y \to X$ is a morphism of varieties over $\mathbb{R}$, we let $i(\mathbb{R}) : Y(\mathbb{R}) \to X(\mathbb{R})$ be the induced map between sets of real points.

Theorem 4.23. *Fix $k \geq 1$ and define $X := \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^{2^{k+1}-1}_{\mathbb{R}}$. There exists a 2^k-dimensional closed $\mathcal{C}^\infty$ submanifold $j : M \hookrightarrow X(\mathbb{R})$ such that $[j] \neq [i(\mathbb{R})] \in MO_{2^k}(X(\mathbb{R}))$ for all 2^k-dimensional closed subvarieties $i : Y \hookrightarrow X$ that are smooth along $Y(\mathbb{R})$.*

Proof. Since $\alpha(2^k+1) = 2$, Theorem 3.6, shows that there is a degree 1 homogeneous polynomial $P \in \mathbb{Z}[x_I]_{|I|=2^k}$ with $s_{2^k}(y) \equiv 2P(c_I(y)) \pmod 4$ for all $y \in MU_{2^{k+1}}$. By Theorem 3.4, one may find $y_0 \in MU_{2^{k+1}}$ such that $s_{2^k}(y_0) \not\equiv 0 \pmod 4$, hence such that $P(c_I(y_0)) \not\equiv 0 \pmod 2$. Letting M be a compact $\mathcal{C}^\infty$ manifold representing $\psi(y_0) \in MO_{2^k}$, Lemma 3.2 shows that $P(w_I(M)) \neq 0 \in \mathbb{Z}/2$. By Whitney's theorem [Whi44, Theorem 5], one may embed $j : M \hookrightarrow X(\mathbb{R})$ in a small ball of $X(\mathbb{R})$.

Let $i : Y \hookrightarrow X$ be as in the statement of Theorem 4.23. Assume for contradiction that $[j] = [i(\mathbb{R})] \in MO_{2^k}(X(\mathbb{R}))$. Let $W \to Y$ be a desingularization of Y which is an isomorphism above $Y(\mathbb{R})$ and let $g : W \to X$ be the induced morphism. Proposition 4.1 (ii) shows that $\deg([Y]^2) = \deg((g_*[W])^2) \equiv \deg(c_{2^k}(N_{W/X})) \pmod 4$, hence that

$$\deg([Y]^2) \equiv \sum_{r=0}^{2^k} \deg(g^* c_r(X) \cdot s_{2^k - r}(W)) \pmod 4. \tag{4.24}$$

Consider the Borel–Haefliger cycle class map $\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}^*(X) \to H^*(X(\mathbb{R}), \mathbb{Z}/2)$ ([BH61], see also [BW20a, §1.6.2]). Since $[j] = [i(\mathbb{R})] \in MO_{2^k}(X(\mathbb{R}))$, one has $\mathrm{cl}_{\mathbb{R}}([Y]) = [M] = 0 \in H^{2^k}(X(\mathbb{R}), \mathbb{Z}/2)$. Set $H_1 := c_1(\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(1)) \in \mathrm{CH}^1(\mathbb{P}^1_{\mathbb{R}})$ and $H_2 := c_1(\mathcal{O}_{\mathbb{P}^{2^{k+1}-1}_{\mathbb{R}}}(1)) \in \mathrm{CH}^1(\mathbb{P}^{2^{k+1}-1}_{\mathbb{R}})$. As $\mathrm{CH}^{2^k}(X)$ is generated by $(H_2)^{2^k}$ and $H_1(H_2)^{2^k-1}$, we compute that the kernel of $\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}^{2^k}(X) \to H^{2^k}(X(\mathbb{R}), \mathbb{Z}/2)$ is generated by $2(H_2)^{2^k}$ and $2H_1(H_2)^{2^k-1}$. As a consequence, $[Y] \in \mathrm{CH}^{2^k}(X)$ is a multiple of 2, and hence $\deg([Y]^2)$ is divisible by 4.

The Euler exact sequences $0 \to \mathcal{O}_{\mathbb{P}^N_{\mathbb{R}}} \to \mathcal{O}_{\mathbb{P}^N_{\mathbb{R}}}(1)^{\oplus N+1} \to T_{\mathbb{P}^N_{\mathbb{R}}} \to 0$ and the Whitney sum formula yield $c(X) = (1+H_1)^2(1+H_2)^{2^k} \in CH^*(X)$. Since $H_1^2 = H_2^{2^k} = 0$, we deduce that $c(X) \equiv 1 \pmod 2$. For $r \geq 1$, let $\gamma_r \in \mathrm{CH}^r(X)$ be such that $c_r(X) = 2\gamma_r$. Since Borel and Haefliger have shown that $\mathrm{cl}_{\mathbb{R}}(c(W)) = w(W(\mathbb{R}))$ ([BH61, §5.18], see also [Kra91, Proposition 3.5.1]), we have $\mathrm{cl}_{\mathbb{R}}(s(W)) = \overline{w}(W(\mathbb{R}))$.

We deduce that, for $r \geq 1$,

$$\begin{aligned} \deg(\mathrm{cl}_{\mathbb{R}}(g^*\gamma_r \cdot s_{2^k-r}(W))) &= \deg(g(\mathbb{R})^*\mathrm{cl}_{\mathbb{R}}(\gamma_r) \cdot \overline{w}_{2^k-r}(W(\mathbb{R}))) \\ &= \deg(j^*\mathrm{cl}_{\mathbb{R}}(\gamma_r) \cdot \overline{w}_{2^k-r}(M)) \\ &= 0 \in \mathbb{Z}/2, \end{aligned} \tag{4.25}$$

where the first equality follows from the functorial properties of $\mathrm{cl}_{\mathbb{R}}$ (see [BW20a, §1.6.2]), the second from the equality of the Stiefel–Whitney numbers of the cobordant maps j and $g(\mathbb{R}) = i(\mathbb{R})$ (see [Con79, Theorem 17.3]), and the third holds since $j^* : H^r(X(\mathbb{R}), \mathbb{Z}/2) \to H^r(M, \mathbb{Z}/2)$ vanishes for $r \geq 1$ because the image of j is included in a small ball of $X(\mathbb{R})$.

Equation (4.25) demonstrates that $\deg(g^*\gamma_r \cdot s_{2^k-r}(W)) \in \mathbb{Z}$ is even, and hence that $\deg(g^*c_r(X) \cdot s_{2^k-r}(W))$ is divisible by 4, for all $r \geq 1$. Plugging the congruences we have obtained into (4.24) shows that $\deg(s_{2^k}(W)) \equiv 0 \pmod 4$, hence that $s_{2^k}(W(\mathbb{C})) \equiv 0 \pmod 4$. Our choice of P implies that $P(c_I(W(\mathbb{C}))) \equiv 0 \pmod 2$.

By [Con79, Theorem 22.4], one has $\psi([W(\mathbb{C})]) = [W(\mathbb{R})] \in MO_{2^k}$. Lemma 3.2 now shows that $P(w_I(W(\mathbb{R}))) = 0 \in \mathbb{Z}/2$, hence that $P(w_I(Y(\mathbb{R}))) \neq P(w_I(M))$. We deduce that $[M] \neq [Y(\mathbb{R})] \in MO_{2^k}$ by cobordism invariance of Stiefel–Whitney numbers. A fortiori, $[j] \neq [i(\mathbb{R})] \in MO_{2^k}(X(\mathbb{R}))$, which is a contradiction. □

Remarks 4.26. (i) Theorem 4.23 implies Theorem 0.8 by [Wal16, Proposition 4.4.4].

(ii) Theorem 4.23 is false for $k = 0$ by [BCR98, Theorem 12.4.11]. The proof fails in this case because $\alpha(2^k + 1) < 2$ precisely for this value of k.

(iii) The simplest particular case of Theorem 4.23 is the following. Embed $\mathbb{P}^2(\mathbb{R})$ in $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ and let $j : \mathbb{P}^2(\mathbb{R}) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^3(\mathbb{R})$ be the induced embedding. Then j is not cobordant to the inclusion of the real locus of a closed subvariety of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^3_{\mathbb{R}}$ which is smooth along its real locus. A fortiori, j may not be isotoped to such a real locus. As in Remark 4.22 (iii), the use of Theorem 3.6 may be replaced by Noether's formula in the proof of this particular case.

(iv) The conclusion of Theorem 4.23, is not explained by a difference between the groups $MO_*^{\mathrm{alg}}(X(\mathbb{R}))$ and $MO_*(X(\mathbb{R}))$, as they coincide by [BT80b, Remark 3 p. 103] or [IS88, Corollary 1 p. 314].

5. Algebraic approximation and algebraic homology

In this section, we fix a smooth projective variety X of dimension $c + d$ over $\mathbb{R}$, and a closed d-dimensional $\mathcal{C}^\infty$ submanifold $j : M \hookrightarrow X(\mathbb{R})$. Recall from §0.3 the definition of the approximation property (A) and of its necessary condition (B) based on cobordism. We now study variants of these properties, our goal being Theorem 5.6.

5.1. A stronger approximation property.

It is natural to consider the following strenghtening of the approximation property (A) considered in §0.3:

Property (A'). For all neighbourhoods $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$ of the inclusion, there exist $\phi \in \mathcal{U}$ and a smooth closed subvariety $Y \subset X$ such that $\phi(M) = Y(\mathbb{R})$.

The following two theorems are analogues of Theorem 0.6 and 0.7 for the property (A'). They are consequences of Theorems 2.7 and 4.19 respectively.

Theorem 5.1. *Properties* (A') *and* (B) *are equivalent if* $d < c$ *and* $d \leq 3$.

Theorem 5.2. *Let $d \geq c$ and $e \geq 1$ be such that $\alpha(c+e) = 2e$. Then there exist X and M such that (A') fails but (B) holds.*

Remarks 5.3. (i) We do not know if it is possible to get rid of the hypothesis that $d \leq 3$ in Theorem 5.1.

(ii) Examples where (B) (and even (A)) holds but (A') fails had already been obtained by Akbulut and King [AK05, Theorem 4], and refined by Kucharz [Kuc09, Theorem 1.1]. Their examples work for all (c, d) with $c \geq 2$ and $d \geq c+2$. The range of pairs (c, d) that we reach in Theorem 5.2 is different.

5.2. **Homology and cobordism obstructions.** Define $H_d^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ to be the image of the Borel–Haefliger cycle class map $\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}, \mathbb{Z}/2)$. Consider the following property.

Property (H). One has $j_*[M] \in H_d^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$.

Property (B) implies (H) since algebraic homology classes are preserved by push-forwards (see for instance [BW20a, §1.6.2]). It follows that (H) is an obstruction to the approximation properties (A) and (A'), that is weaker than (B). In fact, the cobordism obstruction (B) was first used by Bochnak and Kucharz [BK03, Corollary 1.3] to give examples where (A) fails but (H) holds, when $d \geq 3$ and $c \geq 2$.

Lemma 5.4. *If $d \leq 2$, then properties (B) and (H) are equivalent.*

Proof. Since $MO_1 = 0$, an isomorphism

$$(H_{d-2}(X(\mathbb{R}), \mathbb{Z}/2) \otimes MO_2) \oplus H_d(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\sim} MO_d(X(\mathbb{R}))$$

is constructed in [Con79, Theorem 17.2]. It restricts to an isomorphism

$$(H_{d-2}(X(\mathbb{R}), \mathbb{Z}/2) \otimes MO_2) \oplus H_d^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\sim} MO_d^{\mathrm{alg}}(X(\mathbb{R}))$$

by a theorem of Ischebeck and Schülting [IS88, Corollary 1 p. 314], in view of the equality $H_0^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = H_0(X(\mathbb{R}), \mathbb{Z}/2)$. We deduce that the two conditions $j_*[M] \in H_d^{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ and $[j : M \hookrightarrow X(\mathbb{R})] \in MO_d^{\mathrm{alg}}(X(\mathbb{R}))$ are equivalent. □

5.3. **Hypersurfaces.** The following proposition is a well-known improvement of [BCR98, Theorem 12.4.11], which goes back to the work of Benedetti and Tognoli [BT80a, Proposition 1 p. 227] (see also [AK85, Theorem A]).

Proposition 5.5. *If $c = 1$, then properties (A') and (H) are equivalent.*

Proof. Assume that (H) holds. Let $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$ be a neighbourhood of the inclusion. By [BCR98, Theorem 12.4.11], there exist $\psi \in \mathcal{U}$, an open neighbourhood U of $X(\mathbb{R})$ in X and a smooth closed hypersurface $Z \subset U$ with $\psi(M) = Z(\mathbb{R})$. Let $\overline{Z} \subset X$ be the Zariski closure of Z. Since X is smooth, there exist a line bundle $\mathcal{L}$ on X and a section $s \in H^0(X, \mathcal{L})$ with $\overline{Z} = \{s = 0\}$.

Fix a very ample line bundle $\mathcal{O}_X(1)$ on X. Let $(u_1, \dots, u_N)$ be a basis of $H^0(X, \mathcal{O}_X(1))$. The section $v := \sum_{m=1}^N u_m^2 \in H^0(X, \mathcal{O}_X(2))$ vanishes nowhere on $X(\mathbb{R})$. Choose $l \gg 0$ with $\mathcal{M} := \mathcal{L}(2l)$ very ample, let $t \in H^0(X, \mathcal{M})$ be a general small deformation of sv^l, and set $Y := \{t = 0\}$, which is smooth by Bertini.

That Y has the required properties follows from [AR67, §20]. More precisely, the proofs of [AR67, Lemmas 20.3 and 20.4] applied with $X = X(\mathbb{R})$, $Y = \mathcal{M}(\mathbb{R})$, $W \subset Y$ the zero section, $r \geq 1$, and $\mathcal{A} = \mathcal{C}^{r+1}(X(\mathbb{R}), \mathcal{M}(\mathbb{R}))$ show that if t is close to sv^l, then the inclusions $Z(\mathbb{R}) \subset X(\mathbb{R})$ and $Y(\mathbb{R}) \subset X(\mathbb{R})$ are isotopic, by an isotopy which is $\mathcal{C}^\infty$ because so are t and sv^l, and small in the $\mathcal{C}^\infty$ topology (see the use of the implicit section theorem in the proof of [AR67, Lemma 20.3]). □

5.4. **A question of Bochnak and Kucharz.** In [BK03, pp. 685-686], Bochnak and Kucharz ask for which values of c and d are (A) and (H) equivalent. We obtain a full answer to that question, and disprove the expectation raised in [BK03, p. 686] that (A) and (H) are not equivalent for $d = 2$ and $c \geq 3$.

Theorem 5.6. *Properties (H), (B), (A) and (A') are all equivalent in the following cases: if $c \leq 1$, if $d \leq 1$, or if $d = 2$ and $c \geq 3$.*

For all other values of c and d, there exist X and M satisfying (H) but not (A).

Proof. The theorem is trivial if $c = 0$ or $d = 0$. It follows from Proposition 5.5 if $c = 1$. The cases with $d \geq 3$ and $c \geq 2$ are covered by Bochnak and Kucharz in [BK03, Corollary 1.3]. If $d \leq 2$, then (H) is equivalent to (B) by Lemma 5.4. The cases where $d = 1$ and $c \geq 2$, or where $d = 2$ and $c \geq 3$, now follow from Theorem 5.1. Finally, when $c = d = 2$, one may apply Theorem 5.2 □

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