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MODULI SPACES AND ALGEBRAIC CYCLES IN REAL ALGEBRAIC GEOMETRY

by

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ABSTRACT

This thesis intends to make a contribution to the theories of algebraic cycles and moduli spaces over the real numbers. In the study of the subvarieties of a projective algebraic variety, smooth over the field of real numbers, the cycle class map between the Chow ring and the equivariant cohomology ring plays an important role. The image of the cycle class map remains difficult to describe in general; we study this group in detail in the case of real abelian varieties. To do so, we construct integral Fourier transforms on Chow rings of abelian varieties over any field. They allow us to prove the integral Hodge conjecture for one-cycles on complex Jacobian varieties, and the real integral Hodge conjecture modulo torsion for real abelian threefolds.

For the theory of real algebraic cycles, and for several other purposes in real algebraic geometry, it is useful to have moduli spaces of real varieties to our disposal. Insight in the topology of a real moduli space provides insight in the geometry of a real variety that defines a point in it, and the other way around. In the moduli space of real abelian varieties, as well as in the Torelli locus contained in it, we prove density of the set of moduli points attached to abelian varieties containing an abelian subvariety of fixed dimension. Moreover, we provide the moduli space of stable real binary quintics with a hyperbolic orbifold structure, compatible with the period map on the locus of smooth quintics. This identifies the moduli space of stable real binary quintics with a non-arithmetic ball quotient.

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1 INTRODUCTION

This thesis concerns a study of cycles and moduli spaces in real algebraic geometry. We intend to make a contribution towards understanding these concepts, individually as well as in light of one another, by looking at phenomena such as:

- The hyperbolic structure of the moduli space of real binary quintics.
- The distribution of non-simple abelian varieties in a family of real abelian varieties.
- Algebraic curves on complex and real abelian varieties.

The results as well as the methods of these investigations are closely intertwined, as we shall see. The general, overarching theory is *equivariant Hodge theory*: the use of Hodge theory to study cohomology, moduli and cycles of real algebraic varieties.

This introduction is structured as follows. In Section 1.1, we set the stage by explaining what real algebraic geometry is about. We continue to introduce the main players in Section 1.2: these are moduli spaces on the one hand, and algebraic cycles on the other. Finally, in Section 1.3, we discuss our most important results.

1.1 SETTING THE STAGE

Real algebraic geometry concerns the study of algebraic varieties over $\mathbb{R}$, the field of real numbers. As such, the basic objects of the theory are sets of the form

$$X(\mathbb{C}) = \{\alpha \in \mathbb{P}^n(\mathbb{C}) \mid F_i(\alpha) = 0 \; \forall i = 1, \dots, k\}, \quad X(\mathbb{R}) = X(\mathbb{C}) \cap \mathbb{P}^n(\mathbb{R}), \tag{1.1}$$

where the F_i are homogeneous elements of the polynomial ring $\mathbb{R}[X_0, \dots, X_n]$. Studying such polynomial equations is an old and fascinating topic, in which recent developments have led to important progress [BCR98, Man20].

Nowadays, one often thinks of real algebraic geometry as *G-equivariant complex algebraic geometry*. To explain this, we introduce the following:

Definition 1.1. We define, once and for all, G as the Galois group of $\mathbb{C}$ over $\mathbb{R}$.

Consider the algebraic subset $X(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ defined in (1.1). Since the coefficients of the polynomials F_i are real, complex conjugation $x \mapsto \bar{x}$ on the complex projective space $\mathbb{P}^n(\mathbb{C})$ descends to an anti-holomorphic involution $\sigma\colon X(\mathbb{C}) \to X(\mathbb{C})$. In this way, one obtains a functor

$$X \mapsto (X_{\mathbb{C}}, \sigma\colon X(\mathbb{C}) \to X(\mathbb{C}))$$

from the category of projective varieties over $\mathbb{R}$ to the category of projective varieties over $\mathbb{C}$ equipped with an anti-holomorphic G-action. This functor is an *equivalence*.

In particular, to study a real algebraic variety, one may view it either as an $\mathbb{R}$-scheme X and employ algebraic methods, or as a G-equivariant complex analytic space $X(\mathbb{C})$ and use topological and analytical techniques. A guiding principle in this thesis is that especially the combination of both approaches is very powerful.

Led by this principle, we intend to make a contribution to real algebraic geometry by exploring the following two themes:

1. Moduli spaces of certain classes of real algebraic varieties.
2. Real algebraic subvarieties of a fixed ambient real algebraic variety.

These two concepts will be the main players of this thesis. Let us introduce them properly, before we put them into action and explain our results.

1.2 INTRODUCTION TO THE MAIN PLAYERS

1.2.1 *Complex and real moduli spaces*

To understand how the geometry of varieties in some interesting category behaves in family, it can be useful to study the topology of their moduli space. When the base field is $\mathbb{C}$, constructing a moduli space is a well-understood problem. It can be attacked with analytic methods such as Teichmüller theory, or with algebraic methods such as stacks and GIT. For example, if the moduli stack is algebraic and has finite inertia, then a coarse moduli space exists by [KM97].

In real algebraic geometry, it is less straightforward what a real moduli space should be, and how it can be constructed.

Question 1.2. What is a real moduli space?

Intuitively, the real moduli space of a real variety should be a topological space that classifies isomorphism classes of all real varieties that have something in common with the given one (e.g. numerical invariants of some kind). One can also classify equivalence classes of other algebraic objects defined over $\mathbb{R}$, such as sheaves on a real variety, morphisms, or combinations of these. More generally, we can consider sets $S(T)$ of families of selected objects over $\mathbb{R}$-schemes T modulo a certain equivalence relation $\sim$ and build a functor $F\colon (\mathrm{Sch}/\mathbb{R}) \to (\mathrm{Set})$ by the rule $F(T) = S(T)/\sim$. One could define the real moduli space for the moduli problem F as the topological space obtained by giving $F(\mathbb{R})$ the finest topology for which $f_{\mathbb{R}}\colon T(\mathbb{R}) \to F(\mathbb{R})$ is continuous for every $\mathbb{R}$-scheme T and every $f \in F(T)$.

Generally, the set $F(\mathbb{R})$ has more structure than just the structure of a topological space. For example, if the moduli problem F is representable by a scheme X, then its real moduli space $X(\mathbb{R})$ is in fact a real-analytic space. Even in case F is not representable, it is often possible to provide $F(\mathbb{R})$ with some additional structure.

Example 1.3 (Algebraic quotients)**.** Let F be a moduli problem over $\mathbb{R}$ such that $F(T)$ is a certain set of isomorphism classes of families of polarized real varieties (X, ℓ) over T, where ℓ is a G-invariant ample class in the Néron-Severi group of $X_{\mathbb{C}}$. Suppose there is a subscheme H of $\mathrm{Hilb}_{\mathbb{P}^N}$ such that $F(\mathbb{C}) \cong \mathrm{PGL}_{N+1}(\mathbb{C}) \setminus H(\mathbb{C})$. If ℓ is induced by an ample line bundle on X for every $[(X, \ell)] \in F(\mathbb{R})$, then there is a homeomorphism $F(\mathbb{R}) \cong \mathrm{PGL}_{N+1}(\mathbb{R}) \setminus H(\mathbb{R})$. For example, this works for hypersurfaces in $\mathbb{P}^n_{\mathbb{R}}$. Less trivially, this works for polarized real abelian varieties.

Example 1.4 (Analytic quotients)**.** Let $\mathcal{A}_g$ be the stack of principally polarized abelian varieties of dimension g, and let $\mathcal{M}_g$ be the stack of smooth curves of genus g. It is well-known that there exist complex analytic uniformizations

$$\mathcal{A}_g(\mathbb{C}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}) \setminus \mathbb{H}_g, \quad \text{and} \quad \mathcal{M}_g(\mathbb{C}) \cong \Gamma_g \setminus \mathcal{T}_g.$$

Here, $\mathbb{H}_g$ and $\mathcal{T}_g$ are complex manifolds, parametrizing complex abelian varieties and curves endowed with some additional structure, and the covers $\mathbb{H}_g \to \mathcal{A}_g(\mathbb{C})$

and $\mathcal{T}_g \to \mathcal{M}_g(\mathbb{C})$ are obtained forgetting these structures. To obtain real moduli spaces for these moduli problems, one defines sets of anti-holomorphic involutions

$$\{\sigma \colon \mathbb{H}_g \to \mathbb{H}_g\} \quad \text{and} \quad \{\tau \colon \mathcal{T}_g \to \mathcal{T}_g\}$$

in such a way that every principally polarized real abelian variety (resp. real curve) lies in $\mathbb{H}_g^\sigma$ (resp. $\mathcal{T}_g^\tau$) for some σ (resp. τ). For suitably defined subgroups $\mathrm{Sp}_{2g}(\mathbb{Z})(\sigma) \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\Gamma_g(\tau) \subset \Gamma_g$, this gives bijections [GH81, Sil89, SS89]

$$\mathcal{A}_g(\mathbb{R}) \cong \bigsqcup_\sigma \mathrm{Sp}_{2g}(\mathbb{Z})(\sigma) \setminus \mathbb{H}_g^\sigma \quad \text{and} \quad \mathcal{M}_g(\mathbb{R}) \cong \bigsqcup_\tau \Gamma_g(\tau) \setminus \mathcal{T}_g^\tau.$$

We verify in Theorems 2.15 and 2.18 that these bijections are homeomorphisms.

Examples 1.3 and 1.4 suggest that if F arises as the functor attached to an algebraic stack over $\mathbb{R}$, then $F(\mathbb{R})$ should carry the structure of a real-analytic orbifold. In Chapter 2, we show that this indeed the case. We define a functorial orbifold structure on the real locus $\mathscr{X}(\mathbb{R})$ of a smooth, separated Deligne-Mumford stack $\mathscr{X}$ over $\mathbb{R}$. One retrieves the real-analytic space $\mathscr{X}(\mathbb{R})$ when $\mathscr{X}$ is a scheme.

1.2.2 *Complex and real period maps*

We conclude that in complex and in real algebraic geometry, moduli spaces arise naturally via algebraic moduli stacks. As a next step, one may want to equip the moduli space with some additional structure. Hodge theory provides an excellent method for this goal, as we shall now explain.

Consider a projective holomorphic submersion of complex manifolds

$$\begin{tikzcd} & \mathbb{P}^N(\mathbb{C}) \times B \arrow[dr] & \\ X \arrow[ur, hook] \arrow[rr, "\pi"] & & B, \end{tikzcd} \tag{1.2}$$

where the manifold B is connected and equipped with a base point $0 \in B$. Possibly after replacing B by an open subset around 0, for each $t \in B$ there is an isomorphism

$$\mathrm{H}^k(X_t, \mathbb{Z}) \cong \mathrm{H}^k(X_0, \mathbb{Z}).$$

One obtains a family of Hodge structures on $\mathrm{H}^k(X_0, \mathbb{Z})$, parametrized by B. The Hodge filtrations on $F_t^\bullet$ on $\mathrm{H}^k(X_0, \mathbb{C})$ induce a map from B into a flag variety $\mathcal{F}$ and by the factorization of π in (1.2) the image is contained in a certain locally closed subset $\mathcal{D} \subset \mathcal{F}$, the *period domain* [Voi02, §10.1.3]. The induced map

$$P : B \to \mathcal{D}$$

is called a *period map*, and turns out to be holomorphic.

This procedure can be globalized, by taking a suitable cover $\widetilde{B} \to B$ instead of restricting to simply connected opens to trivialize the local system with stalks $\mathrm{H}^k(X_t, \mathbb{Z})$. One obtains a morphism $\widetilde{P}\colon \widetilde{B} \to \mathcal{D}$ that induces a commutative diagram

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\widetilde{P}"] \arrow[d] & \mathcal{D} \arrow[d] \\ B \arrow[r, "P"] & \Gamma \setminus B, \end{tikzcd} \qquad \Gamma = \mathrm{Aut}\left(\mathrm{H}^k(X_0, \mathbb{Z})\right). \tag{1.3}$$

The above construction applies to the complex locus $\mathsf{M}(\mathbb{C})$ of the coarse moduli space M of any separated Deligne-Mumford moduli stack $\mathcal{M}$ which is smooth over $\mathbb{C}$. Indeed, although the complex analytic space $\mathsf{M}(\mathbb{C})$ might be singular and lacking a universal family, for a suitable étale cover $U \to \mathcal{M}$ there is a period map $\widetilde{P}\colon U(\mathbb{C}) \to \mathcal{D}$ that descends to a morphism of analytic spaces $P\colon \mathsf{M}(\mathbb{C}) \to \Gamma \setminus \mathcal{D}$.

For such a smooth moduli stack $\mathcal{M}$, one may wonder if the map $\widetilde{P}\colon U(\mathbb{C}) \to \mathcal{D}$ is immersive (infinitesimal Torelli), if the map $P\colon \mathsf{M}(\mathbb{C}) \to \Gamma \setminus \mathcal{D}$ is injective (global Torelli), and if one can explicitly describe the image of P.

Examples 1.5. These statements are true for the moduli stack $\mathcal{A}_g$ of g-dimensional principally polarized abelian varieties [BL04], as well as for the moduli stack $\mathcal{M}_d$ of degree $2d$-polarized K3 surfaces [Pal85, Huy16].

In contrast with Examples 1.5, it can occur that for the family (1.2), the period map P in diagram (1.3) is actually constant. It may also happen that the period map is an embedding but $\dim(B) < \dim(\mathcal{D})$. In these cases, it is sometimes useful to first associate an auxiliary variety Y_t to the fiber X_t above $t \in B$, carrying some additional structure (such as an embedding $H \subset \mathrm{Aut}(Y_t)$ for some fixed finite group H), and then take periods, namely those of the variety Y_t. For example:

1. If $X_t \subset \mathbb{P}^3_{\mathbb{C}}$ is a cubic surface, one can take Y_t to be the triple cover of $\mathbb{P}^3_{\mathbb{C}}$ ramified along X_t [ACT02a].
2. If $X_t = \{F_t = 0\} \subset \mathbb{P}^1_{\mathbb{C}}$ is a binary quantic [MFK94, Chapter 4], one can take Y_t to be a finite cover of $\mathbb{P}^1_{\mathbb{C}}$ ramified along X_t [DM86], [MO13, Section 5.3].

In this way, one obtains a family $\rho : Y \to B$ whose period domain $\mathcal{D}$ contains a sub-domain $\mathcal{D}' \subset \mathcal{D}$ that parametrizes Hodge structures with additional structure (see [CMSP17, §17.3]). The period map of ρ factors through a map $P\colon B \to \Gamma' \setminus \mathcal{D}'$ that might be closer to being an isomorphism than the original period map was. Such a map P is called an *occult period map*, due to its hidden nature [KR12].

Examples 1.6. The idea of considering the periods of a branched cover of the projective line goes back to Picard [Pic83]; Shimura studied the moduli of such curves in [Shi64]. Deligne, Mostow and Thurston developed on the work of Picard, determining for which stable configurations of points on $\mathbb{P}^1(\mathbb{C})$ the occult period map is an isomorphism [DM86, Mos86, Thu98]. Allcock–Carlson–Toledo identified the complex moduli space of stable cubic surfaces with a four-dimensional ball-quotient [ACT02a, Bea09, CMSP17], extending their construction to cubic threefolds in [ACT11] (c.f. [LS07]). Likewise, Kondō studied complex moduli of curves of genus three and four [Kon00, Kon02], and then genus six with Artebani [AK11].

Remark 1.7. Kudla and Rapoport observed that occult period maps are compatible with families, extending them to morphisms of stacks defined over natural fields of definition [KR12]. The arithmetic these morphisms was studied by Achter [Ach20].

What happens over the real numbers? Let us assume that family (1.2) comes equipped with anti-holomorphic involutions

$$(\tau : X \to X, \sigma : B \to B \mid \sigma(0) = 0) .$$

After replacing B by a G-stable contractible open subset around 0, we can trivialize the local system with stalks $\mathrm{H}^k(X_t, \mathbb{Z})$ in a G-equivariant way. The involution

$$\tau^* : \mathrm{H}^k(X_0, \mathbb{Z}) \to \mathrm{H}^k(X_0, \mathbb{Z})$$

can then be used to define a natural anti-holomorphic involution on $\mathcal{D}$, making the period map G-equivariant. Taking fixed points results in a smooth *real period map*

$$\mathcal{P}^G\colon B^G \to \mathcal{D}^G.$$

As in the complex case, one may, instead of restricting to small opens, use G-equivariant covers to trivialize the monodromy. Such covers can also be used to define real period maps on real moduli spaces. This leads to interesting results.

Examples 1.8. Tate identified the real moduli space of elliptic curves with the disjoint union of two copies of $\mathbb{R}^*$ [Gro80, Chapter 1, Proposition 4.3.1], which was generalized to abelian varieties by Gross–Harris and Silhol [GH81, Sil89]. The case of real K3 surfaces was treated by Nikulin [Nik80, DIK00] (see also [Kha76, Sil89]).

The occult period map construction carries over to the reals as well.

Examples 1.9. *Equivariant occult period maps* are used to uniformize the connected components of the real moduli spaces of cubic surfaces [ACT10], binary sextics [ACT06], curves of genus three [Rie15, HR18], and binary octics [Chu06, Chu11].

This concludes our discussion of the basics of complex and real moduli theory. It is time to meet the other main players:

1.2.3 *Complex and real algebraic cycles*

Any algebraic subvariety of a smooth projective variety induces a class in a suitably defined cohomology group. Important conjectures predict that one can understand the subgroup of algebraic classes in the cohomology of this variety via structures that are *a priori* not directly related to the algebraic cycles themselves (i.e. Hodge theory, Galois representations, etc.). In this section, we explain this in more detail when the base field is either $\mathbb{C}$ or $\mathbb{R}$. Since the situation over $\mathbb{C}$ is better understood than the situation over $\mathbb{R}$, we start our discussion in the complex direction.

Let X be a smooth projective variety over $\mathbb{C}$. A fundamental theorem of Hodge [Hod61] says that for each integer k, there is a functorial *Hodge decomposition*

$$\mathrm{H}^k(X(\mathbb{C}),\mathbb{C}) = \mathrm{H}^k(X(\mathbb{C}),\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X) \ \text{ such that } \ \overline{\mathrm{H}^{p,q}(X)} = \mathrm{H}^{q,p}(X).$$

Define the group of *integral Hodge classes* of degree $2k$ as

$$\mathrm{Hdg}^{2k}(X(\mathbb{C}),\mathbb{Z}) = \left\{ \alpha \in \mathrm{H}^{2k}(X(\mathbb{C}),\mathbb{Z}) \mid \alpha_{\mathbb{C}} \in \mathrm{H}^{k,k}(X) \subset \mathrm{H}^{2k}(X(\mathbb{C}),\mathbb{C}) \right\}.$$

These Hodge classes derive their main interest from the following fact. Let $Z \subset X$ be an irreducible subvariety of dimension i, and $\widetilde{Z} \to Z$ a resolution of singularities [Hir64]. The composition $f\colon \widetilde{Z} \to Z \hookrightarrow X$ induces a Gysin homomorphism

$$f_*\colon \mathrm{H}^0(\widetilde{Z}(\mathbb{C}), \mathbb{Z}) \to \mathrm{H}^{2k}(X(\mathbb{C}), \mathbb{Z}) \qquad (k = \dim(X) - i)$$

which is compatible with the Hodge decomposition. Thus, we obtain a Hodge class $[Z] = f_*(1) \in \mathrm{H}^{2k}(X(\mathbb{C}), \mathbb{Z})$. This construction induces a homomorphism

$$\mathrm{CH}_i(X) \to \mathrm{Hdg}^{2k}(X(\mathbb{C}), \mathbb{Z}) \qquad (k = \dim(X) - i). \tag{1.4}$$

Definition 1.10. For a smooth projective variety X over $\mathbb{C}$, the *integral Hodge conjecture for i-cycles* (IHC_i) is the property that the homomorphism (1.4) is surjective. Thus, X satisfies IHC_i if (1.4) is surjective, and X fails IHC_i if (1.4) is not surjective.

It is known since Atiyah and Hirzebruch [AH62] that there are varieties X for which the property IHC_i does not hold. In some interesting cases, however, the integral Hodge conjecture for i-cycles turns out to be satisfied. In Section 1.3.3 below, we will indicate more precisely what is and what is not known in this direction.

Let us carry the discussion over to the real setting. Let X denote a smooth projective algebraic variety over $\mathbb{R}$. Over the years, much work has been done to establish the right analogue of the cycle class map in real algebraic geometry. Borel and Haefliger [BH61] constructed a cycle class with values in $\mathrm{H}^k(X(\mathbb{R}), \mathbb{Z}/2)$ for any real algebraic subvariety $Z \subset X$ of codimension k, by considering the embedding of real loci $Z(\mathbb{R}) \subset X(\mathbb{R})$. The study of the subgroup

$$\mathrm{H}^k_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2) \subset \mathrm{H}^k(X(\mathbb{R}), \mathbb{Z}/2)$$

formed by these classes is a classical topic in real algebraic geometry [Sil89, BK98, Man17], related to the problem of $\mathcal{C}^\infty$ approximation of submanifolds of $X(\mathbb{R})$ by algebraic subvarieties [BCR98, Ben20], [BW20b, §6.2].

To study the algebraic cycles on X, it is natural to consider a more refined cohomology theory than $\mathrm{H}^\bullet(X(\mathbb{R}), \mathbb{Z}/2)$, forasmuch as the real locus $X(\mathbb{R}) = X(\mathbb{C})^G$ constitutes only a small part of the topological G-structure of $X(\mathbb{C})$. The right choice seems to be provided by G-equivariant cohomology $\mathrm{H}^q_G(X(\mathbb{C}), \mathbb{Z}(j))$

in the sense of Borel (see [Gro57] or [AB84]), where $\mathbb{Z}(j)$ is the abelian group $\mathbb{Z}$ turned into a G-module by declaring that $\sigma(1) = (-1)^j$ for the generator $\sigma \in G$.

The study of equivariant cohomology in real algebraic geometry was initiated by Krasnov [Kra91], and continued by Van Hamel in his thesis [Ham97], who used extensively the formalism of equivariant sheaves and derived categories. As was noted by both [Kra94, Ham97], there is an equivariant cycle class map $\mathrm{CH}_i(X) \to \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$ that fits in a commutative diagram

$$\begin{tikzcd} & \mathrm{CH}_i(X) \arrow[dl] \arrow[d] \arrow[dr] & (k = \dim(X) - i) \\ \mathrm{H}^k(X(\mathbb{R}), \mathbb{Z}/2) & \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k)) \arrow[l] \arrow[r] & \mathrm{H}^{2k}(X(\mathbb{C}), \mathbb{Z}(k)). \end{tikzcd}$$

It turns out that algebraic classes are contained in a certain subgroup

$$\mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0 = \mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k)) \cap \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0 \subset \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$$

that takes into account the Steenrod operations on $\mathrm{H}^\bullet(X(\mathbb{R}), \mathbb{Z}/2)$ as well as the Hodge structure on $\mathrm{H}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))$. See Section 8.2 for details. This leads to:

Definition 1.11 (Benoist–Wittenberg)**.** A smooth projective variety $X_{/\mathbb{R}}$ satisfies the *real integral Hodge conjecture for i-cycles* ($\mathbb{R}\mathrm{IHC}_i$) if the following map is surjective:

$$\mathrm{CH}_i(X) \to \mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0 \qquad (k = \dim(X) - i).$$

Thus, in spite of its name, $\mathbb{R}\mathrm{IHC}_i$ is a property that a smooth projective variety X can satisfy. There are varieties for which it holds, and there are varieties for which it fails. In some sense, the property – if satisfied – reveils deep links between the equivariant complex structure of $X(\mathbb{C})$, the topology of $X(\mathbb{R})$, and the real subvarieties of X. For instance, if $\mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0$ is large, then there should be many subvarieties even though it may be difficult to write down explicit equations.

This concludes our introduction to algebraic cycles and cohomology of complex and real algebraic varieties. Now that the stage is set, and its main players are met, it is time to present our results.

1.3 DISCUSSION OF THE MAIN RESULTS

1.3.1 *Noether-Lefschetz loci for real abelian varieties* *(based on [GF22b])*

Our first result concerns the behaviour of specific algebraic cycles on real abelian varieties in a family. Namely, we have considered the following:

Question 1.12. Let $A \to B$ be a family of polarized real abelian varieties, and let k be a positive integer. Let $S_k(B(\mathbb{R})) \subset B(\mathbb{R})$ be the subset of points $t \in B(\mathbb{R})$ such that A_t contains a real abelian subvariety of dimension k. Does there exist a natural criterion for the density of $S_k(B(\mathbb{R}))$ in $B(\mathbb{R})$?

Viewing the above family as an equivariant variation of Hodge structure, Question 1.12 concerns certain *special loci* in $B(\mathbb{R})$. Indeed, we consider loci of $t \in B(\mathbb{R})$ such that the G-equivariant Hodge structure on the cohomology of $A_t(\mathbb{C})$ is "special" in the sense that it contains a G-stable sub-Hodge structure of rank k. What we ask for, is a density criterion for these special loci. Phrased like this, Question 1.12 might remind of the Green–Voisin criterion for density of Noether-Lefschetz loci [Voi02, Proposition 17.20]. Before we start looking for an answer, let us consider the analogous question for surfaces in $\mathbb{P}^3$.

Let $d \geq 4$ be an integer, and consider the universal degree d smooth hypersurface

$$\mathbb{P}^3 \times \mathscr{B} \supset \mathscr{S} \to \mathscr{B}.$$

The Noether-Lefschetz locus is the set of $t \in \mathscr{B}(\mathbb{C})$ such that $\mathscr{S}_t$ contains a curve which is not a complete intersection. By the main result of [CHM88], the Noether-Lefschetz locus is euclidean dense in $\mathscr{B}(\mathbb{C})$, despite having empty interior [Lef50].

Do we have a similar result over the reals? This question makes sense, since the above universal surface $\mathscr{S} \to \mathscr{B}$ is naturally defined over $\mathbb{R}$. In analogy with the above, one may thus define the *real Noether-Lefschetz locus* as the set of real surfaces S in $\mathbb{P}^3_{\mathbb{R}}$ with $\mathrm{Pic}(S) \ncong \mathbb{Z}$. The situation over $\mathbb{R}$ turns out to be more delicate than the situation over $\mathbb{C}$. Indeed, there exists a component K of the real moduli space of degree four surfaces in $\mathbb{P}^3_{\mathbb{R}}$, such that every S in K satisfies $\mathrm{Pic}(S) \cong \mathbb{Z}$. There is a Green–Voisin criterion over the reals [Ben18, Proposition 1.1], but its hypothesis is more complicated and only applies to one component of $\mathscr{B}(\mathbb{R})$ at a time.

Let us return to abelian varieties. Question 1.12 turns out to have a response which, in light of the above difficulties, is remarkably simple. In [CP90], Colombo and Pirola prove the following theorem. Let $A \to B$ be a holomorphic family of polarized complex abelian varieties over a connected manifold B, and let $S_k(B) \subset B$ be the locus of abelian varieties containing a k-dimensional abelian subvariety. If Condition 3.1 in Chapter 3 holds, then $S_k(B)$ is euclidean dense in B.

Over $\mathbb{R}$, the criterion remains the same:

Theorem 1.13 (Theorem 3.2)**.** *Let $A \to B$ be a family of polarized real abelian varieties. If B is connected and Condition 3.1 holds, then $S_k(B(\mathbb{R}))$ is euclidean dense in $B(\mathbb{R})$.*

To give some applications of Theorem 1.13, consider $\mathcal{A}_g(\mathbb{R})$, the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{R}$, and $\mathcal{M}_g(\mathbb{R})$, the moduli space of smooth, proper and geometrically connected curves of genus g over $\mathbb{R}$. These moduli spaces carry natural topologies for which the Torelli map $t\colon \mathcal{M}_g(\mathbb{R}) \to \mathcal{A}_g(\mathbb{R})$ is continuous, see Section 1.2.1 above.

Theorem 1.14 (Theorem 3.3)**.**

1. *Let $k < g$ be positive integers, and consider the set $S_k(\mathcal{A}_g(\mathbb{R})) \subset \mathcal{A}_g(\mathbb{R})$ of moduli points of real abelian varieties containing a real abelian subvariety of dimension k. Then $S_k(\mathcal{A}_g(\mathbb{R}))$ is dense in $\mathcal{A}_g(\mathbb{R})$.*

2. *Suppose in addition that $k \leq 3 \leq g$, and let $\mathcal{T}_g(\mathbb{R}) = t(\mathcal{M}_g(\mathbb{R}))$ be the Torelli locus in $\mathcal{A}_g(\mathbb{R})$. The set $S_k(\mathcal{A}_g(\mathbb{R})) \cap \mathcal{T}_g(\mathbb{R})$ is dense in $\mathcal{T}_g(\mathbb{R})$.*

3. *Let $V \subset \mathbb{P}\mathrm{H}^0(\mathbb{P}^2_{\mathbb{R}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(d))$ be the moduli space of smooth degree $d \geq 3$ real plane curves. Let $S_1(V)$ be the set of $t \in V$ such that the corresponding curve C_t admits a non-constant map $C_t \to E$ to a real elliptic curve E. Then $S_1(V)$ is dense in V.*

The proof of Theorem 1.13 consists of a detailed study of the differential of the period map for abelian varieties, and its compatibility with several Galois actions (see diagram (3.4)). Theorem 1.14 arises quite naturally as a corollary, due to the results in [CP90] and the fact that the density criteria over $\mathbb{R}$ and $\mathbb{C}$ are the same.

1.3.2 *Real moduli of five points on the line* *(based on [GF21])*

Consider the following question in real moduli theory, phrased informally but whose idea should be clear. Consider a smooth moduli stack $\mathcal{M}$ of smooth projective varieties over $\mathbb{R}$. On the one hand, even if $\mathcal{M}(\mathbb{C})$ is connected, this may not be true for $\mathcal{M}(\mathbb{R})$. In general, the G-equivariant diffeomorphism type of $X(\mathbb{C})$ of real varieties X in one component of $\mathcal{M}(\mathbb{R})$ remains the same by Ehresmann's theorem, whereas varieties in different components of $\mathcal{M}(\mathbb{R})$ may have non-homeomorphic real loci. For example, the moduli space $\mathcal{A}_1(\mathbb{R})$ has two connected components $K_i = \mathbb{R}^*$ $(i \in \{1,2\})$ such that K_i parametrizes elliptic curves whose real locus is the disjoint union of i circles. If $\mathcal{M}(\mathbb{R})$ is not connected, real period maps have to be defined on each component of $\mathcal{M}(\mathbb{R})$ separately (see Section 1.2.2 above).

On the other hand, it is often possible to define a slightly larger moduli stack $\overline{\mathcal{M}} \supset \mathcal{M}$ by allowing mild singularities, whose real locus $\overline{\mathcal{M}}(\mathbb{R})$ *is* connected. In this case, $\overline{\mathcal{M}}(\mathbb{R})$ glues together the various components of $\mathcal{M}(\mathbb{R})$ in a natural way.

Question 1.15*.* Can the real period domains and real period maps be glued as well?

The goal of this section is to explain why, in the case of binary quintics, the answer is *yes*. As for smooth quintics, we have the following (c.f. Examples 1.9):

Theorem 1.16 (Theorem 5.1)**.** *Let $\mathcal{M}_0(\mathbb{R})$ be the real moduli space of smooth binary quintics. For $i \in \{0,1,2\}$, let $\mathscr{M}_i$ be the component of $\mathcal{M}_0(\mathbb{R})$ parametrizing quintics with $2i$ complex and $5-2i$ real points. For each i, the real period map induces an isomorphism of real analytic orbifolds $\mathscr{M}_i \cong P\Gamma_i \setminus \left(\mathbb{R}H^2 - \mathscr{H}_i\right)$. Here $\mathbb{R}H^2$ is the real hyperbolic plane, $\mathscr{H}_i$ a union of geodesic subspaces in $\mathbb{R}H^2$ and $P\Gamma_i$ an arithmetic lattice in $\mathrm{PO}(2,1)$.*

The period map for smooth binary quintics referred to Theorem 1.16 is actually an occult period map (see Section 1.2.2). It is defined by sending a binary quintic $X = \{F = 0\} \subset \mathbb{P}^1_{\mathbb{R}}$ to the G-equivariant Hodge structure with $\mathbb{Z}[\zeta_5]$-action $\mathrm{H}^1(C(\mathbb{C}), \mathbb{Z})$ attached to the degree five cover $C \to \mathbb{P}^1_{\mathbb{R}}$ ramified along X.

To explain the generalization of Theorem 1.16 to moduli of stable quintics, let us consider again a general inclusion of moduli stacks $\mathcal{M} \subset \overline{\mathcal{M}}$ over $\mathbb{R}$ as above. Suppose that a suitable (occult) period map on $\mathcal{M}(\mathbb{C})$ extends to $\overline{\mathcal{M}}(\mathbb{C})$ identifying the latter with an arithmetic quotient of a hermitian symmetric domain. Examples are given by cubic surfaces, configurations of points on $\mathbb{P}^1$, and K3 surfaces [ACT02a, DM86, AE21]. If this happens, then on the one hand, $\overline{\mathcal{M}}(\mathbb{C})$ is a locally symmetric variety, and on the other, the topology of $\overline{\mathcal{M}}(\mathbb{R})$ reveals how one type of real variety deforms into another when it crosses the boundary between two connected components of $\mathcal{M}(\mathbb{R})$. An optimistic approach to positively answer Question 1.15 would be to try to prove that the metric induced on $\mathcal{M}(\mathbb{R})$ via the period map extends to a path metric on the larger moduli space $\overline{\mathcal{M}}(\mathbb{R})$, complete in case $\overline{\mathcal{M}}(\mathbb{C})$ is complete. This may seem like a highly non-trivial thing to do.

A beautiful theorem by Allcock, Carlson and Toledo [ACT10] says that for cubic surfaces, this can in fact be done. As a consequence, the real moduli space of stable cubic surfaces is homeomorphic to a hyperbolic quotient space $P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^4$. Here $\mathbb{R}H^4$ denotes hyperbolic 4-space, $P\Gamma_{\mathbb{R}} \subset \mathrm{PO}(4,1)$ is a discrete subgroup of isometries, and $P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^4$ contains the disjoint union of the connected components $P\Gamma_i \setminus \left(\mathbb{R}H^4 - \mathscr{H}_i\right)$ of the moduli space of real smooth cubic surfaces in its interior. In [ACT06], they establish the analogous result for moduli of stable binary sextics. Apart from these two examples, no other real moduli stacks where known to admit a positive answer to Question 1.15, before we proved our next result.

Let $\mathcal{M}_s(\mathbb{R})$ be the real moduli space of stable binary quintics. We prove (see Theorem 5.2) that there exists a complete hyperbolic metric on $\mathcal{M}_s(\mathbb{R})$ that restricts to the metrics on $\mathscr{M}_i$ induced by Theorem 1.16. With respect to it, $\mathcal{M}_s(\mathbb{R})$ is isometric to the hyperbolic triangle of angles $\pi/3, \pi/5, \pi/10$.

To put it differently: if we define

$$P\Gamma_{3,5,10} = \langle \alpha_1, \alpha_2, \alpha_3 | \alpha_i^2 = (\alpha_1\alpha_2)^3 = (\alpha_1\alpha_3)^5 = (\alpha_2\alpha_3)^{10} = 1 \rangle,$$

then the following holds true. (See Figure 5.1 for a visualization of this theorem.)

Theorem 1.17 (Theorem 5.2)**.** *There is an open embedding $\coprod_i P\Gamma_i \setminus \left(\mathbb{R}H^2 - \mathscr{H}_i\right) \subset P\Gamma_{3,5,10} \setminus \mathbb{R}H^2$ of hyperbolic orbifolds and a homeomorphism $\mathcal{M}_s(\mathbb{R}) \cong P\Gamma_{3,5,10} \setminus \mathbb{R}H^2$ that extend the orbifold isomorphism $\mathcal{M}_0(\mathbb{R}) \cong \coprod_i P\Gamma_i \setminus \left(\mathbb{R}H^2 - \mathscr{H}_i\right)$ of Theorem 1.16.*

The proof of Theorem 1.17 is inspired by the proof of the analogous theorem for cubic surfaces in [ACT10]. In fact, to prove their real uniformization result cited above, Allcock–Carlson–Toledo use their previous result that the complex moduli space of stable cubic surfaces is isomorphic to a four-dimensional ball quotient [ACT02a]. Likewise, we have to prove (see Section 5.2) that the complex moduli space of stable binary quintics is isomorphic to a two-dimensional ball quotient. Then our proof diverges. Allcock–Carlson–Toledo carry out a local calculation in the real moduli space, to prove that the metrics on the components glue along the discriminant to a complete metric on the space of stable surfaces. Instead, we carry out this local calculation on the ball quotient side.

To do so, it seemed natural not to restrict our attention to this two-dimensional ball quotient, but to consider general unitary Shimura varieties instead. In Chapter 4, we develop a method of glueing real hyperbolic quotient spaces in this context. The input is a hermitian lattice Λ of hyperbolic signature over the ring of integers of a CM field; the procedure glues together real ball quotients arising from anti-isometric involutions on Λ along a hyperplane arrangement; the output is again a real ball quotient (or a disjoint union of those), assembling the different pieces in a sometimes non-arithmetic way. See Theorem 4.2 for details. For the right choice of Λ (cohomology of a ramified cover of projective space, c.f. Section 1.2.2), one retrieves the main theorems of [ACT06, ACT10] and obtains the new Theorem 1.17.

1.3.3 *The integral Hodge conjecture for one-cycles on Jacobian varieties (c.f. [BGF22]) (with* Thorsten Beckmann*)*

Next, we consider curves on complex abelian varieties. At first glance it may seem that we are changing directions: the goal is to study conditions under which a complex abelian variety A of dimension g contains sufficiently many curves to generate its group of integral Hodge classes of degree $2g-2$. As such, the nature of this project is discrete rather than continuous; the members of any irreducible family of curves in A all give the same cohomology class in $\mathrm{H}^{2g-2}(A(\mathbb{C}), \mathbb{Z})$. However, it is only a study of the analytic structure of $\mathcal{A}_g(\mathbb{C})$ that allows us to prove density of abelian varieties satisfying the above conditions.

Before getting into details, let us discuss what is known about the integral Hodge conjecture in general. IHC_i holds for every smooth projective X if $i \in \{0, \dim(X)\}$ (trivial), or $i = \dim(X) - 1$ (Lefschetz $(1,1)$). The first counterexamples were provided by torsion classes; the obstructions were topological. Non-torsion non-algebraic Hodge classes can be found on very general hypersurfaces of degree d in $\mathbb{P}^4_{\mathbb{C}}$ for certain d. IHC_{n-2} can fail for rationally connected varieties of dimension n, already for $n = 4$. Conjecturally, these varieties satisfy do satisfy IHC_1, and this is true in dimension three [Voi07, AH62, Tot97, tre92, CTV12, Sch19, Voi02].

We conclude that IHC may hold if one imposes restrictions on the geometry of the variety X. Along the same lines, it is interesting to consider smooth projective threefolds X of Kodaira dimension zero. If such a threefold X satisfies the condition $h^0(X, K_X) > 0$, then X satisfies the integral Hodge conjecture by work of Grabowski, Totaro and Voisin [Gra04, Voi06, Tot21]; this condition on $h^0(X, K_X)$ is necessary by Benoist–Ottem [BO20]. About the integral Hodge conjecture for one-cycles on higher-dimensional varieties X with $K_X = 0$, not much seems to have been known.

This leads us to the starting point of this project, carried out jointly with Thorsten Beckmann. What can be said about one-cycles on abelian varieties? In our investigation of this question, we build on some results of Grabowski, who proved in his thesis [Gra04] that for a positive integer g the following are equivalent:

1. Every complex abelian variety of dimension g satisfies the integral Hodge conjecture for one-cycles.

2. For every principally polarized complex abelian variety (A, θ) of dimension g, the integral cohomology class $\theta^{g-1}/(g-1)!$ is algebraic.

Having established this, Grabowski proves the integral Hodge conjecture for complex abelian threefolds by noting that every principally polarized complex abelian threefold is a product of Jacobians ($\dim(\mathcal{M}_g) = \dim(\mathcal{A}_g)$ for $g = 3$). For $g > 3$, we have $\dim(\mathcal{M}_g) < \dim(\mathcal{A}_g)$, making it hard to generalize his proof.

In fact, the problem is that condition 2 above is a condition on *every* abelian variety in the moduli space $\mathcal{A}_g(\mathbb{C})$. We take the different approach of fixing *one* abelian variety A, and study the integral Hodge conjecture for one-cycles on A.

Theorem 1.18 (Theorem 7.1, with T. Beckmann). *Let (A, θ) be a principally polarized complex abelian variety of dimension g, and suppose that the Hodge class*

$$\frac{\theta^{g-1}}{(g-1)!} \in \mathrm{H}^{2g-2}(A(\mathbb{C}), \mathbb{Z})$$

is algebraic. Then A satisfies the integral Hodge conjecture for one-cycles.

The proof of Theorem 1.18 is inspired by Grabowski's approach to use Fourier transforms. The idea is quite simple, and goes as follows. On cohomology, the Fourier transform of an abelian variety preserves integral classes, so it is natural to ask whether the same holds for the Fourier transform on Chow groups. Suppose that, for some complex abelian variety A, this is indeed the case. That is, there exists a homomorphism $\mathcal{F}_{\widehat{A}} \colon \mathrm{CH}(\widehat{A}) \to \mathrm{CH}(A)$ that commutes with the cycle class map and the Fourier transform on integral cohomology. Because the latter is an isomorphism by [Bea82], one may lift any degree $2g-2$ Hodge class α on A to a degree two Hodge class on $\widehat{A}$. By Lefschetz (1,1), the latter is the class of a line bundle, which $\mathcal{F}_{\widehat{A}}$ maps to a one-cycle in $\mathrm{CH}(A)$ lying above α.

The assumption on which this argument rests, is a compatibility of Fourier transforms with integral Chow groups of abelian varieties. We study this compatibility in detail in Chapter 6, developing a theory of *integral Fourier transforms*. The research in Chapters 6 and 7 was carried out jointly with Thorsten Beckmann.

As a first corollary of Theorem 1.18, note that it applies to Jacobian varieties, for which the minimal class $\theta^{g-1}/(g-1)!$ is well known to be algebraic:

Theorem 1.19 (Theorem 7.2, with T. Beckmann)**.** *Let $C_1, \ldots, C_n$ be smooth projective curves over $\mathbb{C}$. Then $A = \prod_{i=1}^n J(C_i)$ satisfies the integral Hoge conjecture for one-cycles.*

It is also classical that *Hecke orbits* are dense for the euclidean topology of $\mathcal{A}_g(\mathbb{C})$. These are isogeny orbits of polarized abelian varieties in their moduli space (see e.g. [Cha95, CO19]). Since one can control the degree of these isogenies, one obtains:

Theorem 1.20 (Theorem 7.3, with T. Beckmann)**.** *Principally polarized abelian varieties satisfying the integral Hodge conjecture for one-cycles are euclidean dense in $\mathcal{A}_g(\mathbb{C})$.*

Our machinery on integral Fourier transforms (Chapter 6) has similar implications for cycles on abelian varieties in positive characteristic, see Theorems 7.6 and 7.7. Naturally, the next thing to do was to consider cycles on abelian varieties over $\mathbb{R}$.

1.3.4 *The real integral Hodge conjecture for abelian threefolds* *(based on [GF22a])*

Let us have a closer look at Grabowski's elegant proof of the integral Hodge conjecture for complex abelian threefolds. For such a threefold A, the Fourier transform $\mathscr{F}_A$ defines a natural isomorphism $\mathrm{Hdg}^2(A(\mathbb{C}), \mathbb{Z}) \cong \mathrm{Hdg}^4(\widehat{A}(\mathbb{C}), \mathbb{Z})$. By Lefschetz $(1,1)$, one is reduced to checking whether $\mathscr{F}_A$ sends line bundles to classes of one-cycles – for this, one reduces to the line bunde $\mathcal{L} = \mathcal{O}_{J(C)}(\Theta)$ attached to the theta divisor Θ on a Jacobian variety $J(C)$. Since the Fourier transform $\mathscr{F}_{J(C)}$ sends $c_1(\mathcal{L})$ to the class of the curve C inside $J(C)$, this is enough to conclude.

It is natural to ask whether the same line of arguments works over $\mathbb{R}$.

Question 1.21*.* Do real abelian threefolds satisfy the real integral Hodge conjecture?

We remark that the real integral Hodge conjecture for k-cycles holds for every real variety X for $k \in \{0, \dim(X)\}$ by [BW20a], and for $k = \dim(X) - 1$ by [Kra91, Ham97, MvH98]. Thus, to answer Question 1.21, one needs only to consider one-cycles. In general, it is an open problem whether uniruled or Calabi-Yau threefolds satisfy $\mathbb{R}\mathrm{IHC}_1$ [BW20a, Question 2.16]. There are partial results in the uniruled case [BW20b], but for real Calabi-Yau threefolds, not much had been known.

We prove that at least modulo torsion, the answer to Question 1.21 is *yes*.

Theorem 1.22 (Theorem 8.3 and Corollary 8.4)**.** *Let A be an abelian threefold over $\mathbb{R}$. The following map surjective:*

$$\mathrm{CH}_1(A) \to \mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0/(\text{torsion}) = \mathrm{Hdg}^4(A(\mathbb{C}), \mathbb{Z}(2))^G.$$

In particular, if $A(\mathbb{R})$ is connected, then A satisfies the real integral Hodge conjecture.

The proof of Theorem 1.22 uses a real analogue of the equivalence of statements 1 and 2 in Section 1.3.3 (see Theorem 8.20), and a reduction to the Jacobian of a curve with non-empty real locus. The latter rests crucially on density of certain Hecke orbits in the moduli space $\mathcal{A}_g(\mathbb{R})$, a result which we prove in Theorem 8.5. The same density statement implies that Question 1.21 in the principally polarized case reduces to the case of Jacobians (see Theorem 8.7).

Finally, we manage to say something about torsion cohomology classes of degree four on real abelian varieties of any dimension g. The real integral Hodge conjecture for codimension-two cycles predicts that these classes are algebraic, provided that they satisfy a certain topological condition (8.4) introduced in Section 8.2.

To state this result, we need the following fact: for any projective variety X, smooth over $\mathbb{R}$, there exists a canonical filtration $F^\bullet$ on $\mathrm{Hdg}^4_G(X(\mathbb{C}), \mathbb{Z}(2))_0$ coming from a certain spectral sequence, the *Hochschild-Serre* spectral sequence (see (8.5)).

Theorem 1.23 (Theorem 8.8)**.** *Let A be an abelian variety over $\mathbb{R}$. The group $F^3\mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0$ is zero and the group $F^2\mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0$ is algebraic.*

Combined with Theorem 1.22, this theorem has two notable consequences:

1. (Theorem 8.7 and Corollary 8.9). The real integral Hodge conjecture for principally polarized abelian threefolds is equivalent to the surjectivity of the Abel-Jacobi map

$$\mathrm{CH}_1(J(C))_{\mathrm{hom}} \to \mathrm{H}^1(G, \mathrm{H}^3(J(C)(\mathbb{C}), \mathbb{Z}(2)))_0$$

   for the Jacobian $J(C)$ of every real algebraic curve C of genus three whose real locus $C(\mathbb{R})$ is non-empty.

2. (Corollary 8.10). If $A = B \times E$ is the product of a real abelian surface B and a real elliptic curve E whose real locus $E(\mathbb{R})$ is connected, then A satisfies the real integral Hodge conjecture.

Part I

Real moduli spaces

2

REAL MODULI SPACES

2.1 INTRODUCTION

Let $|\mathcal{A}_g(\mathbb{R})|$ be the set of isomorphism classes of principally polarized abelian varieties of dimension g over $\mathbb{R}$. Similarly, define $|\mathcal{M}_g(\mathbb{R})|$ as the set of isomorphism classes of proper, smooth and geometrically connected curves of genus g over $\mathbb{R}$. In [GH81, SS89], Gross, Harris, Seppälä and Silhol provide these spaces with a real semi-analytic structure[1]. In both cases, the real moduli space is a disjoint union of quotients $\Gamma \setminus M$ of a real-analytic manifold M by a properly discontinuous action of a discrete group Γ. Thus, the sets $|\mathcal{A}_g(\mathbb{R})|$ and $|\mathcal{M}_g(\mathbb{R})|$ carry the structure of topological space, semi-analytic variety, and real-analytic orbifold in a natural way.

This seems the right analytic approach to construct the real moduli spaces of abelian varieties and curves. Is there a general approach to construct a suitable real moduli space, starting from an algebraic moduli stack over $\mathbb{R}$? If so, how does it compare to the above analytic construction of the real moduli spaces of abelian varieties and curves? The goal of this chapter is two answer these two questions.

We will consider an algebraic stack $\mathscr{X}$ locally of finite type over $\mathbb{R}$, and define a topology on the set $|\mathscr{X}(\mathbb{R})|$ of isomorphism classes of $\mathscr{X}(\mathbb{R})$ which agrees the euclidean topology on $\mathscr{X}(\mathbb{R})$ when $\mathscr{X}$ is a scheme. If $\mathscr{X}$ admits a coarse moduli space $\mathscr{X} \to M$, then $|\mathscr{X}(\mathbb{R})|$ should be thought of as the real analogue of $M(\mathbb{C})$ with its euclidean topology. The point is that we cannot use $M(\mathbb{R})$, since this set may not be in bijection with $|\mathscr{X}(\mathbb{R})|$. We show that the space $|\mathscr{X}(\mathbb{R})|$ underlies an orbifold structure if $\mathscr{X}$ is a smooth separated Deligne-Mumford stack. We finish the chapter by verifying that if $\mathscr{X}$ is the moduli stack of abelian varieties $\mathcal{A}_g$, or the stack of curves $\mathcal{M}_g$, then the topology on $|\mathscr{X}(\mathbb{R})|$ coincides with the topology arising via the analytic approach of Gross–Harris–Seppälä–Silhol mentioned above.

1 These structures are *not* real-analytic, as opposed to what is claimed in *loc.cit.* – see [Hui99]. Moreover, one has to be aware of the error in the definition of Γ_H on page 180 of [GH81].

2.2 REAL ALGEBRAIC STACKS

In Section 2.2, we equip the set of isomorphism classes $|\mathscr{X}(\mathbb{R})|$ of the real locus $\mathscr{X}(\mathbb{R})$ of a real algebraic stack with a topology, using a scheme U and a morphism $U \to \mathscr{X}$ which is smooth, surjective and essentially surjective on $\mathbb{R}$-points. For smooth Deligne-Mumford stacks, this space $|\mathscr{X}(\mathbb{R})|$ admits an orbifold structure.

Conventions 2.1. For a scheme S, an algebraic stack $\mathscr{X}$ over S and a scheme T over S, we let $|\mathscr{X}(T)|$ denote the set of isomorphism classes of the groupoid $\mathscr{X}(T)$. A smooth (resp. étale) *presentation* of an algebraic stack $\mathscr{X}$ is a smooth (resp. étale) surjective S-morphism $U \to \mathscr{X}$, where U is a scheme over S. A *real algebraic stack* is an algebraic stack locally of finite type over $\mathbb{R}$. By the topology on $X(\mathbb{R})$ for a scheme X locally of finite type over $\mathbb{R}$, we mean the real-analytic topology.

2.2.1 *The topology of the real locus of an algebraic stack*

The goal of Section 2.2.1 is to define a topology on $|\mathscr{X}(\mathbb{R})|$ for any algebraic stack $\mathscr{X}$ locally of finite type over $\mathbb{R}$, in a way that is functorial in $\mathscr{X}$, and that generalizes the real-analytic topology on $\mathscr{X}(\mathbb{R})$ when $\mathscr{X}$ is a scheme.

Definition 2.2 (c.f. [Sak16, AA19])**.** Let $\mathscr{X}$ be an algebraic stack over a scheme S, and let K be a field equipped with a morphism $\mathrm{Spec}(K) \to S$. A presentation $X \to \mathscr{X}$ by an S-scheme X is *K-surjective* if the map $X(K) \to |\mathscr{X}(K)|$ is surjective.

Theorem 2.3 (Laumon–Moret-Bailly)**.** *Let $\mathscr{X}$ be an algebraic stack over a scheme S, and let K be a field equipped with a morphism* $\mathrm{Spec}(K) \to S$*. For any S-morphism $x\colon \mathrm{Spec}(K) \to \mathscr{X}$, there exists an affine scheme X, a smooth morphism $P\colon X \to \mathscr{X}$, and an S-morphism $z\colon \mathrm{Spec}(K) \to X$ such that $P \circ z \cong x$.*

Proof. See [LMB00, Théorème (6.3)]. $\square$

Corollary 2.4. *Let $\mathscr{X}$ be an algebraic stack over a scheme S, and let K be a field equipped with a morphism* $\mathrm{Spec}(K) \to S$*. There exists a K-surjective smooth presentation $U \to \mathscr{X}$.*

Proof. Let $U' \to \mathscr{X}$ be any smooth presentation. For every $[x] \in |\mathscr{X}(K)|$, choose a morphism $x\colon \mathrm{Spec}(K) \to \mathscr{X}$ that represents $[x]$, and define $X_x \to \mathscr{X}$ as in Theorem 2.3. The morphism $U = \left(\bigsqcup_{[x]\in|\mathscr{X}(K)|} X_x\right) \bigsqcup U' \to \mathscr{X}$ satisfies the requirements. $\square$

Proposition 2.5. *Let $\mathscr{X}$ be a real algebraic stack and let $P_i \colon U_i \to \mathscr{X}$ for $i \in \{1,2\}$ be two $\mathbb{R}$-surjective smooth presentations. Consider the two quotient topologies on $|\mathscr{X}(\mathbb{R})|$ given by the two surjections $P_{i,\mathbb{R}} \colon U_i(\mathbb{R}) \to |\mathscr{X}(\mathbb{R})|$. These two topologies are the same.*

Proof. The stack $U_1 \times_{\mathscr{X}} U_2$ is equivalent to an $\mathbb{R}$-scheme W, and the projections $\pi_2 : W \to U_2$ and $\pi_1 : W \to U_1$ are smooth. The morphisms $\pi_{2,\mathbb{R}} : W(\mathbb{R}) \to U_2(\mathbb{R})$ and $\pi_{1,\mathbb{R}} : W(\mathbb{R}) \to U_1(\mathbb{R})$ are surjective: if $x \in U_2(\mathbb{R})$ then $P(x) \in \mathscr{X}(\mathbb{R})$, so by $\mathbb{R}$-surjectivity there is a $y \in U_1(\mathbb{R})$ and an isomorphism $\psi : Q(y) \cong P(x)$ in $\mathscr{X}(\mathbb{R})$; then (x, y, ψ) defines an object in $(U_1 \times_{\mathscr{X}} U_2)(\mathbb{R})$ for which $\pi_{2,\mathbb{R}}(x, y, \psi) = x$ in $U_2(\mathbb{R})$. Since $\pi_{2,\mathbb{R}}$ and $\pi_{1,\mathbb{R}}$ are open by Lemma 2.6 below, the result follows. □

Lemma 2.6. *Let $g : X \to Y$ be a morphism of schemes which are locally of finite type over $\mathbb{R}$. Let $g_{\mathbb{R}} : X(\mathbb{R}) \to Y(\mathbb{R})$ be the induced map of real-analytic spaces. The morphism of analytic spaces $g_{\mathbb{R}}$ is open if the morphism of schemes g is smooth.*

Proof. We can work locally on $X(\mathbb{R})$; let $U \subset X$ and $V \subset Y$ be affine open subschemes with $g(U) \subset V$ such that $g|_U = \pi_2 \circ f$, where $f : U \to \mathbb{A}^d_V$ is étale for some integer $d \geq 0$, and $\pi_2 : \mathbb{A}^d_V \to V$ is the projection on the right factor [Sta18, Tag 054L]. Because $X(\mathbb{C}) \to X$ is continuous [Gro71, §XII, 1.1], the set $U(\mathbb{C})$ is open in $X(\mathbb{C})$, thus $U(\mathbb{R}) = U(\mathbb{C}) \cap X(\mathbb{R})$ is open in $X(\mathbb{R})$. Similarly, $V(\mathbb{R})$ is open in $Y(\mathbb{R})$. Finally, the map $U(\mathbb{R}) \to V(\mathbb{R})$ is open because it factors as the composition

$$U(\mathbb{R}) \xrightarrow{f_{\mathbb{R}}} \mathbb{R}^d \times V(\mathbb{R}) \xrightarrow{\pi_{2,\mathbb{R}}} V(\mathbb{R}),$$

where $\pi_{2,\mathbb{R}}$ is open and $f_{\mathbb{R}}$ a local homeomorphism. The lemma follows. □

Definition 2.7. Let $\mathscr{X}$ be an algebraic stack locally of finite type over $\mathbb{R}$. The *real-analytic topology* on $|\mathscr{X}(\mathbb{R})|$ is the quotient topology on $|\mathscr{X}(\mathbb{R})|$ given by the surjection $U(\mathbb{R}) \to |\mathscr{X}(\mathbb{R})|$ and the real-analytic topology on $U(\mathbb{R})$, for any $\mathbb{R}$-surjective smooth presentation $U \to \mathscr{X}$ (see Definition 2.2 and Corollary 2.4).

In the sequel, $|\mathscr{X}(\mathbb{R})|$ will denote the topological space defined in Definition 2.7.

Corollary 2.8.

1. *The assignment of a topological space to an algebraic stack locally of finite type over the real numbers is functorial.*
2. *If $\rho \colon \mathscr{X} \to \mathscr{Y}$ is a smooth morphism [Sta18, Tag 06FM] of algebraic stacks locally of finite type over $\mathbb{R}$, then the map $\rho_{\mathbb{R}} \colon |\mathscr{X}(\mathbb{R})| \to |\mathscr{Y}(\mathbb{R})|$ is open. In particular, if $\mathscr{U} \subset \mathscr{X}$ is an open substack, then $|\mathscr{U}(\mathbb{R})| \subset |\mathscr{X}(\mathbb{R})|$ is an open subset.*

3. *If a real algebraic stack $\mathscr{X}$ admits a coarse moduli space $\pi : \mathscr{X} \to M$, then the natural map $|\mathscr{X}(\mathbb{R})| \to M(\mathbb{C})$ is continuous.*

Proof. 1. Let $\mathscr{X} \to \mathscr{Y}$ be a morphism of real algebraic stacks, $V \to \mathscr{Y}$ an $\mathbb{R}$-surjective smooth presentation, and $\mathscr{Z} = \mathscr{X} \times_{\mathscr{Y}} V$. Then $|\mathscr{Z}(\mathbb{R})| \to |\mathscr{X}(\mathbb{R})|$ is surjective. If $U \to \mathscr{Z}$ be an $\mathbb{R}$-surjective smooth presentation, then the composition $U \to \mathscr{Z} \to \mathscr{X}$ is an $\mathbb{R}$-surjective smooth presentation of $\mathscr{X}$. Statement 1 follows.

2. Consider an open subset $B \subset |\mathscr{X}(\mathbb{R})|$. Let $\pi\colon V \to \mathscr{Y}$ be an $\mathbb{R}$-surjective smooth presentation and denote the projection $\mathscr{X} \times_{\mathscr{Y}} V \to \mathscr{X}$ by f. Suppose that $\rho\colon \mathscr{X} \to \mathscr{Y}$ is the smooth morphism under consideration, and let $\tilde{\rho}\colon \mathscr{X} \times_{\mathscr{Y}} V \to V$ be its base-change. We have an equality

$$\tilde{\rho}_{\mathbb{R}}(f_{\mathbb{R}}^{-1}(B)) = \pi_{\mathbb{R}}^{-1}(\rho_{\mathbb{R}}(B)) \subset V(\mathbb{R}).$$

Indeed, this can be deduced from the following commutative diagram of sets:

$$\begin{tikzcd}
|(\mathscr{X} \times_{\mathscr{Y}} V)(\mathbb{R})| \arrow[r, two heads] \arrow[rr, bend left, "\tilde{\rho}_{\mathbb{R}}"] \arrow[dr, two heads, "f_{\mathbb{R}}"'] & |\mathscr{X}(\mathbb{R})| \times_{|\mathscr{Y}(\mathbb{R})|} V(\mathbb{R}) \arrow[r] \arrow[d, two heads] & V(\mathbb{R}) \arrow[d, two heads, "\pi_{\mathbb{R}}"] \\
 & |\mathscr{X}(\mathbb{R})| \arrow[r, "\rho_{\mathbb{R}}"] & |\mathscr{Y}(\mathbb{R})| .
\end{tikzcd}$$

Therefore, to prove statement 2, it suffices to treat the case $\mathscr{Y} = V$ is a scheme. This is clear: if $U \to \mathscr{X}$ is any $\mathbb{R}$-surjective smooth presentation, then the composition $U \to \mathscr{X} \to V$ is smooth, hence $U(\mathbb{R}) \to V(\mathbb{R})$ is open by Lemma 2.6.

3. This follows from statement 1 and the continuity of $M(\mathbb{R}) \to M(\mathbb{C})$. □

2.2.2 *Pointwise surjective presentations of Deligne-Mumford stacks*

The goal of this subsection is to prove the following (compare Theorem 2.3):

Theorem 2.9. *Let K be a field, and let $\mathscr{X}$ be a Deligne-Mumford stack over K. For any K-morphism $x\colon \mathrm{Spec}(K) \to \mathscr{X}$, there exists an affine scheme U, an étale morphism $\Psi\colon U \to \mathscr{X}$, and a K-morphism $y\colon \mathrm{Spec}(K) \to U$ such that $\Psi \circ y \cong x$.*

We will deduce Theorem 2.9 from Theorem 2.3. To do so, we will need:

Lemma 2.10 (Laumon–Moret-Bailly)**.** *Let $\mathscr{X}$ be a Deligne-Mumford stack over a scheme S, let X be an affine S-scheme, and let $P\colon X \to \mathscr{X}$ be a smooth morphism. Define*

$\Omega_{X/\mathscr{X}}$ as the conormal sheaf of the diagonal $X \to X \times_{\mathscr{X}} X$. The canonical morphism $\rho\colon \Omega_{X/S} \to \Omega_{X/\mathscr{X}}$ is surjective and $\Omega_{X/\mathscr{X}}$ is an $\mathcal{O}_X$-module locally free of finite rank.

Proof. See [LMB00, (8.2.2) - (8.2.3.2)]. □

Proof of Theorem 2.9. By Theorem 2.3, there exists an affine scheme X, a smooth morphism $P\colon X \to \mathscr{X}$, and a K-morphism $z\colon \mathrm{Spec}(K) \to X$ such that $P \circ z \cong x$. By Lemma 2.10, the $\mathcal{O}_X$-module $\Omega_{X/\mathscr{X}}$ is locally free of finite rank; let r be the rank of $\Omega_{X/\mathscr{X}}$ in a neighborhood around $z \in X$. Consider the sequence (c.f. Lemma 2.10):

$$\mathcal{O}_X \xrightarrow{d} \Omega_{X/K} \xrightarrow{\rho} \Omega_{X/\mathscr{X}}. \tag{2.1}$$

Claim 1: There are sections $f_1, \ldots, f_r \in R$ whose images $\rho d(f_i)(z)$ in $\Omega_{X/\mathscr{X}} \otimes k(z)$ form a basis for the $k(z)$-vector space $\Omega_{X/\mathscr{X}} \otimes k(z)$. (Compare [LMB00, (8.2.4)].)

Indeed, let $R = \mathcal{O}_X(X)$; then $\Omega_{X/K} \otimes k(z)$ is generated by $\{d(f)(z) \mid f \in R\}$, where $d(f) \in \Omega_{X/k}$ is the differential of a section $f \in R$, and $d(f)(z)$ is the image of $d(f)$ in $\Omega_{X/k} \otimes k(z)$. The claim follows from the surjectivity of ρ in (2.1).

We obtain representable K-morphisms

$$f := (f_1, \ldots, f_r) : X \to \mathbb{A}^r_K, \quad \text{and} \quad (P, f)\colon X \to \mathscr{X} \times_K \mathbb{A}^r_K.$$

Claim 2: (P, f) is étale on a neighborhood $X' \subset X$ of z. (Compare [LMB00, (8.2.4.2)].)

To prove Claim 2, let Y be a scheme with surjective étale morphism $Y \to \mathscr{X}$, and let $W = X \times_{\mathscr{X}} Y$. By [Sta18, Tag 01UV] and [Sta18, Tag 01UX], the composition $W \xrightarrow{h} Y \times_K \mathbb{A}^r_K \xrightarrow{\pi} Y$ induces a canonical exact sequence

$$h^*\pi^*\Omega_{\mathbb{A}^r_K/K} \to \Omega_{W/Y} \to \Omega_{W/Y\times\mathbb{A}^r} \to 0.$$

By Claim 1, $h^*\pi^*\Omega_{\mathbb{A}^r_K/K} \otimes k(w) \to \Omega_{W/Y} \otimes k(w)$ is an isomorphism. Thus, by [DG67, IV, §4, 17.11.2], h is étale at $w \in W$. Since $W \to X$ is étale, Claim 2 follows.

Define a closed subscheme $j : U \hookrightarrow X'$ as the following fibre product:

$$\begin{tikzcd}
U \arrow[r, "\Psi"] \arrow[d, "j"] & \mathscr{X} \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[d, "f(z)"] \\
X' \arrow[r, "{(P,f)}"] & \mathscr{X} \times_K \mathbb{A}^r_K \arrow[r] & \mathbb{A}^r_K.
\end{tikzcd}$$

The morphism $\Psi \colon U \to \mathscr{X}$ is étale since (P, f) is étale. Let F be the morphism $\mathscr{X} \to \mathscr{X} \times_K \mathbb{A}^r_K$, and choose an isomorphism $\phi \colon P(z) \cong y$ in $\mathscr{X}(K)$. Then $F(\phi)$ is an isomorphism $F(P(z)) = (P(z), f(z)) \cong (y, f(z))$ in $(\mathscr{X} \times_K \mathbb{A}^r_K)(K)$. By definition of the fibre product of X' and $\mathscr{X}$ over $\mathscr{X} \times_K \mathbb{A}^r_K$ [LMB00, (2.2.2)], the triple

$$\big(z \in X'(K),\ \ x \in \mathscr{X}(K),\ \ F(\phi) \colon (P, f)(z) \cong (y, f(z))\big)$$

induces $y \colon \operatorname{Spec}(K) \to U$ such that $j \circ y = z \in X'(K)$ and $\Psi \circ y \cong x \in \mathscr{X}(K)$. $\square$

Corollary 2.11. *Let K be a field, and let $\mathscr{X}$ be a Deligne-Mumford stack over K. There exists a K-surjective étale presentation $U \to \mathscr{X}$.* $\square$

2.2.3 *The orbifold structure of the real locus of a Deligne-Mumford stack*

Let $\mathscr{X}$ be a smooth separated Deligne-Mumford stack over $\mathbb{R}$. The goal of this section is to use the results of Section 2.2.2 to show that the topological space $|\mathscr{X}(\mathbb{R})|$ (see Definition 2.7) admits a canonical orbifold structure, agreeing with smooth manifold structure of $\mathscr{X}(\mathbb{R})$ in case $\mathscr{X}$ is a scheme. Let $P \colon U \to \mathscr{X}$ be an $\mathbb{R}$-surjective étale presentation; P exists by Corollary 2.11. For $R = U \times_{\mathscr{X}} U$, there are maps $s, t \colon R \to U$ and $c \colon R \times_{s,U,t} R \to R$ turning $(U, R, s, t, c) = (R \rightrightarrows U)$ into a groupoid scheme over $\mathbb{R}$ [LMB00, Proposition (3.8)]. The morphisms $s, t \colon R \to U$ are étale, the morphism $[U/R] \to \mathscr{X}$ is an equivalence [LMB00, (4.3)], and the morphism $j = (t, s) \colon R \to U \times_{\mathbb{R}} U$ is proper since $\mathscr{X}$ is separated over $\mathbb{R}$.

The schemes U and R are smooth over $\mathbb{R}$, thus the real-analytic spaces $U(\mathbb{R})$ and $R(\mathbb{R})$ are real-analytic manifolds. Consequently, the resulting five-tuple

$$\mathscr{X}(\mathbb{R})_P := (U(\mathbb{R}), R(\mathbb{R}), s_{\mathbb{R}}, t_{\mathbb{R}}, c_{\mathbb{R}}) = (R(\mathbb{R}) \rightrightarrows U(\mathbb{R}))$$

is a proper étale Lie groupoid, see [Moe03, §1], [MM03, Chapter 5] or [ALR07, Chapter 1]. Indeed, $j_{\mathbb{R}} \colon R(\mathbb{R}) \to U(\mathbb{R}) \times U(\mathbb{R})$ is proper by [Bou71, Chapitre 1, §10, Théorème 1] because $j_{\mathbb{C}} \colon R(\mathbb{C}) \to U(\mathbb{C}) \times U(\mathbb{C})$ is proper [Gro71, XII, Proposition 3.2]; the maps $s_{\mathbb{R}}$ and $t_{\mathbb{R}}$ are local diffeomorphisms because s and t are étale.

Since $P_{\mathbb{R}} \colon U(\mathbb{R}) \to |X(\mathbb{R})|$ induces a homeomorphism $U(\mathbb{R})/R(\mathbb{R}) = |\mathscr{X}(\mathbb{R})|$, we obtain an orbifold structure [Moe03, §3.2] on the topological space $|\mathscr{X}(\mathbb{R})|$. Let $\mathrm{O}_{\mathbb{R}}(\mathscr{X})$ denote the resulting orbifold [Moe03, §3.3].

Proposition 2.12. *Let $\mathscr{X}$ and $\mathscr{Y}$ be smooth separated Deligne-Mumford stacks over $\mathbb{R}$. The orbifold $\mathrm{O}_{\mathbb{R}}(\mathscr{X})$ does not depend on the choice of $\mathbb{R}$-surjective étale presentation $U \to \mathscr{X}$. A morphism of stacks $\mathscr{X} \to \mathscr{Y}$ induces a morphism of orbifolds $\mathrm{O}_{\mathbb{R}}(\mathscr{X}) \to \mathrm{O}_{\mathbb{R}}(\mathscr{Y})$.*

Proof. From two $\mathbb{R}$-surjective étale presentations $P\colon U \to \mathscr{X}$ and $P'\colon U' \to \mathscr{X}$ we obtain a third one that covers both, namely $P''\colon U'' = U \times_{\mathscr{X}} U' \to \mathscr{X}$. This establishes a Morita equivalence $\mathscr{X}(\mathbb{R})_P \leftarrow \mathscr{X}(\mathbb{R})_{P''} \to \mathscr{X}(\mathbb{R})_{P'}$ [Moe03, §2.4]. Thus, $\mathscr{X}(\mathbb{R})_P$ and $\mathscr{X}(\mathbb{R})_{P'}$ induce equivalent orbifold structures on $|\mathscr{X}(\mathbb{R})|$. Next, let $Q\colon V \to \mathscr{Y}$ and $U \to \mathscr{X} \times_{\mathscr{Y}} V$ be $\mathbb{R}$-surjective étale presentations, and let P be the $\mathbb{R}$-surjective étale presentation $U \to \mathscr{X} \times_{\mathscr{Y}} V \to \mathscr{X}$. The map of groupoids $\mathscr{X}(\mathbb{R})_P \to \mathscr{Y}(\mathbb{R})_Q$ gives a map of orbifolds $\mathrm{O}_{\mathbb{R}}(\mathscr{X}) \to \mathrm{O}_{\mathbb{R}}(\mathscr{Y})$ [Moe03, §3.3]. □

2.3 MODULI OF REAL ABELIAN VARIETIES AND CURVES

In the final part of Chapter 2, we specialize to the stacks $\mathcal{A}_g$ and $\mathcal{M}_g$, of principally polarized abelian varieties of dimension g, respectively smooth, proper and geometrically connected curves of genus g. By the results of Section 2.2, there is a natural topology on the set of isomorphisms classes of the real locus of both stacks; by the results of Section 2.2.3, this topology underlies an orbifold structure. Gross–Harris and Seppälä–Silhol also defined a topology (and even an orbifold structure) on $|\mathcal{A}_g(\mathbb{R})|$ and on $|\mathcal{M}_g(\mathbb{R})|$, see Examples 1.4. Our goal in this Section 2.3 will be to prove that their topology agrees with ours. Although our proof shows that also the two orbifold structures agree, we will not discuss this aspect here.

2.3.1 *Moduli of real abelian varieties*

The set $|\mathcal{A}_g(\mathbb{R})|$ of isomorphism classes of principally polarized real abelian varieties of dimension g can be provided with a real semi-analytic space structure as follows. The result seems to have been proven independently by to Gross–Harris [GH81, Section 9] and Silhol [Sil89, Chapter IV, Section 4], although the former made an error in the definition of the group Γ_H that appears on page 180 of [GH81].

Let us recall Silhol's approach. We only give a sketch here; the reader is referred to Section 8.3.2 for a more detailed account. Following [Sil89, Chapter IV, Definition 4.4], define a subset $\mathscr{T}(g) \subset \mathbb{Z}^2$ as

$$\mathscr{T}(g) = \left\{(r,\alpha) \in \mathbb{Z}^2 : 1 \leq r \leq g,\ \alpha \in \{1,2\} \mid \alpha = 1 \text{ if } r \text{ is odd}\right\} \cup \left\{(0,1)\right\}.$$

Attach to each $\tau = (r,\alpha) \in \mathscr{T}(g)$ a matrix $M(\tau) \in M_g(\mathbb{Z})$ as in [Sil89, Ch.IV, Theorem 4.1] (see the matrices 1 and 2 in Section 8.3.2). Then define

$$\mathrm{GL}_g^\tau(\mathbb{Z}) = \{T \in \mathrm{GL}_g(\mathbb{Z}) : T^t \cdot M(\tau) \cdot T \equiv M(\tau) \bmod 2\} \subset \mathrm{GL}_g(\mathbb{Z}).$$

Consider the genus g Siegel space $\mathbb{H}_g$ and define an anti-holomorphic involution

$$\Sigma_\tau : \mathbb{H}_g \to \mathbb{H}_g, \quad \Sigma_\tau(Z) = M(\tau) - \overline{Z}. \tag{2.2}$$

Let $\mathbb{H}_g^{\Sigma_\tau}$ be the fixed locus of Σ_τ. Then $\mathrm{GL}_g^\tau(\mathbb{Z})$ acts on $\mathbb{H}_g^{\Sigma_\tau}$ via the inclusion

$$f_\tau \colon \mathrm{GL}_g(\mathbb{R}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R}), \quad T \mapsto \begin{pmatrix} T^t & \frac{1}{2}\left(M(\tau)\cdot T^{-1} - T^t \cdot M(\tau)\right) \\ 0 & T^{-1} \end{pmatrix}.$$

By [GH81, Proposition 9.3] or [Sil89, Ch.IV, Theorem 4.6], taking period matrices induces a bijection

$$\left|\mathcal{A}_g(\mathbb{R})\right| \cong \bigsqcup_{\tau \in \mathscr{T}(g)} \mathrm{GL}_g^\tau(\mathbb{Z}) \backslash \mathbb{H}_g^{\Sigma_\tau}. \tag{2.3}$$

2.3.2 *Moduli of real algebraic curves*

In this section, as well as in the rest of this thesis, we shall often use the following:

Definition 2.13. A *real algebraic curve* is a proper, smooth and geometrically connected curve over $\mathbb{R}$.

Seppälä and Silhol provide the set $\left|\mathcal{M}_g(\mathbb{R})\right|$ of real genus g algebraic curves with a topology as follows. Fix a compact oriented $\mathcal{C}^\infty$-surface Σ of genus g and let $\mathcal{T}_g$ be the Teichmüller space of the surface Σ (see e.g. [AC09]).

Definition 2.14. Let $\mathscr{J}(g) \subset \mathbb{Z}^2$ be the set of tuples $(k,\epsilon) \in \mathbb{Z}^2$ that satisfy the following conditions. We have $\epsilon \in \{0,1\}$. If $\epsilon = 1$, then $1 \leq k \leq g+1$ and $k \equiv g+1 \bmod 2$. If $\epsilon = 0$, then $0 \leq k \leq g$.

To every $j = (k(j), \epsilon(j)) \in \mathscr{J}(g)$ one can attach an orientation-reversing involution $\sigma_j : \Sigma \to \Sigma$ of *type* $j \in \mathscr{J}(g)$ [SS89, Definition 1.2]. This means that $k(j) = \#\pi_0(\Sigma^{\sigma_j})$ and that $\epsilon(j) = 0$ if and only if $\Sigma \setminus \Sigma^{\sigma_j}$ is connected. Moreover, every such involution $\sigma_j : \Sigma \to \Sigma$ induces an anti-holomorphic involution $\sigma_j : \mathcal{T}_g \to \mathcal{T}_g$ [SS89, §I.2]. Denote by $N_j = \{g \in \Gamma_g : g \circ \sigma_j = \sigma_j \circ g\}$ the normalizer of $\sigma_j : \mathcal{T}_g \to \mathcal{T}_g$ in the mapping class group Γ_g of Σ. There is a natural bijection [SS89, Theorem 2.1, Definition 2.3]

$$\left|\mathcal{M}_g(\mathbb{R})\right| \cong \bigsqcup_{j \in J} N_j \backslash \mathcal{T}_g^{\sigma_j}. \tag{2.4}$$

2.3.3 *Comparing the real moduli spaces*

Consider the real-analytic topologies on $|\mathcal{A}_g(\mathbb{R})|$ and $|\mathcal{M}_g(\mathbb{R})|$, see Definition 2.7. Our goal in Section 2.3.3 is to prove that (2.3) and (2.4) are homeomorphisms.

Theorem 2.15. *The natural bijection (2.3) is a homeomorphism.*

Proof. Let $\mathcal{S} \to \mathcal{A}_g$ be an $\mathbb{R}$-surjective étale surjection by a scheme $\mathcal{S}$ over $\mathbb{R}$ (see Corollary 2.11). Recall that $\mathcal{A}_g$ is smooth over $\mathbb{R}$ [Jon93, Remark 1.2.5], so that $\mathcal{S}(\mathbb{C})$ is a complex manifold. The composition of $\mathcal{S}(\mathbb{R}) \to \left|\mathcal{A}_g(\mathbb{R})\right|$ with (2.3) defines a surjective *real period map* (c.f. Section 1.2.2):

$$\mathscr{P} : \mathcal{S}(\mathbb{R}) \to \bigsqcup_{\tau \in \mathscr{T}(g)} \mathrm{GL}_g^{\tau}(\mathbb{Z}) \backslash \mathbb{H}_g^{\Sigma_\tau}.$$

Claim 2.16. The real period map $\mathscr{P}$ is continuous and open.

Proof of Claim 2.16. We may work locally on $\mathcal{S}(\mathbb{R})$. Fix a point $0 \in \mathcal{S}(\mathbb{R})$, recall that $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ (Definition 1.1), and consider a G-stable contractible open neighbourhood B of 0 in $\mathcal{S}(\mathbb{C})$ such that $B(\mathbb{R}) = B \cap \mathcal{S}(\mathbb{R})$ is connected. The universal family $\mathcal{X}_g \to \mathcal{A}_g$ induces a holomorphic family of complex abelian varieties $\phi : A \to B$, equipped with two anti-holomorphic involutions

$$(\sigma : A \to A, \chi : B \to B)$$

that commute with ϕ, polarized by some section $E \in R^2\phi_*\mathbb{Z}$, such that

$$E_{\chi(t)}(\sigma_*(x), \sigma_*(y)) = -E_t(x, y)$$

for every $t \in B$ and $x, y \in \mathrm{H}_1(A_t, \mathbb{Z})$, where $\sigma_* \colon \mathrm{H}_1(A_t, \mathbb{Z}) \to \mathrm{H}_1(A_{\chi(t)}, \mathbb{Z})$ is the push-forward of the anti-holomorphic map $\sigma \colon A_t \to A_{\chi(t)}$. In other words, $\underline{\phi} := (\phi, \sigma, \chi, E)$ is a *holomorphic family of polarized real abelian varieties*.

Claim 2.16 will now follow from:

Claim 2.17. The family $\underline{\phi}$ admits a period map $P \colon B \to \mathbb{H}_g$, which is G-equivariant for one of the holomorphic involutions Σ_τ on $\mathbb{H}_g$ (see (2.2)), whose differential dP defines an isomorphism on tangent spaces at each point of B, and such that the induced map $B(\mathbb{R}) \to \mathrm{GL}_g^\tau(\mathbb{Z}) \setminus \mathbb{H}_g^{\Sigma_\tau}$ coincides with the restriction of $\mathscr{P}$ to $B(\mathbb{R})$.

Before we prove Claim 2.17, we show that it implies Claim 2.16. Consider the period map $P \colon B \to \mathbb{H}_g$ in Claim 2.17. The map $P^G \colon B(\mathbb{R}) \to \mathbb{H}_g^{\Sigma_\tau}$ defines an isomorphism on tangent spaces at each point of $B(\mathbb{R})$, thus is open and continuous, hence the restriction of $\mathscr{P}$ to $B(\mathbb{R})$ is open and continuous, proving Claim 2.16. □

Proof of Claim 2.17. Consider the real abelian variety $(A_0, \sigma : A_0 \to A_0)$. Define Λ_0 to be the lattice $\mathrm{H}_1(A_0, \mathbb{Z})$ and consider the alternating form $E_0 : \Lambda_0 \times \Lambda_0 \to \mathbb{Z}$. By [GH81, Section 9], there exists a symplectic basis $\underline{m} = \{m_1, \ldots, m_g; n_1, \ldots, n_g\}$ for Λ_0 and a unique element $\tau \in \mathscr{T}(g)$ such that the matrix corresponding to σ_* with respect to $\underline{m}$ is the matrix

$$T = \begin{pmatrix} I_g & M(\tau) \\ 0 & -I_g \end{pmatrix} \in \mathrm{GL}_{2g}(\mathbb{Z}), \tag{2.5}$$

where $M(\tau) \in M_g(\mathbb{Z})$ corresponds to $\tau \in \mathscr{T}(g)$ as in Section 2.3.1.

The canonical trivialization $R^1\phi_*\mathbb{Z} \cong \mathrm{H}^1(A_0, \mathbb{Z})$ is G-equivariant, and induces the following isomorphism (resp. abelian groups):

$$s : R^1\phi_*\mathbb{Z} \cong \mathrm{H}^1(A_0, \mathbb{Z}) \xrightarrow{E_0} \mathrm{H}_1(A_0, \mathbb{Z}) = \Lambda_0,$$
$$r_t : \mathrm{H}_1(A_t, \mathbb{Z}) \xrightarrow{E_t} \mathrm{H}^1(A_t, \mathbb{Z}) \xrightarrow{s_t} \Lambda_0, \quad \forall t \in B.$$

Thus, for every $t \in B$, the lattice $\mathrm{H}_1(A_t, \mathbb{Z})$ is equipped with the symplectic basis

$$r_t^{-1}(\underline{m}) = \{m(t)_1, \ldots, m(t)_g; n(t)_1, \ldots, n(t)_g\}.$$

We claim that these bases are compatible with the real structure of $\phi : A \to B$, in the sense that for every $t \in B$, the pushforward

$$\sigma_* : \mathrm{H}_1(A_t, \mathbb{Z}) \to \mathrm{H}_1(A_{\chi(t)}, \mathbb{Z})$$

is given by matrix T of equation (2.5) with respect to the bases $r_t^{-1}(\underline{m})$ and $r_{\sigma(t)}^{-1}(\underline{m})$. Indeed, this follows from the fact that the following diagram commutes:

$$\begin{tikzcd}
\Lambda_0 \arrow[d, "\sigma_*"] & \mathrm{H}^1(A_0, \mathbb{Z}) \arrow[l] \arrow[d, "-\tau*"] & \mathrm{H}^1(A_t, \mathbb{Z}) \arrow[l] \arrow[d, "-\sigma_*"] & \mathrm{H}_1(A_t, \mathbb{Z}) \arrow[l] \arrow[d, "\sigma_*"] \\
\Lambda_0 & \mathrm{H}^1(A_0, \mathbb{Z}) \arrow[l] & \mathrm{H}^1(A_{\chi(t)}, \mathbb{Z}) \arrow[l] & \mathrm{H}_1(A_{\chi(t)}, \mathbb{Z}). \arrow[l]
\end{tikzcd} \tag{2.6}$$

The first and the third square commute because if $\sigma : A_t \to A_{\sigma(t)}$ is the anti-holomorphic restriction of τ to the fibre A_t, then $E_t(x, y) = -E_{\chi(t)}(\sigma_*(x), \sigma_*(y))$ for every $x, y \in \mathrm{H}_1(A_t, \mathbb{Z})$. The second diagram commutes because the trivialization $R^1\phi_*\mathbb{Z} \cong \mathrm{H}^1(A_0, \mathbb{Z})$ is G-equivariant.

The family of symplectic bases $\{r_t^{-1}(\underline{m}) \subset \mathrm{H}_1(A_t, \mathbb{Z})\}_{t \in B}$ induces a holomorphic period map $P : B \to \mathbb{H}_g$, which is defined by sending an element $t \in B$ to the period matrix of the abelian variety A_t with respect to the symplectic basis $r_t^{-1}(\underline{m})$. Note that the differential dP defines an isomorphism on each tangent space, because the morphism $\mathcal{S} \to \mathcal{A}_g$ is étale.

Consider the anti-holomorphic involution $\Sigma_\tau : \mathbb{H}_g \to \mathbb{H}_g$ defined in (2.2). We claim that:

1. The composition
$$B(\mathbb{R}) \xrightarrow{P^G} \mathbb{H}_g^{\Sigma_\tau} \to \mathrm{GL}_g^\tau(\mathbb{Z}) \setminus \mathbb{H}_g^{\Sigma_\tau}$$
equals the restriction of $\mathscr{P}$ to the open neighborhood $B(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ of $0 \in B(\mathbb{R})$.

2. The period map P is Galois-equivariant, in the sense that $\Sigma_\tau \circ P = P \circ \sigma$.

The first part follows by definition of the bijection (2.3). So let us prove 2. Fix an element $t \in B$; we need to show that

$$\Sigma_\tau(P(t)) = P(\chi(t)) \in \mathbb{H}_g. \tag{2.7}$$

For this, consider the symplectic basis $r_t^{-1}(\underline{m}) = \{m(t)_i; n(t)_j\}_{ij} \subset \mathrm{H}_1(A_t, \mathbb{Z})$. Let $\{\omega(t)_1, \ldots, \omega(t)_g\} \subset \mathrm{H}^0(A_t, \Omega^1_{A_t})$ be the basis dual to $\{m(t)_1, \ldots, m(t)_g\}$ under the natural pairing

$$\mathrm{H}_1(A_t, \mathbb{Z}) \times \mathrm{H}^0(A_t, \Omega^1_{A_t}) \to \mathbb{C}, \quad (\gamma, \omega) \mapsto \int_\gamma \omega. \tag{2.8}$$

Then $P(t) = \left(\int_{n(t)_i} \omega(t)_j\right)_{ij} \in \mathbb{H}_g$. Moreover, the following diagram commutes by Lemma 3.7 in Chapter 3:

$$\begin{tikzcd} \mathrm{H}_1(A_t, \mathbb{Z}) \arrow[r] \arrow[d, "\tau_*"] & \mathrm{H}^0(A_t, \Omega^1_{A_t})^\vee \arrow[d, "F_{dR}"] \\ \mathrm{H}_1(A_{\sigma(t)}, \mathbb{Z}) \arrow[r] & \mathrm{H}^0(A_{\sigma(t)}, \Omega^1_{A_{\sigma(t)}})^\vee. \end{tikzcd} \tag{2.9}$$

Here, F_{dR} is the anti-linear map induced by the differential $d\sigma : T_e A_t \to T_e A_{\chi(t)}$ of $\sigma : A_t \to A_{\chi(t)}$ and the canonical identification $T_e X = \mathrm{H}^0(X, \Omega^1_X)^\vee$ for any complex abelian variety X, where $e \in X$ is the origin. See also Lemma 3.7. This implies that (2.8) is compatible with σ_* and F_{dR}, and that $F_{dR}\left(\omega(t)_j\right) = \omega(\chi(t))_j$. Therefore,

$$\begin{aligned} \Sigma_\tau(P(t)) &= M(\tau) - \overline{\left(\int_{n(t)_i} \omega(t)_j\right)_{ij}} \\ &= M(\tau) - \left(\int_{\sigma_*(n(t)_i)} F_{dR}\left(\omega(t)_j\right)\right)_{ij} \\ &= M(\tau) - \left(\int_{\sigma_*(n(t)_i)} \omega(\chi(t))_j\right)_{ij}. \end{aligned}$$

If $M(\tau) = (x_{ij})_{ij}$, and $\psi_t : \mathrm{H}_1(A_t, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is the isomorphism identifying $r_t^{-1}(\underline{m})$ with the standard symplectic basis $\{e_1, \ldots, e_g; f_1, \ldots, f_g\} \subset \mathbb{Z}^{2g}$, then, for $T \in \mathrm{GL}_{2g}(\mathbb{Z})$ as in (2.5), we have:

$$\begin{aligned} \sigma_*(n(t)_i) &= \psi^{-1}_{\chi(t)}\left(T \cdot f_i\right) = \psi^{-1}_{\chi(t)}\left(\sum_{k=1}^g x_{ki} e_k - f_i\right) \\ &= \sum_{k=1}^g x_{ki} m(\chi(t))_k - n(\chi(t))_i \in \mathrm{H}_1(A_{\chi(t)}, \mathbb{Z}). \end{aligned}$$

Therefore, we obtain

$$\int_{\sigma_*(n(t)_i)} \omega(\chi(t))_j = \int_{\sum_{k=1}^g x_{ki} m(\chi(t))_k - n(\chi(t))_i} \omega(\chi(t))_j,$$

which further implies that

$$\begin{aligned}\left(\int_{\sigma_*(n(t)_i)} \omega(\chi(t))_j\right)_{ij} &= \left(\int_{x_{ji} m(\chi(t))_j} \omega(\chi(t))_j\right)_{ij} - \left(\int_{n(\chi(t))_i} \omega(\chi(t))_j\right)_{ij} \\ &= (x_{ji})_{ij} - P\left(\chi(t)\right).\end{aligned}$$

Since $\left(x_{ji}\right)_{ij} = \left(x_{ij}\right)_{ij} = M(\tau)$ because $M(\tau)$ is symmetric, the equality (2.7) follows. This finishes the proof of Claim 2.17, and thereby the proof of Claim 2.16. $\square$

Theorem 2.15 follows readily from Claim 2.16. $\square$

Theorem 2.18. *The natural bijection (2.4) is a homeomorphism.*

Proof. The strategy is similar to our strategy in the proof of Theorem 2.15. Let $\mathcal{S} \to \mathcal{M}_g$ be an $\mathbb{R}$-surjective étale presentation by a scheme $\mathcal{S}$ over $\mathbb{R}$, and let $\mathcal{C} \to \mathcal{S}$ be the corresponding family of real algebraic curves of genus g. The composition

$$\mathscr{P} : \mathcal{S}(\mathbb{R}) \to \left|\mathcal{M}_g(\mathbb{R})\right| \cong \bigsqcup_{j \in \mathscr{J}(g)} N_j \setminus \mathcal{T}_g^{\sigma_j}$$

is surjective. As before, it is enough to prove that $\mathscr{P}$ is continuous and open.

For this, note that $\mathcal{S}(\mathbb{C})$ is a complex manifold because $\mathcal{M}_g$ is smooth [DM69, Theorem 5.2]. Fix $0 \in \mathcal{S}(\mathbb{R})$ and consider a G-stable contractible open neighborhood $B \subset \mathcal{S}(\mathbb{C})$ of 0 such that $B(\mathbb{R}) = B \cap \mathcal{S}(\mathbb{R})$ is connected. Let $\phi : C \to B$ be the induced family of Riemann surfaces over B, with anti-holomorphic involutions $(\tau \colon C \to C, \sigma \colon B \to B)$ that commute with ϕ. By [SS89], there exists a Teichmüller structure $[f_0]$, $f_0 : C_0 \xrightarrow{\sim} \Sigma$ and a unique $j \in \mathscr{J}(g)$ such that the composition

$$\Sigma \xrightarrow{f_0^{-1}} C_0 \xrightarrow{\tau} C_0 \xrightarrow{f_0} \Sigma$$

is isotopic to the map $\sigma_j : \Sigma \to \Sigma$ – in other words, such that $[f_0 \circ \tau \circ f_0^{-1}] = [\sigma_j]$.

Since the Kodaira-Spencer morphism $\rho : T_0 B \to \mathrm{H}^1(C_0, T_{C_0})$ is an isomorphism, the family $\phi : C \to B$ is a Kuranishi family for the fiber C_0. The family $\phi : C \to B$

is topologically trivial because B is contractible, thus ϕ can be endowed with a unique Teichmüller structure

$$\{[f_t], f_t : C_t \to \Sigma\}_{t \in B}$$

extending the Teichmüller structure $[f_0]$ on C_0.

For $t \in B$, consider the anti-holomorphic map $\tau : C_t \to C_{\sigma(t)}$. We claim that the two Teichmüller structures $[f_t]$ and $[\sigma_j \circ f_{\sigma(t)} \circ \tau]$ on C_t agree. Indeed, the family

$$\left\{[\sigma_j \circ f_{\sigma(t)} \circ \tau]\right\}_{t \in B}$$

defines a new Teichmüller structure on $\phi : C \to B$, agreeing with $\{[f_t]\}_{t \in B}$ at the point $0 \in B$, therefore agreeing with $\{[f_t]\}_{t \in B}$ everywhere on B [ACG11, XV, §2].

Consequently, the holomorphic map $P : B \to \mathcal{T}_g$ defined as $P(t) = [C_t, [f_t]]$ can be shown to be G-equivariant as follows. The involution $\sigma_j : \mathcal{T}_g \to \mathcal{T}_g$ is defined by sending the class $[C, [f]]$ of a curve C with Teichmüller structure $[f]$ to

$$[C^\sigma, [C^\sigma \xrightarrow{\text{can}} C \xrightarrow{f} \Sigma \xrightarrow{\sigma_j} \Sigma]] \in \mathcal{T}_g,$$

where C^σ is the complex conjugate curve of C and $\text{can} : C^\sigma \to C$ is the canonical anti-holomorphic map (see [Sil89, Chapter I, §1] for these notions). The family $\phi^\sigma : C^\sigma \to B^\sigma$ is isomorphic to the family $\phi : C \to B$ via the maps $\sigma \circ \text{can} : B^\sigma \to B$ and $\tau \circ \text{can} : C^\sigma \to C$, and the composition $\tau \circ \text{can} : C^\sigma \cong C$ restricts to an isomorphism $C_t^\sigma \cong C_{\sigma(t)}$ for each $t \in B$. For $t \in B$, we obtain

$$\sigma_j(P(t)) = \sigma_j([C_t, [f_t]]) = [C_{\sigma(t)}, [C_{\sigma(t)} \xrightarrow{\tau} C_t \xrightarrow{f_t} \Sigma \xrightarrow{\sigma_j} \Sigma]] = [C_{\sigma(t)}, [f_{\sigma(t)}]] = P(\sigma(t)).$$

We conclude that $\sigma_j \circ P = P \circ \sigma$ as desired.

Let $P^G \colon B(\mathbb{R}) \to \mathcal{T}_g^{\sigma_j}$ be the induced morphism on real loci. Since P induces isomorphisms on tangent spaces, the same holds for P^G. With respect to the projection $\pi : \mathcal{T}_g^{\sigma_j} \to N_j \setminus \mathcal{T}_g^{\sigma_j}$, one has

$$\pi \circ P^G = \mathscr{P}|_{B(\mathbb{R})} \colon B(\mathbb{R}) \to N_j \setminus \mathcal{T}_g^{\sigma_j}.$$

Therefore, $\mathscr{P}$ is continuous and open around the point $0 \in \mathcal{S}(\mathbb{R})$. $\square$

3

NOETHER-LEFSCHETZ LOCI FOR REAL ABELIAN VARIETIES

3.1 INTRODUCTION

In the previous Chapter 2, we saw how one can obtain a real moduli space from a real algebraic moduli stack. This generalized the classical approaches to construct the real moduli spaces of abelian varieties and curves. In this chapter, we will continue to consider families of real abelian varieties $\mathcal{A} \to B$. This time, we will look at the distribution of certain Noether-Lefschetz loci in $B(\mathbb{R})$. More precisely, we consider the locus R_k consisting of $t \in B(\mathbb{R})$ such that the fiber A_t contains a real abelian subvariety of fixed dimension. We prove that – even though such a set R_k often has empty interior – in favourable situations, R_k is dense in $B(\mathbb{R})$.

The goal of Chapter 3 will be to prove two results (Theorems 3.2 and 3.3). Let us set up the context. Fix an integer $g \geq 1$. Let $\mathcal{A}$ and B be complex manifolds, and let

$$\left(\psi : \mathcal{A} \to B, \quad s : B \to \mathcal{A}, \quad E \in R^2\psi_*\mathbb{Z}\right) \tag{3.1}$$

be a polarized holomorphic family of g-dimensional complex abelian varieties. The map ψ is a proper holomorphic submersion, and s is a holomorphic section of ψ. Moreover, and $A_t := \psi^{-1}(t)$ is a complex abelian variety of dimension g with origin $s(t)$, polarized by $E_t \in \mathrm{H}^2(A_t, \mathbb{Z})$, for $t \in B$. Assume that ψ admits a real structure in the following sense: $\mathcal{A}$ and B are equipped with anti-holomorphic involutions τ and σ, commuting with ψ and s and compatible with the polarization, in the sense $\tau^*(E) = -E$. For example, this is true when ψ is real algebraic, i.e. induced by a polarized abelian scheme over a smooth $\mathbb{R}$-scheme.

Let $B(\mathbb{R})$ be the set of fixed points under the involution $\sigma : B \to B$. If $t \in B(\mathbb{R})$, then A_t is equipped with an anti-holomorphic involution τ preserving the group law. We shall not distinguish between the category of abelian varieties over $\mathbb{R}$

and the category of complex abelian varieties equipped with an anti-holomorphic involution preserving the group law. Thus, if $t \in B(\mathbb{R})$, then A_t is an abelian variety over $\mathbb{R}$. Define a set $R_k \subset B(\mathbb{R})$ as follows:

$$R_k = \{t \in B(\mathbb{R}) : A_t \textit{ contains an abelian subvariety over } \mathbb{R} \textit{ of dimension } k\}. \tag{3.2}$$

For $t \in B$, the polarization gives an isomorphism $H^{0,1}(A_t) \cong \mathrm{H}^{1,0}(A_t)^\vee$; using the dual of the differential of the period map we obtain a symmetric bilinear form

$$q : \mathrm{H}^{1,0}(A_t) \otimes \mathrm{H}^{1,0}(A_t) \to (T_tB)^\vee. \tag{3.3}$$

Condition 3.1. *There exists an element $t \in B$ and a k-dimensional complex subspace $W \subset \mathrm{H}^{1,0}(A_t)$ such that the complex $0 \to \bigwedge^2 W \to W \otimes \mathrm{H}^{1,0}(A_t) \to (T_tB)^\vee$ is exact.*

The first main theorem of Chapter 3 is the following.

Theorem 3.2. *If B is connected and if Condition 3.1 holds, then R_k is dense in $B(\mathbb{R})$.*

The second main theorem of Chapter 3 is a consequence of Theorem 3.2. It consists of the following three applications of Theorem 3.2. See Definition 2.13 for the definition of *real algebraic curve*.

Theorem 3.3. ***A.*** *Given a positive integer $k < g$, moduli points of abelian varieties over $\mathbb{R}$ containing a k-dimensional abelian subvariety over $\mathbb{R}$ are dense in the moduli space $|\mathcal{A}_g(\mathbb{R})|$ of principally polarized abelian varieties of dimension g over $\mathbb{R}$.*

B. *For $1 \leq k \leq 3 \leq g$, moduli of real algebraic curves C admitting a map $\varphi : C \to A$ to a k-dimensional abelian variety A over $\mathbb{R}$ such that $\varphi(C(\mathbb{C}))$ generates the group $A(\mathbb{C})$ are dense in the moduli space $|\mathcal{M}_g(\mathbb{R})|$ of real algebraic curves of genus g.*

C. *Let $V \subset \mathbb{P}\mathrm{H}^0(\mathbb{P}^2_{\mathbb{R}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(d))$ be the moduli space of smooth degree $d \geq 3$ real plane curves. Let $R_1(V)$ be the set of $t \in V$ such that the corresponding curve C_t admits a non-constant map $C_t \to E$ to a real elliptic curve E. Then $R_1(V)$ is dense in V.*

Remarks 3.4.

1. The topology on the moduli space $|\mathcal{A}_g(\mathbb{R})|$ in Theorem 3.3.A (resp. $|\mathcal{M}_g(\mathbb{R})|$ in Theorem 3.3.B) is the real-analytic topology, see Definition 2.7. It coincides with the topology constructed by Gross–Harris [GH81] (resp. Seppälä–Silhol [SS89]), see Theorem 2.15 (resp. Theorem 2.18).
2. Although well-known in the complex case, Theorem 3.3.A is new in the real case.

Our proofs rely on results in the complex setting proved by Colombo and Pirola in [CP90]. Indeed, Theorem 3.2 is the analogue over $\mathbb{R}$ (with *unchanged* hypothesis) of the following theorem. Define $S_k \subset B$ to be the set of those t in B for which the complex abelian variety A_t contains a complex abelian subvariety of dimension k.

Theorem 3.5 (Colombo-Pirola [CP90])**.** *If B is connected and if Condition 3.1 holds, then S_k is dense in B.* □

Colombo and Pirola in turn were inspired by the Green-Voisin Density Criterion [Voi02, Proposition 17.20]. Indeed, the latter gives a criterion for density of the locus where the fiber contains many Hodge classes for a variation of Hodge structure of weight 2. Theorem 3.5 adapts this result to a polarized variation of Hodge structure of weight 1 (which is nothing but a polarized family of complex abelian varieties). The result is a criterion for the density of the locus where the fiber admits a sub-Hodge structure of rank k.

To be a bit more precise, recall that for a complex manifold U and a rational weight 2 variation of Hodge structure $(H^2_{\mathbb{Q}}, \mathcal{H}, F^1, \nabla)$ on U, the *Noether-Lefschetz locus* $\mathrm{NL}(U) \subset U$ is the locus where the rank of the vector space of Hodge classes is bigger than the general value. If $H^2_{\mathbb{Q}}$ is polarizable then $\mathrm{NL}(U)$ is a countable union of closed algebraic subvarieties of U [CDK95]. The Green-Voisin Density Criterion referred to above decides whether $\mathrm{NL}(U)$ is dense in U. It was first stated in [CHM88] and applied to the universal degree $d \geq 4$ surface $\mathscr{S} \to \mathscr{B}$ in $\mathbb{P}^3$. In this case, $\mathrm{NL}(\mathscr{B})$ is the locus where the Picard group is not generated by a hyperplane section, and the union of the general components of $\mathrm{NL}(\mathscr{B})$ is dense in $\mathscr{B}$ [*loc. cit.*]. Analogously, S_k is a countable union of components of $\mathrm{NL}(B)$ [DL90], and Condition 3.1 implies that this union is dense in B.

Let us carry the discussion over to the real setting. Unfortunately, the Green-Voisin Density Criterion cannot be adapted to the reals without altering the hypothesis. Going back to the universal family $\mathscr{S} \to \mathscr{B}$ of degree $d \geq 4$ surfaces in $\mathbb{P}^3_{\mathbb{C}}$, one observes that this family has a real structure, so that we can define the *real Noether-Lefschetz locus* $\mathrm{NL}(\mathscr{B}(\mathbb{R})) \subset \mathscr{B}(\mathbb{R})$ as the locus of real surfaces S in $\mathbb{P}^3_{\mathbb{R}}$ with $\mathrm{Pic}(S) \neq \mathbb{Z}$. By the above, the Green-Voisin Density Criterion is fulfilled hence $\mathrm{NL}(\mathscr{B})$ is dense in $\mathscr{B}$, whereas density of $\mathrm{NL}(\mathscr{B}(\mathbb{R}))$ in $\mathscr{B}(\mathbb{R})$ may fail: for every degree 4 surface in $\mathbb{P}^3_{\mathbb{R}}$ whose real locus is a union of 10 spheres, $\mathrm{Pic}(S) = \mathbb{Z}$, and so $\mathrm{NL}(\mathscr{B}(\mathbb{R})) \cap K = \emptyset$ for any connected component K of surfaces of such a topological type [Ben18, Remark 1.5]. There is an alternate criterion [Ben18, Proposition 1.1], but the hypothesis is more complicated thus harder to fulfill, and only implies density of $\mathrm{NL}(\mathscr{B}(\mathbb{R}))$ in one component of $\mathscr{B}(\mathbb{R})$ at a time. It is therefore remarkable that for the real analogue of density of S_k in B, none of these problems occur. Theorem 3.2 shows that the complex density criterion can be carried over to the reals *without changing it*. Condition 3.1 does not involve the real structures at all, applying to any real structure on the family. The result is density of $S_k \subset B$ *and* $R_k \subset B(\mathbb{R})$. It is for this reason that the applications of Theorem 3.2 are generous: the statements in Theorem 3.3, as well as their proofs, are direct analogues of some applications of Theorem 3.5 in [CP90].

Finally, we remark that Colombo-Pirola's Theorem 3.5 has been generalized by Ching-Li Chai [Cha98], who considers a variation of rational Hodge structures over a complex analytic variety and rephrases and answers the following question in the context of Shimura varieties: when do the points corresponding to members having extra Hodge cycles of a given type form a dense subset of the base? It should be interesting to investigate whether such a generalization can be carried over to the real numbers as well.

3.2 ABELIAN SUBVARIETIES IN A FAMILY

In this section, we shall prove Theorem 3.2.

Let

$$(\psi : \mathcal{A} \to B, \quad s : B \to \mathcal{A}, \quad E \in R^2\psi_*\mathbb{Z})$$

be as in Section 3.1. Let $\mathbb{V}_{\mathbb{Z}} = R^1\psi_*\mathbb{Z}$ be the $\mathbb{Z}$-local system attached to ψ. The holomorphic vector bundle $\mathcal{H} = \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_B$ is endowed with a filtration by the holomorphic subbundle $F^1\mathcal{H} = \mathcal{H}^{1,0} \subset \mathcal{H}$, of fiber

$$(\mathcal{H}^{1,0})_t = \mathrm{H}^{1,0}(A_t) \subset \mathrm{H}^1(A_t, \mathbb{C}) = \mathcal{H}_t.$$

Denote by $\mathcal{H}_{\mathbb{R}}$ the real C^∞-subbundle of $\mathcal{H}$ whose fibers are $(\mathcal{H}_{\mathbb{R}})_t = \mathrm{H}^1(A_t, \mathbb{R})$. Define $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Recall that $\mathcal{A}$ and B are endowed with anti-holomorphic involutions τ and σ such that $\psi \circ \tau = \sigma \circ \psi$. The map τ induces, for each $t \in B$, an anti-holomorphic isomorphism $\tau : A_t \cong A_{\sigma(t)}$. The pullback of τ gives an isomorphism $\tau^* : \mathcal{H}_{\sigma(t)} \to \mathcal{H}_t$ inducing an involution of differentiable manifolds

$$F_\infty : \mathcal{H} \to \mathcal{H}$$

over the involution $\sigma : B \to B$. Composing F_∞ fiberwise with complex conjugation provides an involution of differentiable bundles

$$F_{dR} : \mathcal{H} \to \mathcal{H}$$

over $\sigma\colon B \to B$ which respects the Hodge decomposition by [Sil89, I, Lemma 2.4].

Let $\mathscr{G}(k, \mathcal{H}^{1,0})$ be the complex Grassmannian bundle of complex k-planes in $\mathcal{H}^{1,0}$ over B, and $\mathscr{G}(2k, \mathcal{H}_{\mathbb{R}})$ the real Grassmannian bundle of real $2k$-planes in $\mathcal{H}_{\mathbb{R}}$ over B. We see that G acts on these bundles via the morphisms $F_{dR} : \mathcal{H}^{1,0} \to \mathcal{H}^{1,0}$ and $\tau^* : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$. Since the diffeomorphism $\mathcal{H}_{\mathbb{R}} \to \mathcal{H}^{1,0}$, defined as the composition of morphisms

$$\mathcal{H}_{\mathbb{R}} \hookrightarrow \mathcal{H} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1} \to \mathcal{H}^{1,0},$$

is G-equivariant, it induces a G-equivariant morphism of differentiable manifolds

$$\mathscr{G}(k, \mathcal{H}^{1,0}) \to \mathscr{G}(2k, \mathcal{H}_{\mathbb{R}}).$$

Let $0 \in B(\mathbb{R})$ and choose a G-stable contractible neighbourhood U of 0 in B. Trivialize $\mathbb{V}_{\mathbb{Z}}$ over U, which trivializes $\mathscr{G}(2k, \mathcal{H}_{\mathbb{R}})$ over U, and consider the morphism Φ defined as the composition

$$\begin{aligned} \mathscr{G}(k, \mathcal{H}^{1,0})|_U &\to \mathscr{G}(2k, \mathcal{H}_{\mathbb{R}})|_U \\ &\cong U \times \mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R})) \to \mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R})). \end{aligned}$$

Let j be the canonical map $\mathrm{Grass}_{\mathbb{Q}}(2k, \mathrm{H}^1(A_0, \mathbb{Q})) \to \mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R}))$. We obtain the following diagram:

$$\begin{tikzcd} \mathscr{G}(k, \mathcal{H}^{1,0})|_U \arrow[r, "\Phi"] \arrow[d, "f"] & \mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R})) \\ U & \mathrm{Grass}_{\mathbb{Q}}(2k, \mathrm{H}^1(A_0, \mathbb{Q})). \arrow[u, "j"] \end{tikzcd} \tag{3.4}$$

Write $U(\mathbb{R}) = U \cap B(\mathbb{R})$. Diagram (3.4) provides a parametrization of polarized real abelian varieties containing a k-dimensional real abelian subvariety:

Proposition 3.6. *1. The morphisms f, Φ and j in diagram (3.4) are G-equivariant.*

2. We have

$$f(\Phi^{-1}(j(\mathrm{Grass}_{\mathbb{Q}}(2k, \mathrm{H}^1(A_0, \mathbb{Q}))^G))) = R_k \cap U(\mathbb{R}),$$

where $R_k \subset B(\mathbb{R})$ is the set defined in (3.2).

Proof. 1. The fact that f and j are G-equivariant is immediate from the description of F_{dR}. As for the morphism Φ, it suffices to show that the trivialization

$$\mathcal{H}_{\mathbb{R}} \cong U \times \mathrm{H}^1(A_0, \mathbb{R})$$

is G-equivariant. This map is induced by the restriction $r : \mathbb{V}_{\mathbb{R}}|_U \to (\mathbb{V}_{\mathbb{R}})_0$, which is an isomorphism of local systems; but r is unique if we require that r induces the identity on $(\mathbb{V}_{\mathbb{R}})_0$.

2. Let $t \in U(\mathbb{R})$ and consider the polarized real abelian variety (A_t, τ). We have $A_t \cong V/\Lambda$, where $V \cong \mathrm{H}^{1,0}(A_t)^*$ and $\Lambda \cong \mathrm{H}_1(A_t, \mathbb{Z})$. It follows that A_t contains a complex abelian subvariety X of dimension k if and only if there exists a k-dimensional $\mathbb{C}$-vector subspace $W_1 \subset \mathrm{H}^{1,0}(A_t)$ and a k-dimensional $\mathbb{Q}$-vector subspace $W_2 \subset \mathrm{H}^1(A_t, \mathbb{Q})$ such that, under the canonical real isomorphism $\mathrm{H}^{1,0}(A_t) \cong \mathrm{H}^1(A_t, \mathbb{R})$, the space W_1 is identified with $W_2 \otimes \mathbb{R}$. In this case, $L = W_2^* \cap \Lambda$ is a lattice in W_1^*, and $X = W_1^*/L$. Then observe that the k-dimensional complex abelian subvariety $X \subset A_t$ is a k-dimensional real abelian subvariety $(X, \tau|_X)$ of (A_t, τ) if and only if $\tau(X) = \tau(W_1^*/L) = W_1^*/L = X$. The latter is equivalent to $F_{dR}(W_2) = W_2$ by Lemma 3.7 below, and we are done. □

Lemma 3.7. *Let A be a complex torus, let $\Lambda = \mathrm{H}_1(A, \mathbb{Z})$ and consider the Hodge decomposition $\Lambda_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$. For an anti-holomorphic involution $\sigma : A \to A$ such that $\sigma(e) = e$, let $F_\infty(\sigma) : \Lambda \to \Lambda$ be the pushforward of σ, and let $F_{dR}(\sigma) : V^{-1,0} \to V^{-1,0}$ correspond to the differential $d\sigma_e : T_eA \to T_eA$. This defines a bijection between:*

(i) The set of real structures $\sigma : A \to A$.

(ii) The set of involutions $F_\infty : \Lambda \to \Lambda$ such that $F_{\infty,\mathbb{C}}\left(V^{-1,0}\right) = V^{0,-1}$.

(iii) The set of anti-linear involutions $F_{dR} : V^{-1,0} \to V^{-1,0}$ such that $F_{dR}(\Lambda) = \Lambda$.

Proof. If $\sigma : A \to A$ is as in (i), then $F_\infty(\sigma)_\mathbb{C}$ interchanges the factors of the Hodge decomposition by [Sil89, Chapter I, Lemma (2.4)], so $\sigma \mapsto F_\infty(\sigma)$ is a well-defined map $(i) \to (ii)$. For $F_\infty : \Lambda \to \Lambda$ as in (ii), the restriction to $V^{-1,0}$ of the composition of $F_{\infty,\mathbb{C}}$ with complex conjugation defines a map $F_{dR} = \mathrm{conj} \circ F_{\infty,\mathbb{C}} : V^{-1,0} \to V^{-1,0}$ as in (iii), which equals $F_{dR}(\sigma)$ when $F_\infty = F_\infty(\sigma)$. Then $F_\infty \mapsto \mathrm{conj} \circ F_{\infty,\mathbb{C}}$ determines the bijection between (ii) and (iii). Finally, the map $\sigma \mapsto F_{dR}(\sigma)$, $(i) \to (iii)$ is the map giving the natural correspondence between anti-holomorphic involutions on A preserving e and anti-holomorphic involutions on its universal cover T_eA that preserve 0 and are compatible with $\pi_1(A, e) = \mathrm{H}_1(A, \mathbb{Z})$-orbits. $\square$

3.2.1 *Proving the density theorem*

For a smooth manifold W on which a compact Lie group H acts by diffeomorphisms, the set of fixed points W^H has a natural manifold structure that makes it a submanifold of W [Aud04, Corollary I.2.3]. Φ in Diagram (3.4) is G-equivariant by Proposition 3.6.1; denote by Φ^G the induced morphism on fixed spaces. We obtain a diagram of $\mathcal{C}^\infty$-manifolds:

$$\begin{tikzcd} \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})} \arrow[r, "\Phi^G"] \arrow[d, "f"] & \mathrm{Grass}_\mathbb{R}(2k, \mathrm{H}^1(A_0, \mathbb{R}))^G \\ U(\mathbb{R}) & \mathrm{Grass}_\mathbb{Q}(2k, \mathrm{H}^1(A_0, \mathbb{Q}))^G. \arrow[u, "j"] \end{tikzcd} \tag{3.5}$$

Recall that we obtained the equality

$$f(\Phi^{-1}(j(\mathrm{Grass}_\mathbb{Q}(2k, \mathrm{H}^1(A_0, \mathbb{Q}))^G))) = R_k \cap U(\mathbb{R})$$

in Proposition 3.6.2. Recall also the symmetric bilinear form

$$q : \mathrm{H}^{1,0}(A_t) \otimes \mathrm{H}^{1,0}(A_t) \to (T_tU)^\vee$$

from Section 3.1, given by the differential of the period map and the isomorphism $H^{0,1}(A_t) \cong \mathrm{H}^{1,0}(A_t)^\vee$ which the polarization induces. Finally, recall the notation (Equation (3.2), §3.1)

$$R_k = \{t \in B(\mathbb{R}) : A_t \textit{ contains an abelian subvariety over } \mathbb{R} \textit{ of dimension } k\}.$$

Let us fix notation and introduce the aim of this section.

Notation 3.8. For $t \in B$ and $W \in \mathrm{Grass}_{\mathbb{C}}(k, \mathrm{H}^{1,0}(A_t))$, let $W^{\perp}$ denote the orthogonal complement of W in $\mathrm{H}^{1,0}(A_t)$ with respect to the polarization E_t. Define the sets $\mathscr{E}_k, \mathscr{F}_k \subset \mathscr{G}(k, \mathcal{H}^{1,0})$ and $\mathscr{E}_k(\mathbb{R}), \mathscr{F}_k(\mathbb{R}), \mathscr{R}_{k,U} \subset \mathscr{G}(k, \mathcal{H}^{1,0})^G$ as follows:

$$\mathscr{E}_k = \{(t, W) \in \mathscr{G}(k, \mathcal{H}^{1,0}) : 0 \to W \otimes W^{\perp} \to (T_t B)^{\vee} \text{ is exact}\}$$

$$\mathscr{F}_k = \{(t, W) \in \mathscr{G}(k, \mathcal{H}^{1,0}) : 0 \to \wedge^2 W \to W \otimes \mathrm{H}^{1,0}(A_t) \to (T_t B)^{\vee} \text{ is exact}\}$$

$$\mathscr{E}_k(\mathbb{R}) = \mathscr{E}_k \cap \mathscr{G}(k, \mathcal{H}^{1,0})^G$$

$$\mathscr{F}_k(\mathbb{R}) = \mathscr{F}_k \cap \mathscr{G}(k, \mathcal{H}^{1,0})^G$$

$$\mathscr{R}_{k,U} = \Phi^{-1}(j(\mathrm{Grass}_{\mathbb{Q}}(2k, \mathrm{H}^1(A_0, \mathbb{Q}))^G)).$$

Note that $f(\mathscr{R}_{k,U}) = R_k \cap U(\mathbb{R})$. Our strategy to prove Theorem 3.2 is the following:

Proposition 3.9. *Let $\mathscr{E}_k, \mathscr{F}_k, \mathscr{R}_{k,U} \subset \mathscr{G}(k, \mathcal{H}^{1,0})$ be as above. One has inclusions*

$$\mathscr{F}_k(\mathbb{R})|_{U(\mathbb{R})} \hookrightarrow \mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})} \hookrightarrow \overline{\mathscr{R}_{k,U}} \hookrightarrow \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})}.$$

If $\mathscr{F}_k$ is not empty then $\mathscr{F}_k(\mathbb{R})$ is dense in $\mathscr{G}(k, \mathcal{H}^{1,0})^G$. Consequently, if $\mathscr{F}_k \neq \emptyset$, then $\mathscr{R}_{k,U}$ is dense in $\mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})}$; in particular, then $R_k \cap U(\mathbb{R})$ is dense in $U(\mathbb{R})$.

Before proving Proposition 3.9 we remark that it implies Theorem 3.2.

Proof of Theorem 3.2. It suffices to show that for each $x \in B(\mathbb{R})$ and any G-stable contractible open neighborhood $x \in U \subset B$, the set $R_k \cap U \cap B(\mathbb{R})$ is dense in $U \cap B(\mathbb{R})$. Let $x \in U \subset B$ be such a real point and open neighborhood. Condition 3.1 is satisfied if and only if $\mathscr{F}_k$ is non-empty; by Proposition 3.9, we are done. □

Proof of Proposition 3.9. We first prove the inclusions. We have $\mathscr{F}_k \subset \mathscr{E}_k$: if $(t, W) \in \mathscr{F}_k$ then $\mathrm{Ker}(W \otimes \mathrm{H}^{1,0}(A_t) \to (T_t B)^{\vee}) \subset W \otimes W$. Hence $\mathscr{F}_k(\mathbb{R})|_{U(\mathbb{R})} \subset \mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}$.

For the second inclusion, consider again the map

$$\Phi : \mathscr{G}(k, \mathcal{H}^{1,0})|_U \to \mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R}))$$

as in diagram (3.4). Define a set $Y \subset \mathscr{G}(k, \mathcal{H}^{1,0})$ as follows:

$$Y = \{x \in \mathscr{G}(k, \mathcal{H}^{1,0})|_U \mid \textit{ the rank of } d\Phi \textit{ is maximal at } x\} \subset \mathscr{G}(k, \mathcal{H}^{1,0})|_U.$$

Then $Y = \mathscr{E}_k|_U$ by [CP90, §1]. Let Z be the set of points $x \in \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})}$ such that the differential

$$d(\Phi^G) : T_x\mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})} \to T_{\Phi(x)}\mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R}))^G$$

of Φ^G at x has maximal rank. We claim that

$$Z = Y \cap \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})}.$$

Indeed, for a point $x \in \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})}$, the rank of $d(\Phi^G)$ is maximal at x if and only if the rank of $d\Phi$ is maximal at x. The latter is true because

$$T_x\mathscr{G}(k, \mathcal{H}^{1,0}) = (T_x\mathscr{G}(k, \mathcal{H}^{1,0}))^G \otimes_{\mathbb{R}} \mathbb{C} = T_x\mathscr{G}(k, \mathcal{H}^{1,0})^G \otimes_{\mathbb{R}} \mathbb{C},$$

and similarly for $T_{\Phi(x)}\mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R}))$, and because the differential $d\Phi$ at the point $x \in \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})}$ is the complexification of the differential $d(\Phi^G)$ at x.

Consequently,

$$Z = Y \cap \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})} = \mathscr{E}_k \cap \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})} = \mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}.$$

It follows that the morphism

$$\Phi^G : \mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})} \to \mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R}))^G$$

is open when restricted to the set $\mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}$. Since $j(\mathrm{Grass}_{\mathbb{Q}}(2k, \mathrm{H}^1(A_0, \mathbb{Q}))^G)$ is dense in $\mathrm{Grass}_{\mathbb{R}}(2k, \mathrm{H}^1(A_0, \mathbb{R}))^G$ by Lemma 3.10 below, the set

$$\Phi^{-1}(j(\mathrm{Grass}_{\mathbb{Q}}(2k, \mathrm{H}^1(A_0, \mathbb{Q}))^G)) \cap \mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}$$

is dense in $\mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}$. The equality

$$\Phi^{-1}(j(\mathrm{Grass}_{\mathbb{Q}}(2k, \mathrm{H}^1(A_0, \mathbb{Q}))^G)) \cap \mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})} = \mathscr{R}_{k,U} \cap \mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}$$

implies that $\mathscr{R}_{k,U} \cap \mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}$ is dense in $\mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}$. In particular, $\mathscr{E}_k(\mathbb{R})|_{U(\mathbb{R})}$ is contained in the closure of $\mathscr{R}_{k,U}$ in $\mathscr{G}(k, \mathcal{H}^{1,0})^G|_{U(\mathbb{R})}$. The inclusions are thus proved.

It remains to prove the assertion that non-emptiness of $\mathscr{F}_k$ implies density of $\mathscr{F}_k(\mathbb{R})$ in $\mathscr{G}(k, \mathcal{H}^{1,0})^G$. Write $\mathscr{G} = \mathscr{G}(k, \mathcal{H}^{1,0})$ and $\mathscr{G}(\mathbb{R}) = \mathscr{G}^G$. Let $Z_k \subset \mathscr{G}$ be the complement of $\mathscr{F}_k$ in $\mathscr{G}$. Observe that Z_k equals the set of pairs $(t, W) \in \mathscr{G}$ such that the injection

$$\iota \colon \bigwedge^2 W \to \operatorname{Ker}(W \otimes \mathrm{H}^{1,0}(A_t) \to (T_t B)^\vee)$$

is not an isomorphism. This latter map ι varies holomorphically; therefore, locally on $\mathscr{G}$, the set of points where the rank is not maximal is determined by the vanishing of minors in a matrix with holomorphic coefficients. Since $\mathscr{G}$ is connected, it follows that Z_k is nowhere dense in $\mathscr{G}$.

If $n = \dim_{\mathbb{C}}(\mathscr{G})$, then $\mathscr{G}(\mathbb{R})$ is a closed real submanifold of $\mathscr{G}$ of real dimension n, and we claim that $Z_k(\mathbb{R})$ is nowhere dense in $\mathscr{G}(\mathbb{R})$. Suppose for contradiction that $Z_k(\mathbb{R})$ contains a non-empty open set $V \subset \mathscr{G}(\mathbb{R})$. Locally around $t \in V$, $t \in V \subset \mathscr{G}(\mathbb{R})$ is the inclusion of the real locus $0 \in B(\mathbb{R}) \subset \mathbb{R}^n$ of a euclidean open ball $0 \in B \subset \mathbb{C}^n$, such that, for some holomorphic functions $f_i : B \to \mathbb{C}$, one has

$$B \cap Z_k = \{f_1 = \ldots = f_m = 0\}.$$

Since the f_i vanish on $B(\mathbb{R})$, they vanish on B, hence $B \subset Z_k$ which is absurd. $\square$

Lemma 3.10. *Let $n, k \in \mathbb{Z}_{\geq 0}$. Let V be an n-dimensional vector space over $\mathbb{Q}$ and consider a linear transformation $F \in \mathrm{GL}(V)_{\mathbb{Q}}$ which is diagonalizable over $\mathbb{Q}$. For $L \in \{\mathbb{Q}, \mathbb{R}\}$, denote by $\mathbb{G}(k, V)^F(L)$ the set of k-dimensional L-subvector spaces $W \subset V \otimes_{\mathbb{Q}} L$ for which $F(W) = W$. Then $\mathbb{G}(k, V)^F(\mathbb{Q})$ is dense in $\mathbb{G}(k, V)^F(\mathbb{R})$.*

Proof. For $F = \mathrm{id}$ this is an elementary fact. In order to deduce the general case from this, let $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}^*$ be the eigenvalues of F, and denote by

$$V_i := V^{F = \lambda_i} \subset V, \quad i \in I := \{1, \ldots, r\}$$

the corresponding eigenspaces. Eigenspaces are preserved under scalar extension, so $(V \otimes_{\mathbb{Q}} \mathbb{R})^{F_{\mathbb{R}} = \lambda_i} = V_i \otimes_{\mathbb{Q}} \mathbb{R}$. But a k-dimensional $\mathbb{R}$-subvector space $W \subset V_{\mathbb{R}}$ satisfies $F(W) = W$ if and only if $W = \bigoplus_{i \in I} W_i$ with W_i a $\mathbb{R}$-subvector space of $V_i \otimes_{\mathbb{Q}} \mathbb{R}$ for each $i \in I$ and $\sum_{i \in I} \dim W_i = k$, and W is defined over $\mathbb{Q}$ if and only if each W_i is. This means that, under the canonical diffeomorphism

$$\mathbb{G}(k, V)^F(\mathbb{R}) \cong \bigsqcup_{\sum k_i = k} \prod_{i \in I} \mathbb{G}(k_i, V_i)(\mathbb{R}), \tag{3.6}$$

the rational subspace $\mathbb{G}(k,V)^F(\mathbb{Q})$ is identified with $\bigsqcup_{k_i}\prod_{i\in I}\mathbb{G}(k_i,V_i)(\mathbb{Q})$. $\square$

3.3 REAL DEFORMATION SPACES

3.3.1 *Density in the complex deformation space*

Let ψ be a holomorphic family $\psi : \mathcal{A} \to B$ of complex abelian varieties of dimension g, polarized by a section $E \in R^2\psi_*\mathbb{Z}$. For $t \in B$, denote by $\omega_t \in \mathrm{H}^1(A_t, \Omega^1_{A_t})$ the Kähler class corresponding to the polarization $E_t \in \mathrm{H}^2(A_t,\mathbb{Z})$. Let $t \in B$ and define $X = \psi^{-1}(t)$. Suppose that ψ is a universal local deformation of (X,ω_t); in other words, that the Kodaira-Spencer map

$$\rho : T_tB \to \mathrm{H}^1(X,T_X)_\omega \tag{3.7}$$

is an *isomorphism*. Here, $\mathrm{H}^1(X,T_X)_\omega$ denotes the kernel of the composition

$$\mathrm{H}^1(X,T_X) \to \mathrm{H}^2(X,T_X\otimes\Omega^1_X) \to \mathrm{H}^2(X,\mathcal{O}_X)$$

of cup-product with ω_t with the map on cohomology induced by $T_X\otimes\Omega^1_X \to \mathcal{O}_X$.

Proposition 3.11. *Condition 3.1 is satisfied for the point $t \in B$ and any element*

$$W \in \mathrm{Grass}_{\mathbb{C}}(k, \mathrm{H}^{1,0}(A_t)).$$

Proof. By a theorem of Griffiths [Voi02, Théorème 17.7], the dual $q^\vee$ of the bilinear form

$$q : \mathrm{H}^1(X,\mathcal{O}_X)^\vee \otimes \mathrm{H}^1(X,\mathcal{O}_X)^\vee = \mathrm{H}^{1,0}(X)\otimes H^{1,0}(X) \to (T_tB)^\vee$$

(see equation (3.3) in §3.1) factors as

$$T_tB \to \mathrm{H}^1(X,T_X) \to \mathrm{Hom}(\mathrm{H}^0(X,\Omega^1_X),\mathrm{H}^1(X,\mathcal{O}_X)) \xrightarrow{\sim} \mathrm{H}^1(X,\mathcal{O}_X)\otimes\mathrm{H}^1(X,\mathcal{O}_X).$$

The second arrow is an isomorphism. The third arrow is induced by the polarization. Remark that the following diagram commutes, and that the first row is exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(X, T_X)_\omega \arrow[r] & \mathrm{H}^1(X, T_X) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^2(X, \mathcal{O}_X) \arrow[r] & 0 \\ 0 \arrow[r] & T_t B \arrow[u, "\rho"] \arrow[r, "q^\vee"] & \mathrm{H}^1(X, \mathcal{O}_X) \otimes \mathrm{H}^1(X, \mathcal{O}_X) \arrow[r] & \bigwedge^2 \mathrm{H}^1(X, \mathcal{O}_X) \arrow[u, "\wr"] \arrow[r] & 0. \end{tikzcd} \tag{3.8}$$

We conclude that the Kodaira-Spencer map ρ in (3.7) is an isomorphism if and only if the second row in (3.8) is exact if and only $q^\vee$ induces an isomorphism

$$q^\vee : T_t B \xrightarrow{\sim} \mathrm{Sym}^2 \mathrm{H}^1(X, \mathcal{O}_X). \tag{3.9}$$

Identify $(T_t B)^\vee$ with $\mathrm{Sym}^2 \mathrm{H}^{1,0}(X)$ and q with $\mathrm{H}^{1,0}(X) \otimes \mathrm{H}^{1,0}(X) \to \mathrm{Sym}^2 \mathrm{H}^{1,0}(X)$, and observe that for any complex vector space V and any k-dimensional subspace $W \subset V$, the natural sequence

$$0 \to \bigwedge^2 W \to W \otimes V \to \mathrm{Sym}^2(V)$$

is exact. $\square$

3.3.2 *Real structures on deformation spaces*

The goal of this section is to prove that for a universal local deformation of a complex manifold X, any real structure on X extends uniquely to a real structure on the local deformation.

Let X be a compact complex manifold, possibly polarized by the first Chern class $\omega = c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$ of an ample line bundle $\mathcal{L}$, and let

$$\pi : \mathscr{X} \to B \ni 0$$

be a universal local deformation of X (resp. of (X, ω)), where B is a complex analytic space. Let $\sigma : X \to X$ be an anti-holomorphic involution, compatible with ω in case X is polarized. In the following proposition and its proof, only the germ of π in $0 \in B$ plays a role and all statements should be read in this sense.

Proposition 3.12 (compare [CF03], Section 4)**.** *The real structure $\sigma : X \to X$ extends uniquely to a real structure on the universal local deformation $\pi : \mathscr{X} \to B$. In other words, possibly after restricting $(B, 0)$, there is a unique couple of anti-holomorphic involutions $(\tau : B \to B, \mathscr{T} : \mathscr{X} \to \mathscr{X})$ such that $\pi \circ \mathscr{T} = \tau \circ \pi$, $\tau(0) = 0$ and $\mathscr{T}|_X = \sigma : X \to X$.*

Proof. Consider the complex conjugate analytic spaces X^σ, B^σ and $\mathscr{X}^\sigma$ of X, B and $\mathscr{X}$ respectively (see [Sil89, Chapter I, Definition 1.1] for the definition of complex conjugate analytic variety; the definition for general analytic spaces is similar). There is an induced local deformation

$$\pi^\sigma \colon \mathscr{X}^\sigma \to B^\sigma \ni 0$$

of the manifold $(\mathscr{X}^\sigma)_0 = (\mathscr{X}_0)^\sigma = X^\sigma$. This local deformation is universal.

The anti-holomorphic involution $\sigma : X \to X$ induces a biholomorphic function $\phi : X^\sigma \to X$ such that $\phi \circ \phi^\sigma = \mathrm{id} : X \to X$, hence the fibers of π and π^σ above 0 are isomorphic via ϕ. By the universal properties of π and π^σ, this means that, possibly after restricting B around 0, there is a unique pair of biholomorphisms $(f : B^\sigma \to B, g : \mathscr{X}^\sigma \to \mathscr{X})$ making the following diagram cartesian:

$$\begin{tikzcd} \mathscr{X}^\sigma \arrow[r, "g"] \arrow[d, "\pi^\sigma"] & \mathscr{X} \arrow[d, "\pi"] \\ B^\sigma \arrow[r, "f"] & B. \end{tikzcd} \tag{3.10}$$

Moreover, $f(0) = 0$ and $g|_{X^\sigma} : X^\sigma \to X$ is the map ϕ. Applying the functor $(-)^\sigma$ to this diagram, we obtain a pair of cartesian diagrams

$$\begin{tikzcd} \mathscr{X} \arrow[r, "g^\sigma"] \arrow[d, "\pi"] & \mathscr{X}^\sigma \arrow[r, "g"] \arrow[d, "\pi^\sigma"] & \mathscr{X} \arrow[d, "\pi"] \\ B \arrow[r, "f^\sigma"] & B^\sigma \arrow[r, "f"] & B \end{tikzcd} \tag{3.11}$$

such that $(f \circ f^\sigma)(0) = 0$, and such that the map

$$(g \circ g^\sigma)|_X \colon X \to X^\sigma \to X$$

equals $\phi \circ \phi^\sigma = \mathrm{id}$. We must have $f \circ f^\sigma = \mathrm{id}$ and $g \circ g^\sigma = \mathrm{id}$. By composing $f^\sigma : B \to B^\sigma$ (resp. $g^\sigma : \mathscr{X} \to \mathscr{X}^\sigma$) with the canonical anti-holomorphic map $B^\sigma \to B$ (resp. $\mathscr{X}^\sigma \to \mathscr{X}$), we obtain the desired involutions τ and $\mathscr{T}$. $\square$

3.4 ANALYTIC FAMILIES OF JACOBIANS

The goal of this section is to prove that a real structure on a family of curves extends canonically to a real structure on the corresponding family of Jacobians.

Let B be a simply connected complex manifold and let $\pi : \mathscr{X} \to B$ be a family of compact Riemann surfaces of genus g. The relative Jacobian

$$\psi \colon J_{\mathscr{X}} = \underline{\mathrm{Pic}}^0(\mathscr{X}/B) \to B \tag{3.12}$$

of π arises from following exact sequence of sheaves on $\mathscr{X}$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\mathscr{X}} \xrightarrow{\exp(2\pi i z)} \mathcal{O}^*_{\mathscr{X}} \longrightarrow 0. \tag{3.13}$$

Indeed, sequence (3.13) induces an inclusion $R^1\pi_*\mathbb{Z} \to R^1\pi_*\mathcal{O}_{\mathscr{X}}$, where we identify $R^1\pi_*\mathbb{Z}$ with its étalé space and $R^1\pi_*\mathcal{O}_{\mathscr{X}}$ with its corresponding holomorphic vector bundle; define $J_{\mathscr{X}} = R^1\pi_*\mathcal{O}_{\mathscr{X}}/R^1\pi_*\mathbb{Z}$.

We claim that family (3.12) is a polarized holomorphic family of g-dimensional complex abelian varieties. Indeed, $\psi : J_{\mathscr{X}} \to B$ admits a section $s : B \to J_{\mathscr{X}}$, corresponding to the line bundle $\mathcal{O}_{J_{\mathscr{X}}} \in \mathrm{Pic}^0(\mathscr{X})$ (recall that $J_{\mathscr{X}}$ represents the relative Picard functor of degree 0). For $0 \in B$, we have the Riemann form

$$E_0 = -\langle\ ,\ \rangle : \bigwedge^2 \mathrm{H}_1(\mathscr{X}_0, \mathbb{Z}) \to \mathbb{Z}.$$

The trivialization $R^2\pi_*\mathbb{Z} \cong \mathrm{H}^2(J_{\mathscr{X}_0}, \mathbb{Z})$ implies that

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(\wedge^2 \mathrm{H}_1(\mathscr{X}_0, \mathbb{Z}), \mathbb{Z}) &= \mathrm{Hom}_{\mathbb{Z}}(\wedge^2 \mathrm{H}_1(\mathrm{Jac}(\mathscr{X}_0), \mathbb{Z}), \mathbb{Z}) \\ &= \mathrm{H}^2(J_{\mathscr{X}\,0}, \mathbb{Z}) \cong \mathrm{H}^0(B, R^2\pi_*\mathbb{Z}).\end{aligned}$$

In this way, E_0 extends to a polarization on the Jacobian family (3.12).

Lemma 3.13. *Consider the polarized family of Jacobian varieties*

$$(\psi : J_{\mathscr{X}} \to B, s : B \to J_{\mathscr{X}}, E \in R^2\psi_*\mathbb{Z})$$

defined above. If the relative curve π admits a real structure $(\sigma : B \to B, \sigma' : \mathscr{X} \to \mathscr{X})$, then the polarized relative Jacobian $\psi : J_{\mathscr{X}} \to B$ admits a real structure compatible with σ, in the sense that there exists an anti-holomorphic involution $\Sigma : J_{\mathscr{X}} \to J_{\mathscr{X}}$ such that

$$(i) \quad \psi \circ \Sigma = \sigma \circ \psi, \qquad (ii) \quad \Sigma \circ s = s \circ \sigma, \qquad (iii) \quad \Sigma^*(E) = -E.$$

Proof. Write $\mathbb{V}_{\mathbb{Z}} = R^1\pi_*\mathbb{Z}$ and $\mathcal{H} = \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_B$. Then $\mathcal{H}$ is endowed with a filtration by the holomorphic subbundle $F^1\mathcal{H} \subset \mathcal{H}$, and we have

$$\mathcal{H}/F^1\mathcal{H} = \mathcal{H}^{0,1} = R^1\pi_*\mathcal{O}_{\mathscr{X}}.$$

By the strategy in Section 3.2, the real structure (σ, σ') on $\pi : \mathscr{X} \to B$ induces an anti-holomorphic involution $F_{dR} : \mathcal{H} \to \mathcal{H}$ compatible with σ and preserving $\mathcal{H}^{0,1}$ and $\mathbb{V}_{\mathbb{Z}}$. Since $J_{\mathscr{X}} = \mathcal{H}^{0,1}/\mathbb{V}_{\mathbb{Z}}$, the involution F_{dR} induces an anti-holomorphic involution $\Sigma : J_{\mathscr{X}} \to J_{\mathscr{X}}$. By construction of F_{dR} in Section 3.2, we have $\psi \circ \Sigma = \sigma \circ \psi$. For $t \in B$, Σ induces an anti-holomorphic map $\Sigma : \mathrm{Jac}(\mathscr{X}_{\sigma(t)}) \to \mathrm{Jac}(\mathscr{X}_t)$. We need to prove that $\Sigma^*(E_t) = -E_{\sigma(t)}$ for the map

$$\Sigma^* : \mathrm{H}^2(\mathrm{Jac}(\mathscr{X}_t), \mathbb{Z}) \to \mathrm{H}^2(\mathrm{Jac}(\mathscr{X}_{\sigma(t)}), \mathbb{Z}).$$

This follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{H}^1(\mathscr{X}_t, \mathbb{R}) \otimes \mathrm{H}^1(\mathscr{X}_t, \mathbb{R}) \arrow[r, "\cup"] \arrow[d, "(\sigma')^* \otimes (\sigma')^*"] & \mathrm{H}^2(\mathscr{X}_t, \mathbb{R}) \arrow[r, "\sim"] \arrow[d, "(\sigma')^*"] & \mathbb{R} \arrow[d, "-1"] \\
\mathrm{H}^1(\mathscr{X}_{\sigma(t)}, \mathbb{R}) \otimes \mathrm{H}^1(\mathscr{X}_{\sigma(t)}, \mathbb{R}) \arrow[r, "\cup"] & \mathrm{H}^2(\mathscr{X}_{\sigma(t)}, \mathbb{R}) \arrow[r, "\sim"] & \mathbb{R}.
\end{tikzcd}$$

The square on the left commutes because pullback commutes with cup-product, and the square on the right commutes because $\sigma' : \mathscr{X}_{\sigma(t)} \xrightarrow{\sim} \mathscr{X}_t$ is anti-holomorphic thus reverses the orientation since $\dim(\mathscr{X}_t) = \dim(\mathscr{X}_{\sigma(t)}) = 1$ is odd. □

3.5 DENSITY IN REAL MODULI SPACES

The goal of this section is to prove Theorem 3.3 by applying the results of the previous sections.

3.5.1 *Density in* $|\mathcal{A}_g(\mathbb{R})|$

Proof of Theorem 3.3.A. Let $\tau \in \mathscr{T}(g)$ and $Z \in \mathbb{H}_g^{\Sigma_\tau}$ (see Section 2.3.1). Let

$$(X, \omega \in \mathrm{H}^2(X, \mathbb{Z}))$$

be a principally polarized complex abelian variety with symplectic basis with period matrix Z. Then (X, ω) admits a unique real structure $\sigma : X \to X$, compatible with ω and the symplectic basis [GH81, Section 9]. There exists a Σ_τ-invariant connected open neighborhood $B \subset \mathbb{H}_g$ of $Z \in \mathbb{H}_g$, and a universal local deformation

$$\pi : \mathscr{X} \to B \ni Z$$

of the polarized complex abelian variety (X, ω). By Proposition 3.12, possibly after restricting B around Z, the real structure $\sigma : X \to X$ extends uniquely to a real structure on the polarized family π, which, by uniqueness, is compatible with the real structure $\Sigma_\tau \colon B \to B$. By Proposition 3.11, Condition 3.1 is satisfied. By Theorem 3.2, the subset

$$R_k \cap B^{\Sigma_\tau} \subset B^{\Sigma_\tau}$$

is dense in B^{Σ_τ}. It follows that R_k is dense in $\mathbb{H}_g^{\Sigma_\tau}$. □

3.5.2 *Density in* $|\mathcal{M}_g(\mathbb{R})|$

Proof of Theorem 3.3.B. Suppose that $g \geq 3$, let $j \in \mathscr{J}(g)$ and consider a point $0 \in \mathcal{T}_t^{\sigma_j}$ (see Section 2.3.2). Let $(X, [f])$ be a complex Teichmüller curve of genus g that gives rise to the point 0. By [SS89], there is a unique real structure $\sigma : X \to X$ which is compatible with the Teichmüller structure $[f]$ and the involution $\sigma_j : \Sigma \to \Sigma$. Moreover, there exists a σ_j-invariant simply connected open subset $B \subset \mathcal{T}_g$ of 0 in the Teichmüller space $\mathcal{T}_g$, and a Kuranishi family

$$\pi : \mathscr{X} \to B \ni 0$$

of the Riemann surface X. By Proposition 3.12, up to restricting B around 0, the real structure $\sigma : X \to X$ extends uniquely to a real structure

$$(\tau \colon B \to B, \mathcal{T} \colon \mathscr{X} \to \mathscr{X})$$

on the Kuranishi family π such that $\tau(0) = 0$. By uniqueness, $\tau : B \to B$ coincides with σ_j. By Lemma 3.13, the real structure $(\tau, \mathscr{T})$ induces a real structure

$$(\tau \colon B \to B, \Sigma \colon J_{\mathscr{X}} \to J_{\mathscr{X}})$$

on the Jacobian $J_{\mathscr{X}} \to B$ of the curve $\pi : \mathscr{X} \to B$.

Let $k \in \{1, 2, 3\}$. Observe that, by Theorem 3.2, it suffices to prove that Condition 3.1 holds in B. That is, we need to show that there exists an element $t \in B$ and a k-dimensional complex subspace

$$W \subset \mathrm{H}^{1,0}(\mathrm{Jac}(\mathscr{X}_t)) = \mathrm{H}^{1,0}(\mathscr{X}_t)$$

such that the sequence

$$0 \to \bigwedge^2 W \to W \otimes H^{1,0}(\mathscr{X}_t) \to (T_t B)^\vee$$

is exact. The family $\pi : \mathscr{X} \to B$ is a universal local deformation of $\mathscr{X}_t$ for each $t \in B$, hence $T_t B \cong \mathrm{H}^1(\mathscr{X}_t, T_{\mathscr{X}_t})$. By [Voi02, Lemme 10.22], the dual of

$$q : H^{1,0}(\mathscr{X}_t) \otimes H^{1,0}(\mathscr{X}_t) \to (T_t B)^\vee$$

is nothing but the cup-product

$$\mathrm{H}^0(K_{\mathscr{X}_t}) \otimes \mathrm{H}^0(K_{\mathscr{X}_t}) \to \mathrm{H}^0(K_{\mathscr{X}_t}^{\otimes 2}).$$

We are reduced to the claim that for each $k \in \{1, 2, 3\}$, there exists $t \in B$ and a k-dimensional subspace $W \subset \mathrm{H}^0(K_{\mathscr{X}_t})$ such that the following sequence is exact:

$$0 \to \wedge^2 W \to W \otimes \mathrm{H}^0(K_{\mathscr{X}_t}) \to \mathrm{H}^0(K_{\mathscr{X}_t}^{\otimes 2}). \tag{3.14}$$

In [CP90, *Proof of Theorem (3)*], Colombo and Pirola consider the moduli space of complex genus $g \geq 3$ curves

$$\mathcal{M}_g(\mathbb{C}) = \Gamma_g \setminus \mathcal{T}_g$$

to prove the complex analogue of Theorem 3.3.B. They show that there exists a moduli point $p = [C] \in \mathcal{M}_g(\mathbb{C})$ and a k-dimensional complex subspace $W \subset \mathrm{H}^0(K_C)$ such that (3.14) is exact. This implies that Condition 3.1 is satisfied for some point $t \in \mathcal{T}_g$. Since Condition 3.1 is open for the Zariski topology on $\mathcal{T}_g$, it is dense for the euclidean topology, hence Condition 3.1 holds for some $t \in B$. □

3.5.3 *Real plane curves covering an elliptic curve*

Proof of Theorem 3.3.C. Fix $d \in \mathbb{Z}_{\geq 3}$. Define $N = \binom{d+2}{2}$ and let

$$\mathscr{B}(\mathbb{C}) \subset \mathrm{H}^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(d)) \cong \mathbb{C}^N$$

be the Zariski open subset of non-zero degree d homogeneous polynomials F that define smooth plane curves $\{F = 0\} \subset \mathbb{P}^2(\mathbb{C})$. Consider the universal plane curve

$$\mathscr{B}(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \supset \mathscr{S}(\mathbb{C}) \xrightarrow{\pi} \mathscr{B}(\mathbb{C}).$$

The map π is induced by a morphism of real varieties $\mathscr{S} \to \mathscr{B}$. For a projective, flat morphism of locally Noetherian schemes with integral geometric fibers, the relative Picard scheme exists [Gro62, §V, 3.1]. We obtain an abelian scheme

$$\underline{\mathrm{Pic}}^0_{\mathscr{S}/\mathscr{B}} \to \mathscr{B}$$

of relative dimension $g = (d-1)(d-2)/2$ over $\mathbb{R}$. By Theorem 3.2, it suffices to prove the existence of an element $t \in \mathscr{B}(\mathbb{C})$ for which there exists a non-zero $v \in \mathrm{H}^{1,0}(\mathrm{Jac}(\mathscr{S}_t(\mathbb{C})))$ such that the map

$$\langle v \rangle \otimes \mathrm{H}^{1,0}(\mathrm{Jac}(\mathscr{S}_t(\mathbb{C}))) \to (T_t\mathscr{B}(\mathbb{C}))^\vee$$

is injective. This is done in [CP90, *Proof of Proposition (6)*]. In fact, the element $t \in \mathscr{B}(\mathbb{C})$ attached to the degree d Fermat curve $F = X_0^d + X_1^d + X_2^d$ satisfies this criterion (compare [Kim91, Proposition 3]). □

4

GLUEING REAL BALL QUOTIENTS

4.1 INTRODUCTION

In the previous Chapters 2 and 3, we often encountered (and made use of) the fact that for each connected component of the moduli space of real abelian varieties, there is a natural morphism that identifies this component with the quotient $\Gamma \setminus M$ of a real-analytic manifold by a properly discontinuous group action. The same goes for the moduli space of real algebraic curves. In the next Chapter 5, we will see that something similar is true for the real moduli space $\mathcal{M}_0(\mathbb{R})$ of smooth binary quintics. One difference stands out: each component $\mathscr{M}_i$ of $\mathcal{M}_0(\mathbb{R})$ identifies only with an open subset $\Gamma_i \setminus (M - \mathscr{H}_i)$ of a certain quotient $\Gamma_i \setminus M$, instead of the entire quotient space. In this case, $M = \mathbb{R}H^2$ is the real hyperbolic plane, and the open sets are obtained by removing unions $\mathscr{H}_i$ of lower-dimensional geodesic subspaces.

This difference actually works in our favour. The larger moduli space $\mathcal{M}_s(\mathbb{R})$ of stable real binary quintics is connected, which raises the dream that:

1. One can, one way or another, glue the quotient spaces $\Gamma_i \setminus \mathbb{R}H^2$ into some larger hyperbolic quotient $\Gamma \setminus \mathbb{R}H^2$; and that
2. The various isomorphisms $\mathscr{M}_i \cong \Gamma_i \setminus (\mathbb{R}H^2 - \mathscr{H}_i)$ extend to an isomorphism $\mathcal{M}_s(\mathbb{R}) \cong \Gamma \setminus \mathbb{R}H^2$.

It turns out that this dream can be realized. Chapter 5 will be devoted to Step 2. In the current Chapter 4, we focus on Step 1.

To carry out Step 1, it seemed natural not to restrict our attention to the two-dimensional case, but to glue real ball quotients in any dimension. Unitary Shimura varieties turned out to provide a suitable framework for doing so. We build upon work of Allcock, Carlson and Toledo [ACT10]. Let us outline the construction.

Let K be a CM field of degree $2g$ over $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$, and let Λ be a finite free $\mathcal{O}_K$-module equipped with a hermitian form $h : \Lambda \times \Lambda \to \mathcal{O}_K$. Suppose that h has signature $(n, 1)$ with respect to an embedding $\tau : K \to \mathbb{C}$ and is definite other infinite places of K. Let $\mathbb{C}H^n$ be the space of negative lines in $\Lambda \otimes_{\mathcal{O}_K,\tau} \mathbb{C}$ and $P\Gamma = \mathrm{Aut}(\Lambda, h)/\mu_K$ where $\mu_K \subset \mathcal{O}_K^*$ is the group of finite units in $\mathcal{O}_K$. Let $P\mathscr{A}$ be the quotient of the set of anti-unitary involutions $\alpha : \Lambda \to \Lambda$ by μ_K.

Consider the hyperplane arrangement $\mathscr{H} = \cup_{h(r,r)=1} \langle r_{\mathbb{C}} \rangle^{\perp} \subset \mathbb{C}H^n$ and assume

Condition 4.1. *Different hyperplanes intersect orthogonally or not at all, c.f. [ACT02b].*

For example, this holds under a condition on the CM field K (see Theorem 4.50) satisfied when K is cyclotomic or quadratic (see Lemma 4.52). (In fact, condition 4.1 is *always* satisfied if one is willing to adapt the definition of $\mathscr{H}$, see Remark 4.53.)

We claim that there is a canonical way to glue the different copies

$$\mathbb{R}H^n_\alpha := (\mathbb{C}H^n)^\alpha \subset \mathbb{C}H^n, \quad \alpha \in P\mathscr{A}$$

of the real hyperbolic space $\mathbb{R}H^n$ along the hyperplane arrangement $\mathscr{H}$. See Remark 4.20 for the precise formulation of the equivalence relation. This gives a topological space which we denote by Y, acted upon by $P\Gamma$. Define $P\Gamma_\alpha \subset P\Gamma$ to be stabilizer of $\mathbb{R}H^n_\alpha$. The precise goal of Chapter 4 is to prove the following theorem.

Theorem 4.2. *The topological space $P\Gamma \setminus Y$ admits a metric that makes it a complete path metric space. With respect it, the natural map $P\Gamma \setminus Y \to P\Gamma \setminus \mathbb{C}H^n$ is a local isometry. In fact, the metric underlies a real hyperbolic orbifold structure on $P\Gamma \setminus Y$, such that*

$$\coprod_{\alpha \in P\Gamma \setminus P\mathscr{A}} [P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})] \subset P\Gamma \setminus Y$$

is an open suborbifold and such that for each connected component $C \subset P\Gamma \setminus Y$ there is a lattice $P\Gamma_C \subset \mathrm{PO}(n, 1)$ and an isomorphism of real hyperbolic orbifolds $C \cong [P\Gamma_C \setminus \mathbb{R}H^n]$.

For some moduli stacks of smooth hypersurfaces $\mathcal{M}_0$ one can apply Theorem 4.2 to the hermitian lattice Λ that arises as the cohomology of the cover of projective space ramified along a member of the moduli space. Let $\mathcal{M}_s$ be the stack of GIT stable hypersurfaces. If the discriminant $\Delta = \mathcal{M}_s(\mathbb{C}) - \mathcal{M}_0(\mathbb{C})$ is a normal crossings divisor and the period map induces an isomorphism of analytic spaces

$$\mathcal{M}_s(\mathbb{C}) \cong P\Gamma \setminus \mathbb{C}H^n$$

identifying Δ with $P\Gamma \setminus \mathscr{H}$, then there is a real period homeomorphism

$$\mathcal{M}_s(\mathbb{R}) \cong P\Gamma \setminus Y.$$

For cubic surfaces and binary sextics, this is the content of [ACT10, ACT06]. For binary quintics, this yields the main Theorem 5.2 of Chapter 5 (via Theorem 5.22).

Remarks 4.3.
1. The lattice $P\Gamma_C$ attached to a component $C \subset P\Gamma \setminus Y$ can be non-arithmetic. Indeed, such is the case for $K = \mathbb{Q}(\zeta_5)$ and $h = \operatorname{diag}(1, 1, \frac{1-\sqrt{5}}{2})$ by Remark 5.3 and Theorem 5.34, and for $K = \mathbb{Q}(\zeta_3)$ and $h = \operatorname{diag}(1, \ldots, 1, -1)$ for $n = 3$ [ACT06] and $n = 4$ [ACT10].
2. Our glueing construction relies on Condition 4.1, saying that the hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^n$ is an *orthogonal arrangement* in the sense of [ACT02b]. Condition 4.1 is in turn implied by the following condition, satisfied by quadratic and cyclotomic CM fields (see Section 4.4):

Condition 4.4. *The different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ (see e.g. [Neu99, Chapter III]) is generated by an element $\eta \in \mathcal{O}_K - \mathcal{O}_F$ such that $\eta^2 \in \mathcal{O}_F$.*

Remarks 4.5.
1. In fact, there is always a canonical orthogonal arrangement $\mathcal{H} \subset \mathbb{C}H^n$ attached to h in such a way that $\mathcal{H} = \mathscr{H}$ when Condition 4.4 holds, see Remark 4.53. Moreover, one can glue the different copies $\mathbb{R}H^n_\alpha$ of real hyperbolic n-space along the hyperplane arrangement $\mathcal{H}$ obtaining a complete hyperbolic orbifold as in Theorem 4.2, but we will not prove this.
2. In [Shi75], Shimura studied real points of an arithmetic quotient of a bounded symmetric domain. Gromov and Piatetski-Shapiro developed a construction of glueing hyperbolic quotients in [GPS87]. The difference between their glueing method and ours is explained in [ACT10, Section 13, Remark (1)].

4.2 THE GLUEING CONSTRUCTION

Consider a hermitian form h on a finite free module over the ring of integers of a CM field K with hyperbolic signature for some embedding of K into $\mathbb{C}$. As is well-known, there is a complex ball quotient $P\Gamma \setminus \mathbb{C}H^n$ attached to h in a canonical way. We prove in this section that there is also a natural real ball quotient $P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^n$ attached to h (or a disjoint union of those). As in the complex case, $P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^n$ is sometimes a moduli space for real varieties. To define this space, one considers the real hyperbolic spaces $(\mathbb{C}H^n)^\alpha$ attached to anti-unitary involutions $\alpha : \Lambda \to \Lambda$, glues them along a hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^n$, takes the quotient by $P\Gamma$, and defines a complete real hyperbolic orbifold structure on the result.

4.2.1 *The set-up*

Let g and n be positive integers. Let K be a CM field of degree $2g$ over $\mathbb{Q}$ and let $F \subset K$ be its totally real subfield. Let $\mathcal{O}_K$ (resp. $\mathcal{O}_F$) be the ring of integers of K (resp. F) and let $\sigma \in \mathrm{Gal}(K/F)$ be the non-trivial element. We will often write

$$\sigma\colon K \to K, \quad \sigma(x) = \overline{x}.$$

Fix a set of embeddings

$$\Psi = \{\tau_i\colon K \to \mathbb{C}\}_{1 \le i \le g} \quad | \quad \Psi \cup \Psi\sigma = \{\tau_i, \tau_i\sigma\}_{1 \le i \le g} = \mathrm{Hom}(K, \mathbb{C}). \tag{4.1}$$

Let Λ be a free $\mathcal{O}_K$-module of rank $n+1$ equipped with a hermitian form

$$h\colon \Lambda \times \Lambda \to \mathcal{O}_K$$

of signature (r_i, s_i) with respect to τ_i. In other words, h is linear in its first argument and σ-linear in its second, and the complex vector space $\Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C}$ admits a basis $\{e_i\}$ such that $(h^{\tau_i}(e_i, e_j))_{ij}$ is a diagonal matrix with r_i diagonal entries equal to 1 and s_i diagonal entries equal to -1. Here

$$h^{\tau_i}\colon \Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C} \times \Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C} \to \mathbb{C}$$

is the hermitian form attached to h and the embedding τ_i. Define

$$\tau = \tau_1 : K \to \mathbb{C}, \quad \text{and} \quad V = \Lambda \otimes_{\mathcal{O}_K, \tau} \mathbb{C},$$

$$\text{and assume that} \qquad (r_i, s_i) \quad = \quad \begin{cases} (n, 1) & \text{if } i = 1, \\ (n+1, 0) & \text{if } 2 \le i \le g. \end{cases}$$

Let m be the largest positive integer for which the m-th cyclotomic field $\mathbb{Q}(\zeta_m)$ can be embedded in K, where $\zeta_m = e^{2\pi i/m} \in \mathbb{C}$. Let $\zeta \in K$ be a primitive m-th root of unity in K, and define

$$\mu_K = \langle \zeta \rangle \subset \mathcal{O}_K^* \subset \mathcal{O}_K.$$

Moreover, define Γ to be the unitary group of Λ, and $P\Gamma$ as its quotient by μ_K:

$$\Gamma = U(\Lambda)(\mathcal{O}_K) = \mathrm{Aut}_{\mathcal{O}_K}(\Lambda, h) \quad \text{and} \quad P\Gamma = \Gamma/\mu_K.$$

A norm one vector $r \in \Lambda$ is called a *short root*. Let $\mathscr{R} \subset \Lambda$ be the set of short roots. For $r \in \mathscr{R}$, define isometries $\phi_r^i \colon V \to V$ as follows:

$$\phi_r(x) = x - (1-\zeta)h(x,r) \cdot r, \quad \phi_r^i(x) = x - (1-\zeta^i)h(x,r) \cdot r, \quad i \in (\mathbb{Z}/m)^*.$$

Note that $\phi_r^i \in \Gamma$ for $r \in \mathscr{R}$, and that $\phi_r^i = \phi_r \circ \cdots \circ \phi_r$ (i times). In particular, $\phi_r^m = \mathrm{id}$. Let $\mathbb{P}(V)$ be the projective space of lines in V, and let

$$\mathbb{C}H^n = \{\ell = [v] \in \mathbb{P}(V) \mid h(v,v) < 0\} \subset \mathbb{P}(V)$$

be the space of negative lines in V. Define

$$H_r = \{x \in \mathbb{C}H^n : h(x,r) = 0\} \text{ for } r \in \mathscr{R}, \quad \text{and} \quad \mathscr{H} = \bigcup_{r \in \mathscr{R}} H_r \quad \subset \quad \mathbb{C}H^n.$$

Lemma 4.6. *The family of hyperplanes $(H_r)_{r \in \mathscr{R}}$ is locally finite, so that the hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^n$ is a divisor of $\mathbb{C}H^n$.*

Proof. See [Bea09, Lemma 5.3]. □

Define an $\mathcal{O}_F$-linear map $\alpha : \Lambda \to \Lambda$ to be *anti-unitary* if for all $x, y \in \Lambda$ and $\lambda \in \mathcal{O}_K$, one has $\alpha(\lambda x) = \sigma(\lambda) \cdot \alpha(x)$ and $h(\alpha(x), \alpha(y)) = \sigma(h(x,y)) \in \mathcal{O}_K$. Define Γ' to be the group of unitary and anti-unitary $\mathcal{O}_F$-linear bijections $\Lambda \xrightarrow{\sim} \Lambda$. Let $\mathscr{A} \subset \Gamma'$ be the set of anti-unitary involutions $\alpha \colon \Lambda \to \Lambda$. Then

$$\mu_K \subset \Gamma \subset \Gamma' \quad - \quad \text{define} \quad P\Gamma' = \Gamma'/\mu_K. \tag{4.2}$$

Let $\lambda \in K^*$. Observe that

$$\left(\lambda \in \mathcal{O}_K^* \text{ and } |\lambda|^2 = \lambda \cdot \sigma(\lambda) = 1\right) \quad \Longleftrightarrow \quad \lambda \in \mu_K. \tag{4.3}$$

Indeed, we have, for any embedding $\varphi\colon K \to \mathbb{C}$, that

$$|\varphi(\lambda)|^2 = \varphi(\lambda) \cdot \overline{\varphi(\lambda)} = \varphi(\lambda) \cdot \varphi(\sigma(\lambda)) = \varphi(\lambda \cdot \sigma(\lambda)),$$

where $\overline{\varphi(\lambda)} = \varphi(\sigma(\lambda))$ by [Mil20, Proposition 1.4]. Moreover, we have $|\varphi(\lambda)| = 1$ for each $\varphi\colon K \to \mathbb{C}$ if and only if λ is a root of 1, see [Mil08, Corollary 5.6].

Lemma 4.7. *Let* $\mathrm{Isom}(\mathbb{C}H^n)$ *be the group of isometries* $f\colon \mathbb{C}H^n \xrightarrow{\sim} \mathbb{C}H^n$. *The natural homomorphism* $P\Gamma' \to \mathrm{Isom}(\mathbb{C}H^n)$ *is injective.*

Proof. Let $g \in \Gamma'$ be an element that induces the identity on $\mathbb{C}H^n$. Let $\{e_i\}_{i=1,\dots,n+1}$ be a basis for Λ. Consider $\{e_i\}$ as a basis of V. There exist $\lambda_i \in \mathbb{C}^*$ such that $g(e_i) = \lambda_i \cdot e_i$. We must have $\lambda_1 = \ldots = \lambda_{n+1}$. If $g \in \Gamma$, this means that $g = \lambda \in \mathbb{C}^* \subset \mathrm{GL}(V)$. The basis $\{e_i\}$ induces a $\mathbb{C}$-linear isomorphism $V \cong \mathbb{C}^{n+1}$, hence λ acts on $\mathcal{O}_K^{n+1} \subset \mathbb{C}^{n+1}$ by multiplication, thus $\lambda \in \mathcal{O}_K^*$. For $r \in \mathscr{R}$, we have $1 = h(r,r) = h(g(r), g(r)) = |\lambda|^2$, thus $\lambda \in \{x \in \mathcal{O}_K^* \mid |x| = 1\} = \mu_K$ (see (4.3)).

Suppose that $g \notin \Gamma$. Then g induces the map $\sum_i \mu_i e_i \mapsto \lambda \cdot \sum_i \overline{\mu_i} e_i$ on V, which is absurd since then $g \neq \mathrm{id} \in \mathrm{Isom}(\mathbb{C}H^n)$. □

The group μ_K acts on $\mathscr{A}$ by multiplication; define

$$P\mathscr{A} = \mu_K \setminus \mathscr{A}, \quad \text{and} \quad C\mathscr{A} = P\Gamma \setminus P\mathscr{A},$$

where $P\Gamma$ acts on $P\mathscr{A}$ by conjugation. Any $\alpha \in P\mathscr{A}$ defines an anti-holomorphic involution

$$\alpha : \mathbb{C}H^n \to \mathbb{C}H^n; \quad \text{define} \quad \mathbb{R}H^n_\alpha = (\mathbb{C}H^n)^\alpha \subset \mathbb{C}H^n.$$

For any element $\alpha \in \mathscr{A}$, the quadratic form $h|_{V^\alpha}$ on the real vector space $V^\alpha = \Lambda^\alpha \otimes_{\mathcal{O}_F, \tau|_F} \mathbb{C}$ has hyperbolic signature. The following lemma is readily proved:

Lemma 4.8. *For* $\alpha \in \mathscr{A}$, *let* $\mathbb{P}(V^\alpha)$ *be the real projective space of lines in* V^α, *and let* $\mathbb{R}H(V^\alpha) \subset \mathbb{P}(V^\alpha)$ *be the space of negative lines in* V^α. *The canonical isomorphism* $\mathbb{P}(V^\alpha) \cong \mathbb{P}(V)^\alpha$ *restricts to an isomorphism* $\mathbb{R}H(V^\alpha) \cong \mathbb{R}H^n_\alpha$. □

We conclude that $\mathbb{R}H^n_\alpha \subset \mathbb{C}H^n$ is isometric to the real hyperbolic space of dimension n. Finally, we define

$$P\Gamma_\alpha = \mathrm{Stab}_{P\Gamma}(\mathbb{R}H^n_\alpha) \subset P\Gamma \qquad \text{(the stabilizer of } \mathbb{R}H^n_\alpha \text{ in } P\Gamma).$$

4.2.2 *Preliminary results*

We assume, in the entire Section 4.2, that the following condition is satisfied:

Condition 4.9. *If $r, t \in \mathscr{R}$ are such that $H_r \neq H_t$ and $H_r \cap H_t \neq \emptyset$, then $h(r,t) = 0$.*

Example 4.10. Theorem 4.50 of Section 4.4 shows that Condition 4.9 if the following conditions are satisfied:

1. The different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by a single element $\eta \in \mathcal{O}_K$ such that $\sigma(\eta) = -\eta$ and $\Im(\tau_i(\eta)) > 0$ for every i.
2. The CM type (K, Ψ) is primitive.

We will see that condition 1 is automatically satisfied for quadratic and cyclotomic CM fields K, see Lemma 4.52.

Note that Condition 4.9 implies that if $H_{r_1}, \ldots, H_{r_k}$ for $r_i \in \mathscr{R}$ are mutually distinct, and if their common intersection is non-empty, then $\cap_{i=1}^k H_{r_i} \subset \mathbb{C}H^n$ is a totally geodesic subspace of codimension k. Note also that for any $r \in \mathscr{R}$, the element $\phi_r \in \Gamma$ generates a finite subgroup $\langle \phi_r \rangle \subset \Gamma$ of order m, and that the restriction of the quotient map $\pi \colon \Gamma \to P\Gamma$ to this subgroup $\langle \phi_r \rangle \subset \Gamma$ is injective. We will abuse notation, by letting $\phi_r \in P\Gamma$ denote the image of $\phi_r \in \Gamma$ in $P\Gamma$.

Definition 4.11. Let $\mathcal{H} = \{H_r \mid r \in \mathscr{R}\}$. For $x \in \mathbb{C}H^n$, define

$$\mathcal{H}(x) = \{H \in \mathcal{H} \mid x \in H\}, \quad G(x) = \left\{\phi_r^i \in P\Gamma \text{ for } r \in \mathscr{R}, i \in \mathbb{Z}/m \mid x \in H_r\right\}.$$

The hyperplanes $H \in \mathcal{H}(x)$ are called the *nodes* of x. We say that x *has k nodes* if the cardinality of $\mathcal{H}(x)$ is k.

Lemma 4.12. *Let $x \in \mathbb{C}H^n$ and suppose that x has k nodes. Then $G(x) \cong (\mathbb{Z}/m)^k$.*

Proof. Let $r, t \in \mathscr{R}$. Then, for $z \in \Lambda$, one has

$$\begin{aligned}\phi_r^i(\phi_t^j(z)) &= \phi_r^i\left(z - (1-\zeta^j)h(z,t)\cdot t\right)\\ &= z - (1-\zeta^j)h(z,t)\cdot t - (1-\zeta^i)h\left(\left(z - (1-\zeta^j)h(z,t)\cdot t\right), r\right)\cdot r\\ &= z - (1-\zeta^j)h(z,t)\cdot t - (1-\zeta^i)h(z,r)\cdot r + (1-\zeta^i)(1-\zeta^j)h(z,t)h(t,r)\cdot r.\end{aligned} \tag{4.4}$$

Now suppose that $H_r, H_t \in \mathcal{H}(x)$, with $H_r \neq H_t$. By Condition 4.9, we have $h(r,t) = 0$; by (4.4), this implies that $\phi_r^i \circ \phi_t^j = \phi_t^j \circ \phi_r^i$ for each $i, j \in \mathbb{Z}/m$. We conclude that the group $G(x)$ is abelian.

Next, suppose that $H_r = H_t \in \mathcal{H}(x)$. To finish the proof, it suffices to show that $\phi_t = \phi_r^i$ for some $i \in \mathbb{Z}/m$. To prove this, we introduce Lemma 4.13 below. $\square$

Lemma 4.13. *Let $r \in \mathscr{R}$. Let $\phi : \mathbb{C}H^n \to \mathbb{C}H^n$ be an isometry such that $\phi^m = \mathrm{id}$ and such that ϕ restricts to the identity on $H_r \subset \mathbb{C}H^n$. Then $\phi = \phi_r^i$ for some $i \in \mathbb{Z}/m$.*

Proof. Let $\mathbb{H}^n_{\mathbb{C}}$ be the hyperbolic space attached to the standard hermitian space $\mathbb{C}^{n,1}$ of dimension $n+1$. It is classical that

$$\mathrm{Stab}_{U(n,1)}(\mathbb{H}^{n-1}_{\mathbb{C}}) = U(n) \times U(1).$$

Thus, any $\phi \in U(n,1)$ that fixes $\mathbb{H}^{n-1}_{\mathbb{C}}$ pointwise lies in $U(1) = \{z \in \mathbb{C}^* : |z|^2 = 1\}$. If $\phi^m = \mathrm{id}$, then ϕ lies in the unique subgroup of $U(1)$ that has order m. $\square$

We will also consider $G(x)$ as a subgroup of Γ sometimes, which we may do, since for each $[g] \in G(x)$ there is a unique $g \in \Gamma$ such that $\pi(g) = [g]$ for $\pi \colon \Gamma \to P\Gamma$. Lemma 4.7 allows us to view $P\mathscr{A}$ as a subset of $\mathrm{Isom}(\mathbb{C}H^n)$, and also to view the groups $G(x) \subset P\Gamma \subset P\Gamma'$ (for any $x \in \mathbb{C}H^n$) as subgroups of $\mathrm{Isom}(\mathbb{C}H^n)$. Define

$$\widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha.$$

We will glue the different hyperbolic spaces $\mathbb{R}H^n_\alpha$, by defining an equivalence relation $\sim$ on $\widetilde{Y}$. To define this equivalence relation, we need a couple of results.

Lemma 4.14. *Let $\alpha \in \mathscr{A}$ and $r \in \mathscr{R}$. Then $\alpha \circ \phi_r^i = \phi_{\alpha(r)}^{-i} \circ \alpha$.*

Proof. Indeed, for $x \in \Lambda$, we have

$$\begin{aligned}
\alpha(\phi_r^i(x)) &= \alpha\left(x - (1-\zeta^i)h(x,r)\cdot r\right) \\
&= \alpha(x) - (1-\zeta^{-i})\overline{h(x,r)}\cdot \alpha(r) \\
&= \alpha(x) - (1-\zeta^{-i})h(\alpha(x),\alpha(r))\cdot\alpha(r) \\
&= \phi_{\alpha(r)}^{-i}(\alpha(x)).
\end{aligned}$$

$\square$

Lemma 4.15. *Let $x \in \mathbb{R}H^n_\alpha$ and write $\mathcal{H}(x) = \{H_{r_1}, \ldots, H_{r_k}\}$ for some $r_i \in \mathscr{R}$. Then for each $i \in \{1, \ldots, k\}$ there is a unique $j \in \{1, \ldots, k\}$ such that $\alpha(H_{r_i}) = H_{\alpha(r_i)} = H_{r_j}$.*

Proof. Indeed, we have, for any $\beta \in \mathscr{A}$ and $r \in \mathscr{R}$, that

$$\beta(H_r) = H_{\beta(r)}.$$

Since $x \in H_{r_i}$, we have $x = \alpha(x) \in \alpha(H_{r_i}) = H_{\alpha(r_i)}$ for every i. In particular, we have $H_{\alpha(r_i)} \in \mathcal{H}(x)$ (see Definition 4.11), so that $H_{\alpha(r_i)} = H_{r_j}$ for some j. □

Lemma 4.16. *Let $r, t \in \mathscr{R}$ and $i, j \in \mathbb{Z}/m - \{0\}$ be such that $\phi^i_r = \phi^j_t \in \Gamma$. Then $i = j$ and $a \cdot t = b \cdot r$ for some $a, b \in \mathcal{O}_K - \{0\}$ such that $|a|^2 = |b|^2$.*

Proof. Suppose for contradiction that there are no such $a, b \in \mathcal{O}_K - \{0\}$. Since $K = \mathrm{Frac}(\mathcal{O}_K)$, this implies that $r, t \in V$ are linearly independent over K, and hence over $\mathbb{C}$, which contradicts the equality

$$\zeta^i \cdot r = \phi^i_r(r) = \phi^j_t(r) = r - (1 - \zeta^j)h(r, t) \cdot t \in V.$$

Thus, there exist $a, b \in \mathcal{O}_K - \{0\}$ with $a \cdot t = b \cdot r$. We have $|a|^2 = h(a \cdot t, a \cdot t) = h(b \cdot r, b \cdot r) = |b|^2$. Write $\lambda = b/a \in K^*$; then $t = \lambda \cdot r$ with $|\lambda|^2 = 1$, so that

$$\zeta^i \cdot r = \phi^i_r(r) = \phi^j_t(r) = \phi^j_{\lambda \cdot r}(r) = r - (1 - \zeta^j)h(r, \lambda \cdot r) \cdot \lambda \cdot r = \zeta^j \cdot r \in V,$$

from which we conclude that $i = j$. □

Definition 4.17. Let $\alpha \in P\mathscr{A}$ and $x \in \mathbb{R}H^n_\alpha$. Write $\mathcal{H}(x) = \{H_{r_1}, \ldots, H_{r_k}\}$, see Definition 4.11. By Lemma 4.15, the involution α induces an involution on the set $\mathcal{H}(x)$. Define $\alpha \colon I \to I$ as the resulting involution on the set $I = \{1, \ldots, k\}$.

Proposition 4.18. *Let $\alpha \in P\mathscr{A}$ and $x \in \mathbb{R}H^n_\alpha$. Write $\mathcal{H}(x) = \{H_{r_1}, \ldots, H_{r_k}\}$ (Definition 4.11) and let $g = \phi^{i_1}_{r_1} \circ \cdots \circ \phi^{i_k}_{r_k} \in G(x)$ for some $i_\nu \in \mathbb{Z}/m$. The following are equivalent:*

1. *We have $g \circ \alpha \in P\mathscr{A}$. (In other words, $g \circ \alpha$ is an involution.)*
2. *For each $\nu \in I$, we have $i_\nu \equiv i_{\alpha(\nu)} \bmod m$.*

Proof. Lift $\alpha \in P\mathscr{A}$ to an element $\alpha \in \mathscr{A}$. We claim that, for each $i \in I$, we have

$$\phi_{\alpha(r_i)} = \phi_{r_{\alpha(i)}}.$$

Indeed, by Lemma 4.15, for each $i \in I$, we have $\alpha(H_{r_i}) = H_{\alpha(r_i)} = H_{r_j} \in \mathcal{H}(x)$ for some $j \in I$. By definition of the involution $\alpha\colon I \to I$, we have $j = \alpha(i)$. Therefore, $\phi_{\alpha(r_i)} = \phi_{r_{\alpha(i)}}^{b_i}$ for some $b_i \in \mathbb{Z}/m$, see Lemma 4.13. By Lemma 4.16, we have $b_i = 1$.

By Lemma 4.14 and by the claim above, we obtain $\phi_{r_\nu}^{i_\nu} \circ \alpha = \alpha \circ \phi_{\alpha(r_\nu)}^{-i_\nu} = \alpha \circ \phi_{r_{\alpha(\nu)}}^{-i_\nu}$ for each $\nu \in I$, which implies that

$$\phi_{r_1}^{i_1} \circ \cdots \circ \phi_{i_k}^{i_k} \circ \alpha = \alpha \circ \phi_{r_{\alpha(1)}}^{-i_1} \circ \cdots \circ \phi_{i_{\alpha(k)}}^{-i_k}.$$

Therefore,

$$\begin{aligned}\left(\phi_{r_1}^{i_1} \circ \cdots \circ \phi_{i_k}^{i_k} \circ \alpha\right)^2 &= \phi_{r_1}^{i_1} \circ \cdots \circ \phi_{i_k}^{i_k} \circ \phi_{r_{\alpha(1)}}^{-i_1} \circ \cdots \circ \phi_{i_{\alpha(k)}}^{-i_k} \\ &= \phi_{r_{\alpha(1)}}^{i_{\alpha(1)}} \circ \cdots \circ \phi_{i_{\alpha(k)}}^{i_{\alpha(k)}} \circ \phi_{r_{\alpha(1)}}^{-i_1} \circ \cdots \circ \phi_{i_{\alpha(k)}}^{-i_k} \\ &= \phi_{r_{\alpha(1)}}^{i_{\alpha(1)}-i_1} \circ \cdots \circ \phi_{i_{\alpha(k)}}^{i_{\alpha(k)}-i_k} \in G(x),\end{aligned}$$

and this is the identity in $G(x)$ if and only if $i_\nu \equiv i_{\alpha(\nu)} \bmod m$. □

4.2.3 *The glueing construction*

Definition 4.19. Define a relation $R \subset \widetilde{Y} \times \widetilde{Y}$ as follows. An element

$$(x_\alpha, y_\beta) \in \mathbb{R}H^n_\alpha \times \mathbb{R}H^n_\beta \subset \widetilde{Y} \times \widetilde{Y}$$

is an element of R if the following conditions are satisfied:

1. With respect to the inclusions $\mathbb{R}H^n_\alpha \subset \mathbb{C}H^n$ and $\mathbb{R}H^n_\beta \subset \mathbb{C}H^n$, we have $x_\alpha = y_\beta \in \mathbb{C}H^n$.
2. If $\alpha \neq \beta$, then
   a) $x_\alpha = y_\beta$ lies in $\mathscr{H}$; and
   b) $\beta = g \circ \alpha \in P\mathscr{A}$ for some $g \in G(x_\alpha) = G(y_\beta)$ (c.f. Lemma 4.7).

Remark 4.20. Conditions 1 and 2 in Definition 4.19 say that we are identifying points of $\mathbb{R}H^n_\alpha \cap \mathscr{H}$ and $\mathbb{R}H^n_\beta \cap \mathscr{H}$ that have the same image in $\mathbb{C}H^n$. But we do not glue all such points: the real structures α and β are required to differ by complex reflections in the hyperplanes that pass through x. In fact, we will see below (see Lemma 4.25) that the glueing can be rephrased as follows: we glue $\mathbb{R}H^n_\alpha$

and $\mathbb{R}H^n_\beta$ along their intersection, provided that this intersection is contained in $\mathscr{H}$ in such a way that for some (equivalently, any) $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$, the real structures α and β differ by reflections in hyperplanes that pass through x.

Lemma 4.21. *R is an equivalence relation.*

Proof. Consider three elements $x_\alpha, y_\beta, z_\gamma \in \widetilde{Y}$. The fact that $x_\alpha \sim x_\alpha$ is clear.

Suppose that $x_\alpha \sim y_\beta$. If $\alpha = \beta$ then $x_\alpha = y_\beta \in \widetilde{Y}$ hence $y_\beta \sim x_\alpha$. If $\alpha \neq \beta$ then $x_\alpha = y_\beta \in \mathscr{H} \subset \mathbb{C}H^n$, and $\beta = g \circ \alpha$ for $g \in G(x_\alpha) = G(y_\beta)$ as in Definition 4.19. Since $\alpha = g^{-1} \circ \beta$ with $g^{-1} \in G(x_\alpha)$, this shows that $y_\beta \sim x_\alpha$.

Suppose that $x_\alpha \sim y_\beta$ and $y_\beta \sim z_\gamma$; we claim that $x_\alpha \sim z_\gamma$. We may and do assume that α, β and γ are different, which implies that $x_\alpha = y_\beta = z_\gamma \in \mathscr{H}$, that $\gamma = h \circ \beta$ for some $h \in G(y_\beta)$, and that $\beta = g \circ \alpha$ for some $g \in G(x_\alpha)$. We obtain $\gamma = h \circ \beta = h \circ g \circ \alpha$ for $h \circ g \in G(x_\alpha) = G(y_\beta) = G(z_\gamma)$. □

Definition 4.22. Define Y to be the quotient of $\widetilde{Y}$ by the equivalence relation R, and equip it with the quotient topology. We shall prove (Lemma 4.23) that the group $P\Gamma$ acts on Y. We call $P\Gamma \setminus Y$ the *glued space* attached to the hermitian $\mathcal{O}_K$-lattice (Λ, h).

Lemma 4.23. *The action of $P\Gamma$ on $\mathbb{C}H^n$ induces an action of $P\Gamma$ on $\widetilde{Y}$, compatible with the equivalence relation R, so that $P\Gamma$ acts on Y. Moreover, $P\Gamma \setminus \widetilde{Y} = \coprod_{\alpha \in C\mathscr{A}} P\Gamma \setminus \mathbb{R}H^n_\alpha$.*

Proof. If $\phi \in P\Gamma$, then $\phi\left(\mathbb{R}H^n_\alpha\right) = \mathbb{R}H^n_{\phi\alpha\phi^{-1}}$ hence $P\Gamma$ acts on $\widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha$, and

$$P\Gamma \setminus \widetilde{Y} = P\Gamma \setminus \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha = \coprod_{\alpha \in C\mathscr{A}} P\Gamma \setminus \mathbb{R}H^n_\alpha.$$

Now suppose that $x_\alpha \sim y_\beta \in \widetilde{Y}$ and $f \in P\Gamma$. Then $f(x_\alpha) \in \mathbb{R}H^n_{f\alpha f^{-1}}$ and $f(y_\beta) \in \mathbb{R}H^n_{f\beta f^{-1}}$. We claim that

$$f(x_\alpha)_{f\alpha f^{-1}} \sim f(y_\beta)_{f\beta f^{-1}}.$$

For this, we may and do assume that $x_\alpha \neq y_\beta$, hence $x_\alpha = y_\beta \in \mathscr{H}$ and $\beta = g \circ \alpha$ for some $g \in G(x_\alpha)$ as in Definition 4.19. In particular, $f(x_\alpha) = f(y_\beta)$. Since $f \circ \phi^i_r \circ f^{-1} = \phi^i_{f(r)}$ for each $r \in \mathscr{R}$ and $i \in \mathbb{Z}/m$, and $h(x, r) = 0$ if and only if $h(f(x), f(r)) = 0$, we have $fG(x)f^{-1} = G(f(x))$ for each $x \in \mathbb{C}H^n$. We are done:

$$f\beta f^{-1} = f(g \circ \alpha)f^{-1} = fgf^{-1} \circ (f\alpha f^{-1}), \quad fgf^{-1} \in G(f(x_\alpha)).$$

□

4.3 THE HYPERBOLIC ORBIFOLD STRUCTURE

We are now in position to state the main theorem of Chapter 4:

Theorem 4.24. 1. *The glued space $P\Gamma \setminus Y$ admits a metric that makes it a complete path metric space. The natural map $P\Gamma \setminus Y \to P\Gamma \setminus \mathbb{C}H^n$ is a local isometry.*

2. *Each point $x \in P\Gamma \setminus Y$ admits an open neighborhood $U \subset P\Gamma \setminus Y$ which is isometric to the quotient of an open subset $V \subset \mathbb{R}H^n$ by a finite group of isometries. Therefore, the glued space $P\Gamma \setminus Y$ has a real hyperbolic orbifold structure.*

3. *One has $\coprod_{\alpha \in C\mathscr{A}} [P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})] \subset P\Gamma \setminus Y$ as an open suborbifold.*

4. *The connected components of the real-hyperbolic orbifold $P\Gamma \setminus Y$ are uniformized by $\mathbb{R}H^n$. This means that for each component $C \subset P\Gamma \setminus Y$ there exists a lattice $P\Gamma_C \subset \mathrm{PO}(n,1)$ and an isomorphism of real hyperbolic orbifolds $C \cong [P\Gamma_C \setminus \mathbb{R}H^n]$. In other words,*

$$P\Gamma \setminus Y \cong \coprod_{C \in \pi_0(P\Gamma \setminus K)} [P\Gamma_C \setminus \mathbb{R}H^n] \,.$$

The remaining part of Section 4.2 is devoted to the proof of Theorem 4.24. It can happen that $P\Gamma \setminus Y$ is connected: such is the case when $K = \mathbb{Q}(\zeta_3)$ and $h = \mathrm{diag}(1,1,1,1,-1)$, see [ACT10]. When $K = \mathbb{Q}(\zeta_3)$ and $h = \mathrm{diag}(1,1,1,-1)$, then $P\Gamma \setminus Y$ has two components, see [ACT07, Remark 6]. In Chapter 5 we show that some (d,n), there is a homeomorphism between $P\Gamma \setminus Y$ and the space of stable hypersurfaces of degree d in $\mathbb{P}^n_{\mathbb{R}}$, restricting to an orbifold isomorphism between $\coprod_\alpha [P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})]$ and the locus of smooth hypersurfaces (Theorem 5.22).

4.3.1 *The path metric on the glued space*

We start with a lemma. We will need it in the proof of Lemma 4.27 below, which will in turn be used to define a path metric on $P\Gamma \setminus Y$ making it locally isometric to quotients of $\mathbb{R}H^n$ by finite groups of isometries. It also serves as a sanity check: once there exists an element $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ such that $x_\alpha \sim x_\beta$, then one glues the entire copy $\mathbb{R}H^n_\alpha$ to the copy $\mathbb{R}H^n_\beta$ along their intersection in $\mathbb{C}H^n$.

Lemma 4.25. 1. *Let $g = \prod_{\nu=1}^k \phi_{r_\nu}^{i_\nu} \in \Gamma$ for some set $\{r_\nu\} \subset \mathscr{R}$ of mutually orthogonal short roots r_ν, where $i_\nu \not\equiv 0 \bmod m$ for each ν. Then $(\mathbb{C}H^n)^g = \cap_{\nu=1}^k H_{r_\nu}$.*

2. *Let $\alpha, \beta \in P\mathscr{A}$ and $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ such that $x_\alpha \sim x_\beta$. Then $y_\alpha \sim y_\beta$ for every $y \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$.*

3. *The natural map $\widetilde{Y} \to \mathbb{C}H^n$ descends to a $P\Gamma$-equivariant map $\mathscr{P} : Y \to \mathbb{C}H^n$.*

Proof. 1. Let $y \in V$ be representing an element in $(\mathbb{C}H^n)^\phi$. Since the r_i are orthogonal, and $g(y) = \lambda$ for some $\lambda \in \mathbb{C}^*$, we have

$$g(y) = \prod_{\nu=1}^{k} \phi_{r_\nu}^{i_\nu}(y) = y - \sum_{\nu=1}^{k} \left(1 - \zeta^{i_\nu}\right) h(y, r_\nu) r_\nu = \lambda y, \tag{4.5}$$

hence $(1-\lambda)y = \sum_{i=1}^{\ell} \left(1 - \zeta^{i_\nu}\right) h(y, r_\nu) r_\nu \in V$. But y spans a negative definite subspace of V while the r_ν span a positive definite subspace, so that we must have $1 - \lambda = 0 = \sum_{\nu=1}^{k} \left(1 - \zeta^{i_\nu}\right) h(y, r_\nu) r_\nu$. Since the r_ν are mutually orthogonal, they are linearly independent; since $\zeta^{i_\nu} \neq 1$ we find $h(y, r_\nu) = 0$ for each ν. Conversely, if $x \in \cap H_{r_\nu}$, then $\phi_{r_\nu}^{i_\nu}(x) = x$ for each ν.

2. Since $x_\alpha \sim x_\beta$, there exists $g \in G(x)$ such that $\beta = g \circ \alpha$. Write $\mathcal{H}(x) = \{H_{r_1}, \ldots, H_{r_k}\}$. Let $y \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$. Then $\alpha(y) = \beta(y) = y$ implies that $g(y) = y$. In particular, $y \in \cap_\nu H_{r_\nu}$ by part 1, which implies that $\mathcal{H}(x) \subset \mathcal{H}(y)$, which in turn implies that $G(x) \subset G(y)$. We conclude that $g \in G(y)$. Hence $y_\alpha \sim y_\beta$.

3. If $x_\alpha \sim y_\beta$, then $x = y \in \mathbb{C}H^n$. □

By Lemma 4.25, we obtain continuous maps

$$\mathscr{P} : K \to \mathbb{C}H^n, \quad \text{and} \quad \overline{\mathscr{P}} : P\Gamma \setminus Y \to P\Gamma \setminus \mathbb{C}H^n.$$

Our next goal is to prove that each point $x \in Y$ has a neighbourhood $V \subset Y$ that maps homeomorphically onto a finite union $\cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n$. Hence x has an open neighourhood $x \in U \subset V$ that identifies with an open set in a union of copies of $\mathbb{R}H^n$ in $\mathbb{C}H^n$ under the map $\mathscr{P}$. This allows us to define a metric on Y by pulling back the metric on $\mathbb{C}H^n$.

Lemma 4.26. *Each compact set $Z \subset \mathbb{C}H^n$ meets only finitely many $\mathbb{R}H^n_\alpha$, $\alpha \in P\mathscr{A}$.*

Proof. Recall the subgroup $P\Gamma' \subset \mathrm{Isom}(\mathbb{C}H^n)$ (see (4.2) and Lemma 4.7). We have that $P\Gamma'$ acts properly discontinuously on $\mathbb{C}H^n$. So if S is the set of $\alpha \in P\mathscr{A}$ such that $\alpha Z \cap Z \neq \emptyset$, then S is finite. In particular, Z meets only finitely many $\mathbb{R}H^n_\alpha$. □

Fix a point $f \in Y$ and a point $x_\alpha \in \widetilde{Y}$ lying above f. Let $\alpha_1, \dots, \alpha_\ell$ be the elements in $P\mathscr{A}$ such that $x_{\alpha_i} \sim x_\alpha$ for each $i \in I := \{1, \dots, \ell\}$ (since the group $G(x)$ is finite by Lemma 4.12, these are finite in number).

Let $p : \widetilde{Y} \to Y$ be the quotient map, and define

$$Y_f = p\left(\coprod_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right) \subset Y. \tag{4.6}$$

We prove that Y is locally isometric to opens in unions of real hyperbolic subspaces of $\mathbb{C}H^n$. Indeed, we have the following:

Lemma 4.27. 1. *The set Y_f is closed in Y.*

2. *We have $\mathscr{P}\left(Y_f\right) = \cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n$, and the map*

$$\mathscr{P}_f : Y_f \to \cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}$$

induced by $\mathscr{P}$ is a homeomorphism.

3. *The set $Y_f \subset Y$ contains an open neighborhood U_f of f in Y.*

Proof. 1. One has

$$p^{-1}\left(Y_f\right) = p^{-1}\left(p\left(\coprod_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right)\right) = \bigcup_{i=1}^{\ell} p^{-1}\left(p\left(\mathbb{R}H^n_{\alpha_i}\right)\right).$$

Therefore, it suffices to show that $p^{-1}\left(p\left(\mathbb{R}H^n_{\alpha_i}\right)\right)$ is closed in Y. But notice that $x_\beta \in p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right)$ if and only if $x \in \mathbb{R}H^n_\alpha$ and $x_\alpha \sim x_\beta$, which implies (Lemma 4.25) that $\mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta \subset p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right)$. Hence for any $\alpha \in P\mathscr{A}$, one has

$$p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right) = \coprod_{\beta \sim \alpha} \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta,$$

where $\beta \sim \alpha$ if and only if there exists $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ such that $x_\alpha \sim x_\beta$. It follows that $p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right) \cap \mathbb{R}H^n_\beta$ is closed in $\mathbb{R}H^n_\beta$ for every $\beta \in P\mathscr{A}$. But the $\mathbb{R}H^n_\beta$ are open in $\widetilde{Y}$ and cover $\widetilde{Y}$, so that $p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right)$ is closed in $\widetilde{Y}$.

2. We have

$$\mathscr{P}_f(Y_f) = \mathscr{P}\left(p\left(\coprod_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right)\right) = \widetilde{\mathscr{P}}\left(\coprod_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right) = \bigcup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n.$$

To prove injectivity, let $x_{\alpha_i}, y_{\alpha_j} \in \widetilde{Y}$ and suppose that $x = y \in \mathbb{C}H^n$. Then indeed, $x_{\alpha_i} \sim y_{\alpha_j}$ because $\sim$ is an equivalence relation by Lemma 4.21.

Let $Z \subset \mathbb{C}H^n$ be a compact set. Write

$$\widetilde{\mathscr{P}} : \widetilde{Y} \to \mathbb{C}H^n$$

for the canonical map. Remark that Z meets only finitely many of the $\mathbb{R}H^n_\alpha$ for $\alpha \in P\mathscr{A}$, see Lemma 4.26. Each $Z \cap \mathbb{R}H^n_\alpha$ is closed in Z since $\mathbb{R}H^n_\alpha$ is closed in $\mathbb{C}H^n$, so each $Z \cap \mathbb{R}H^n_\alpha$ is compact. We conclude that $\widetilde{\mathscr{P}}^{-1}(Z) = \coprod Z \cap \mathbb{R}H^n_\alpha$ is compact. In particular, $\widetilde{\mathscr{P}}$ is closed [Lee13, Theorem A.57].

Finally, we prove that $\mathscr{P}_f$ is closed. Let $Z \subset Y_f$ be a closed set. Then Z is closed in Y by part 1, hence $p^{-1}(Z)$ is closed in $\widetilde{Y}$, hence $\widetilde{\mathscr{P}}\left(p^{-1}(Z)\right)$ is closed in $\mathbb{C}H^n$, so that

$$\mathscr{P}_f(Z) = \mathscr{P}(Z) = \widetilde{\mathscr{P}}\left(p^{-1}(Z)\right) = \left(\widetilde{\mathscr{P}}\left(p^{-1}(Z)\right)\right) \cap \left(\cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right)$$

is closed in $\cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}$.

3. Let $x = \mathscr{P}(f) \in \mathbb{C}H^n$. Since $\mathbb{C}H^n$ is locally compact, there exists a compact set $Z \subset \mathbb{C}H^n$ and an open set $U \subset \mathbb{C}H^n$ with $x \in U \subset Z$. Since Z is compact, it meets only finitely many of the $\mathbb{R}H^n_\beta \subset \mathbb{C}H^n$ (Lemma 4.26). Consequently, the same holds for U; define $V = \mathscr{P}^{-1}(U) \subset Y$. Define

$$\mathscr{B} = \{\beta \in P\mathscr{A} : U \cap \mathbb{R}H^n_\beta \neq \emptyset\}.$$

Also define, for $\beta \in P\mathscr{A}$, $Z_\beta = p\left(\mathbb{R}H^n_\beta\right) \subset Y$. Then

$$f \in V \subset \bigcup_{\beta \in \mathscr{B}} Z_\beta = \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x}} Z_\beta \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) \neq x}} Z_\beta.$$

Since each Z_β is closed in Y by the proof of part 1, there is an open $V' \subset V$ with

$$f \in V' \subset \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x}} Z_\beta = \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x \\ x_\beta \sim x_\alpha}} Z_\beta \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x \\ x_\beta \not\sim x_\alpha}} Z_\beta$$

Hence again there exists an open subset $V'' \subset V'$ with

$$f \in V'' \subset \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x)=x \\ x_\beta \sim x_\alpha}} Z_\beta \subset \bigcup_{\substack{\beta \in P\mathscr{A} \\ \beta(x)=x \\ x_\beta \sim x_\alpha}} Z_\beta = Y_f.$$

Therefore, $U_f := V'' \subset Y$ satisfies the requirements. □

We need one further lemma:

Lemma 4.28. *The topological space Y is Hausdorff.*

Proof. Let $f, f' \in Y$ be elements such that $f \neq f'$. First suppose that $f \notin Y_{f'}$. Since $Y_{f'}$ is closed in Y by Lemma 4.27, there is an open neighbourhood U of f such that $U \cap U_{f'} \subset U \cap Y_{f'} = \emptyset$.

Next, suppose that $f \in Y_{f'}$. Lift f and f' to elements $x_\alpha, y_\beta \in \widetilde{Y}$. Assume first that $x = y$. This means that $\mathscr{P}(f) = \mathscr{P}(f')$. Since $\mathscr{P} : Y_{f'} \to \mathbb{C}H^n$ is injective, this implies that $f = f'$, contradiction. So we have $x \neq y \in \mathbb{C}H^n$. But $\mathbb{C}H^n$ is Hausdorff, so there are open subsets $(U \subset \mathbb{C}H^n, V \subset \mathbb{C}H^n)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Then $\mathscr{P}^{-1}(U) \cap \mathscr{P}^{-1}(V) = \emptyset$. □

We then obtain:

Proposition 4.29. *Y is naturally a path metric space which is piecewise isometric to $\mathbb{R}H^n$.*

Proof. From Lemma 4.27, we deduce that for each $f \in Y$ there exists an open neighborhood $f \in U_f \subset Y$ such that $\mathscr{P}$ induces a homeomorphism $Y \supset U_f \xrightarrow{\sim} \mathscr{P}(U_f) \subset \mathbb{C}H^n$. Pull back the metric on $\mathscr{P}(U_f)$ to obtain a metric on U_f. Then define a metric on Y as the largest metric which is compatible with the metric on each open set U_f and which preserves the lengths of paths. □

Proposition 4.30. *The path metric on Y descends to a path metric on $P\Gamma \setminus Y$.*

Proof. The metric on Y descends in any case to a pseudo-metric on $P\Gamma \setminus Y$, and by [Gro99, Chapter 1], this is a metric if $P\Gamma$ acts by isometries on Y with closed orbits. This is true: the fact that $P\Gamma$ acts isometrically on Y comes from the $P\Gamma$-equivariance of $\mathscr{P} : Y \to \mathbb{C}H^n$ (Lemma 4.25) together with the construction of the metric on Y (Proposition 4.29). To check that the $P\Gamma$-orbits are closed in Y, let $f \in Y$ with representative $x_\alpha \in \widetilde{Y}$. By equivariance of $p : \widetilde{Y} \to Y$, we have $p^{-1}(P\Gamma \cdot f) = P\Gamma \cdot (p^{-1}f)$, so since p is a quotient map, it suffices to show that

$$P\Gamma \cdot \left(p^{-1}f\right) = P\Gamma \cdot \cup_{x_\beta \sim x_\alpha} x_\beta = \cup_{x_\beta \sim x_\alpha} P\Gamma \cdot x_\beta$$

is closed in $\widetilde{Y}$, thus that each orbit $P\Gamma \cdot x_\beta$ is closed in $\widetilde{Y}$. Since $P\Gamma$ is discrete, it suffices to show that $P\Gamma$ acts properly on $\widetilde{Y}$. So let $Z \subset \widetilde{Y}$ be any compact set: we claim that $\{g \in P\Gamma : gZ \cap Z \neq \emptyset\}$ is a finite set. Indeed, for each $g \in P\Gamma$, one has $\widetilde{\mathscr{P}}(gZ \cap Z) \subset g\widetilde{\mathscr{P}}(Z) \cap \widetilde{\mathscr{P}}(Z)$, and the latter is non-empty for only finitely many $g \in P\Gamma$, by properness of the action of $P\Gamma$ on $\mathbb{C}H^n$.

Since the metric on Y is a path metric, so is the metric on $P\Gamma \setminus Y$ [Gro99]. □

4.3.2 *The orbifold structure on the glued space*

The next step is to prove that the glued space $P\Gamma \setminus Y$ (see Definition 4.22) is locally isometric to quotients of open sets in $\mathbb{R}H^n$ by finite groups of isometries.

Definition 4.31. Let $f \in Y$ with representative $x_\alpha \in \widetilde{Y}$. Thus, x is an element in $\mathbb{C}H^n$, and $\alpha \in P\mathscr{A}$ is the class of an anti-unitary involution such that $\alpha(x) = x$.

1. The *nodes* of f are by definition the nodes of x_α (see Definition 4.11). Thus, these are the hyperplanes $H \in \mathcal{H}(x)$, i.e. the hyperplanes $H_r \in \mathcal{H}$ defined by short roots $r \in \mathscr{R}$ such that $x \in H_r$ (equivalently, such that $h(x,r) = 0$).

2. The number of nodes of f is the cardinality of $\mathcal{H}(x)$.

3. The anti-unitary involution $\alpha \in P\mathscr{A}$ induces an involution on the set $\mathcal{H}(x)$ by Lemma 4.15. Let $H \in \mathcal{H}(x)$ be a node. We call H a *real node* of f if $\alpha(H) = H$. We call $(H, \alpha(H))$ a *pair of complex conjugate nodes* of f if $\alpha(H) \neq H$.

4. If k is the number of nodes of f, we will generally write $k = 2a + b$, with a the number of pairs of complex conjugate nodes of f, and b the number of real nodes of f.

Fix again a point $f \in Y$ and a point $x_\alpha \in \widetilde{Y}$ lying above f. Let $k = 2a + b$ be the number of nodes of f. Thus $x \in \mathbb{R}H^n_\alpha$, and there exist $r_1, \ldots, r_k \in \mathscr{R}$ such that

$$\mathcal{H}(x) = \{H_{r_1}, \ldots, H_{r_k}\}, \quad G(x) = \langle \phi_{r_1}, \ldots, \phi_{r_\ell} \rangle \cong (\mathbb{Z}/m)^k.$$

For $\beta \in P\mathscr{A}$, observe that $x_\beta \sim x_\alpha$ if and only if $\alpha \circ \beta \in G(x)$. We relabel the r_i so that they satisfy the following condition:

$$\begin{aligned} \alpha(H_{r_i}) &= H_{r_{i+1}} \text{ for } i \text{ odd and } i \leq 2a, \\ \alpha(H_{r_i}) &= H_{r_{i-1}} \text{ for } i \text{ even and } i \leq 2a, \text{ and} \\ \alpha(H_{r_i}) &= H_{r_i} \text{ for } i \in \{2a+1, \dots, k\}. \end{aligned} \tag{4.7}$$

In other words, H_{r_i} is a real root if and only if $i > 2a$, and $\left(H_{r_i}, H_{r_{i+1}}\right)$ is a pair of complex conjugate roots if and only if $i < 2a$ is odd.

Lemma 4.32. *Continue with the notation from above.*

1. *Let $\beta \in P\mathscr{A}$ be such that $x_\beta \sim x_\alpha$. Then*

$$\beta = \prod_{i=1}^{a} \left(\phi_{r_{2i-1}} \circ \phi_{r_{2i}}\right)^{j_i} \circ \prod_{i=2a+1}^{k} \phi_{r_i}^{j_i} \circ \alpha$$

for some $j_1, \dots, j_a, j_{2a+1}, \dots, j_k \in \mathbb{Z}/m$. In particular, there are m^{a+b} such β.

2. *There is an isometry $\mathbb{C}H^n \xrightarrow{\sim} \mathbb{B}^n(\mathbb{C})$ identifying x with the origin, ϕ_{r_i} with the map*

$$\mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C}), \quad (t_1, \dots, t_i, \dots, t_n) \mapsto (t_1, \dots, \zeta t_i, \dots, t_n),$$

and α with the map defined by

$$t_i \mapsto \begin{cases} \bar{t}_{i+1} & \text{for } i \text{ odd and } i \leq 2a \\ \bar{t}_{i-1} & \text{for } i \text{ even and } i \leq 2a \\ \bar{t}_i & \text{for } i > 2a. \end{cases} \tag{4.8}$$

Proof. 1. This follows readily from Proposition 4.18.

2. Since the H_{r_i} are orthogonal by Condition 4.9, and their intersection contains x, we can find coordinates $t_1, \dots, t_{n+1}$ on V that induce an identification $(V, h) \cong \mathbb{C}^{n,1} := \left(\mathbb{C}^{n+1}, H\right)$ with $H(x,x) = |x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2$, in such a way that $H_{r_i} \subset V$ is identified with the hyperplane $\{t_i = 0\} \subset \mathbb{C}^{n+1}$ and $x \in \cap_i H_{r_i}$ with the point $(0, 0, \dots, 0, 1)$. We will do this in the following way. Define

$$T = \langle x \rangle \oplus \langle r_1 \rangle \oplus \cdots \oplus \langle r_k \rangle \subset V, \quad W = T^\perp = \{w \in V \mid h(w,t) = 0 \ \ \forall t \in T\}.$$

For each $i \in I = \{1, \dots, k\}$, we have $\alpha(r_i) = \lambda_i \cdot r_{\alpha(i)}$ for some $\lambda_i \in K$ (see Lemma 4.15, Definition 4.17, and Lemmas 4.13 and 4.12). Observe that $\alpha(W) = W$. Since $W \subset \langle x \rangle^\perp$, the hermitian space $(W, h|_W)$ is positive definite. Let $\{w_1, \dots, w_{n-k}\} \subset W$ be an orthonormal basis such that $\alpha(w_i) = w_i$, which exists by the elementary

Lemma 4.33. *Let (W, h) a non-degenerate hermitian vector space of dimension $n \geq 1$ and let $\alpha \colon W \to W$ be an anti-linear involution with $h(\alpha(x), \alpha(y)) = \overline{h(x,y)}$ for $x, y \in W$. For each positive integer $m \leq n$, there exists a linearly independent set $\{w_i\}_{i=1}^m \subset W$ such that $h(w_i, w_j) = \pm\delta_{ij}$, and such that $\alpha(w_i) = w_i$ for each $i = 1, \dots, m$.* □

Let $\{e_i\}_{i=1}^{n+1}$ be the standard basis of $\mathbb{C}^{n+1}$, and define a $\mathbb{C}$-linear isomorphism

$$\Phi \colon V \xrightarrow{\sim} \mathbb{C}^{n+1}, \quad \left(\frac{x}{h(x,x)} \mapsto e_{n+1}, \quad r_i \mapsto e_i, \quad w_i \mapsto e_i \right). \tag{4.9}$$

By (4.7), we have that $\alpha(r_i) = \lambda_i \cdot r_{i+1}$ for i odd and $i \leq 2a$, that $\alpha(r_i) = \lambda_i \cdot r_{i-1}$ for i even and $i \leq 2a$, and that $\alpha(r_i) = \lambda_i \cdot r_i$ for $i > 2a$. We conclude that the anti-linear involution on $\mathbb{C}^{n+1}$ induced by α and (4.9) corresponds to the matrix

$$\alpha = \begin{pmatrix} 0 & \alpha_1 & \dots & 0 & \dots & & \dots & 0 \\ \alpha_2 & 0 & 0 & 0 & & \dots & & \vdots \\ 0 & 0 & 0 & \alpha_4 & & & & \\ 0 & 0 & \alpha_3 & 0 & & & & \\ \vdots & 0 & \ddots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & 0 & \vdots & & & \alpha_n & 0 \\ 0 & 0 & \dots & 0 & & \dots & 0 & \alpha_{n+1} \end{pmatrix}$$

where each α_i is an anti-linear involution $\mathbb{C} \to \mathbb{C}$, and $\alpha_i = \alpha_{i+1}$ for $i < 2a$ odd. If $\alpha_i(1) = \mu_i \in \mathbb{C}^*$, then $\mu_i^{-1} \cdot \alpha_i = \mathrm{conj} \colon \mathbb{C} \to \mathbb{C}$ (complex conjugation). Since $|\mu_i| = 1$, there exists $\rho_i \in \mathbb{C}$ such that $\mu_i = \overline{\rho_i}/\rho_i$ and $|\rho_i| = 1$. This gives $\mu_i^{-1} \cdot \alpha_i = \rho_i \cdot \alpha_i \cdot \rho_i^{-1} = \mathrm{conj} \colon \mathbb{C} \to \mathbb{C}$. The composition

$$V \xrightarrow{\Phi} \mathbb{C}^{n+1} \xrightarrow{\mathrm{diag}(\rho_i)} \mathbb{C}^{n+1}$$

induces an isomorphism $\mathbb{C}H^n \cong \mathbb{B}^n(\mathbb{C})$ with the required properties. □

Definition 4.34.
- Define $A_f = \mathrm{Stab}_{P\Gamma}(f)$ to be the subgroup of $P\Gamma$ fixing $f \in Y$. This contains the group $G(x) \cong (\mathbb{Z}/m)^k$.
- Define B_f as the subgroup of $G(x)$ generated by the order m complex reflections associated to the real nodes of f, rather than all the nodes. Hence $B_f = \langle \phi_{r_i} \rangle_{i>2a} \cong (\mathbb{Z}/m)^b$.

Recall the quotient map $p \colon \widetilde{Y} \to Y$, the definition (4.6) of Y_f, and Lemma 4.27.

Lemma 4.35. *The stabilizer A_f of $f \in Y$ preserves the subset $Y_f \subset Y$.*

Proof. Let $\psi \in A_f$, with $f = p(x_\alpha) \in Y$, $x \in \cap_i H_{r_i}$. Then $\psi(x)_{\psi\alpha\psi^{-1}} \sim x_\alpha$. Now let $p(y_\beta) \in Y_f$. Then $\beta(x) = x$ and $x_\alpha \sim x_\beta$. Hence $x_\alpha \sim \psi \cdot x_\alpha \sim \psi \cdot x_\beta = \psi(x)_{\psi\beta\psi^{-1}}$. This implies that $\psi\beta\psi^{-1} \circ \alpha \in G(x)$, so that $p\left(\psi(y)_{\psi\beta\psi^{-1}}\right) \in Y_f$. □

We also need the following lemma. Write $m = 2^a k$ with $k \neq 0 \bmod 2$.

Lemma 4.36. *Let $T = \{t \in \mathbb{C} : t^m \in \mathbb{R}\}$. Then $G = \langle \zeta_m \rangle$ acts on T by multiplication. Each element in T/G has a unique representative of the form $\zeta_{2^{a+1}}^{\epsilon} \cdot r$, $r \geq 0$ and $\epsilon \in \{0, 1\}$.*

Proof. Therefore, we have $a \geq 1$. Next, observe that $t = r\zeta_{2m}^j$ for some $j \in \mathbb{Z}$ and $r \in \mathbb{R}$ if and only if $t^m \in \mathbb{R}$. One easily shows that since $\gcd(2,k) = 1$, we have $\zeta_{2^{a+1}} \cdot \zeta_{2^a k} = (\zeta_{2^{a+1}k})^{k+2}$. Raising both sides to the power $b = (k+2)^{-1} \in (\mathbb{Z}/m)^*$ gives $\zeta_{2m} = \zeta_{2^{a+1}}^b \cdot \zeta_m^b$. Consequently, $t^m \in \mathbb{R}$ if and only if $t = r \cdot \zeta_{2^{a+1}}^{bj} \cdot \zeta_m^{bj}$ for some $r \in \mathbb{R}$. Finally, $\zeta_{2^{a+1}}^u \cdot \zeta_{2^a}^v = \zeta_{2^{a+1}}^{u+2v}$ hence $\langle \zeta_{2^{a+1}} \rangle / \langle \zeta_{2^a} \rangle \cong \mathbb{Z}/2$. □

We obtain the key to Theorem 4.24.

Proposition 4.37. *Keep the above notations, and consider the set $Y_f \subset Y$ (see (4.6)).*

1. *If f has no nodes, then $G(x) = B_f$ is trivial, and $Y_f = \mathbb{R}H_\alpha^n \cong \mathbb{B}^n(\mathbb{R})$.*
2. *If f has only real nodes, then $B_f \setminus Y_f$ is isometric to $\mathbb{B}^n(\mathbb{R})$.*
3. *If f has a pairs of complex conjugate nodes ($k = 2a$), and no other nodes, then $B_f \setminus Y_f = Y_f$ is the union of m^a copies of $\mathbb{B}^n(\mathbb{R})$, any two of which meet along a $\mathbb{B}^{2c}(\mathbb{R})$ for some integer c with $0 \leq c \leq a$.*
4. *If f has $2a$ complex conjugate nodes and b real nodes, then there is an isometry between $B_f \setminus Y_f$ and the union of m^a copies of $\mathbb{B}^n(\mathbb{R})$ identified along common $\mathbb{B}^{2c}(\mathbb{R})'s$, that is, the set Y_f of case 3 above.*

5. *In each case, A_f acts transitively on the indicated copies of $\mathbb{B}^n(\mathbb{R})$. If $\mathbb{B}^n(\mathbb{R})$ is any one of them, and $\Gamma_f = (A_f/B_f)_{\mathbb{B}^n(\mathbb{R})}$ its stabilizer, then the natural map*

$$\Gamma_f \setminus \mathbb{B}^n(\mathbb{R}) \to \big(A_f/B_f\big) \setminus \big(B_f \setminus Y_f\big) = A_f \setminus Y_f$$

is an isometry of path metrics.

Proof. 1. This is clear.

2. Suppose then that f has k real nodes. Then in the local coordinates t_i of Lemma 4.32.2, we have that $\alpha : \mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C})$ is defined by $\alpha(t_i) = \bar{t}_i$. Part 1 of the same lemma shows that any $\beta \in P\mathscr{A}$ fixing x such that $x_\alpha \sim x_\beta$ is of the form

$$\mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C}), \quad (t_1, \ldots, t_i, \ldots, t_n) \mapsto (\bar{t}_1\zeta^{j_1}, \ldots, \bar{t}_k\zeta^{j_k}, \bar{t}_{k+1}, \ldots, \bar{t}_n).$$

Since f has k real nodes and no complex conjugate nodes, we have (writing $j = (j_1, \ldots, j_k)$ and $\alpha_j = \prod_{i=1}^k \phi_{r_i}^{j_i} \circ \alpha$):

$$Y_f \cong \bigcup_{j_1,\ldots,j_k=1}^{m} \mathbb{R}H^n_{\alpha_j} \cong \{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_1^m, \ldots, t_k^m, t_{k+1}, \ldots, t_n \in \mathbb{R}\}\,.$$

Each of the 2^k subsets

$$K_{f,\epsilon_1,\ldots,\epsilon_k} := \Big\{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : \zeta_{2^{a+1}}^{-\epsilon_1} t_1, \ldots, \zeta_{2^{a+1}}^{-\epsilon_k} t_k \in \mathbb{R}_{\geq 0} \text{ and } t_{k+1}, \ldots, t_n \in \mathbb{R}\Big\}\,,$$

indexed by $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}$, is isometric to the closed region in $\mathbb{B}^n(\mathbb{R})$ bounded by k mutually orthogonal hyperplanes. By Lemma 4.36, their union U is a fundamental domain for B_f, in the sense that it maps homeomorphically and piecewise-isometrically onto $B_f \setminus Y_f$. Under its path metric, $U = \cup K_{f,\epsilon_1,\ldots,\epsilon_k}$ is isometric to $\mathbb{B}^n(\mathbb{R})$ by the following map:

$$U \to \mathbb{B}^n(\mathbb{R}), \quad (t_1, \ldots, t_k) \mapsto \big((-\zeta_{2^{a+1}})^{-\epsilon_1} t_1, \ldots, (-\zeta_{2^{a+1}})^{-\epsilon_k} t_k, t_{k+1}, \ldots, t_n\big)\,.$$

This identifies $B_f \setminus Y_f$ with the standard $\mathbb{B}^n(\mathbb{R}) \subset \mathbb{B}^n(\mathbb{C})$.

3. Now suppose f has $k = 2a$ nodes $H_{r_1}, \ldots, H_{r_{2a}}$. There are now m^a anti-isometric involutions α_{j_i} fixing x and such that $x_{\alpha_{j_i}} \sim x_\alpha$: they are given in the coordinates t_i as follows, taking $j = (j_1, \ldots, j_a) \in (\mathbb{Z}/m)^a$:

$$\alpha_j : (t_1, \ldots, t_n) \mapsto (\bar{t}_2\zeta^{j_1}, \bar{t}_1\zeta^{j_1}, \ldots, \bar{t}_{2a}\zeta^{j_a}, \bar{t}_{2a-1}\zeta^{j_a}, \bar{t}_{2a+1}, \ldots, \bar{t}_n).$$

So any fixed-point set $\mathbb{R}H^n_{\alpha_j}$ is identified with

$$\mathbb{B}^n(\mathbb{R})_{\alpha_j} := \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i = \bar{t}_{i-1}\zeta^{j_i} \text{ for } 1 \le i \le 2a \text{ even}, t_i \in \mathbb{R} \text{ for } i > 2a\right\}.$$

All these m^a copies of $\mathbb{B}^n(\mathbb{R})$ meet at the origin of $\mathbb{B}^n(\mathbb{C})$; in fact, for $j \neq j'$, the space $\mathbb{B}^n(\mathbb{R})_{\alpha_j}$ meets the space $\mathbb{B}^n(\mathbb{R})_{\alpha_{j'}}$ in a $\mathbb{B}^{2c}(\mathbb{R})$ if c is the number of pairs (j_i, j_i') with $j_i = j_i'$.

4. Now we treat the general case. In the local coordinates t_i, any anti-unitary involutions fixing x and equivalent to α is of the form

$$\begin{aligned}\alpha_j :&(t_1, \dots, t_n) \mapsto \\ &(\bar{t}_2\zeta^{j_1}, \bar{t}_1\zeta^{j_1}, \dots, \bar{t}_{2a}\zeta^{j_a}, \bar{t}_{2a-1}\zeta^{j_a}, \bar{t}_{2a+1}\zeta^{j_{2a+1}}, \dots, \bar{t}_k\zeta^{j_k}, \bar{t}_{k+1}, \dots, \bar{t}_n)\end{aligned}$$

for some $j = (j_1, \dots, j_a, j_{2a+1}, \dots, j_k) \in (\mathbb{Z}/m)^{a+b}$. We now have $B_f \cong (\mathbb{Z}/m)^b$ acting by multiplying the t_i for $2a+1 \le i \le k$ by powers of ζ, and there are m^{a+b} anti-unitary involutions α_j. We have

$$Y_f \cong \bigcup_{j_1,\dots,j_k=1}^{m} \mathbb{R}H^n_{\alpha_j} \cong \\ \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_2^m = \bar{t}_1^m, \dots, t_{2a}^m = \bar{t}_{2a-1}^m, t_{2a+1}^m, \dots, t_k^m, t_{k+1}, \dots, t_n \in \mathbb{R}\right\}.$$

We look at subsets $K_{f,\epsilon_1,\dots,\epsilon_k} \subset Y_f$ again, this time defined as

$$\begin{aligned}K_{f,\epsilon} =&K_{f,\epsilon_1,\dots,\epsilon_k} \\ &= \{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid \\ &\quad t_i^m = \bar{t}_{i-1}^m \ i \le 2a \text{ even}, \zeta_{2^{a+1}}^{-\epsilon_i} t_i \in \mathbb{R}_{\ge 0} \ 2a < i \le k, t_i \in \mathbb{R}, i > k\Big\}.\end{aligned}$$

As before, we have that the natural map $U := \bigcup_\epsilon K_{f,\epsilon} \to B_f \setminus Y_f$ is an isometry. Define

$$\widetilde{Y}_f = \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i^m = \bar{t}_{i-1}^m \text{ for } i \le 2a \text{ even}, t_i \in \mathbb{R}, \text{ for } i > 2a\right\}.$$

Under its path metric, $U = \cup_\epsilon K_{f,\epsilon_1,\ldots,\epsilon_k}$ is isometric to $\widetilde{Y}_f$ by the following map:

$$U \to \widetilde{Y}_f, \quad (t_1, \ldots, t_k) \mapsto \\ \left(t_1, \ldots, t_{2a}, (-\zeta_{2^{a+1}})^{-\epsilon_1} t_{2a+1}, \ldots, (-\zeta_{2^{a+1}})^{-\epsilon_k} t_k, t_{k+1}, \ldots, t_n\right).$$

Hence $B_f \setminus Y_f \cong \widetilde{Y}_f$; but since $\widetilde{Y}_f$ is what Y_f was in case 3, we are done.

5. The transitivity of A_f on the copies of $\mathbb{B}^n(\mathbb{R})$ follows from the fact that $G(x) \subset A_f$ contains transformations multiplying $t_1, \ldots, t_{2a}$ by powers of ζ, hence $t_i \mapsto \zeta^u t_i, t_{i-1} \mapsto t_{i-1}$ maps those t_{i-1}, t_i with $t_i = \bar{t}_{i-1}\zeta^{j_i}$ to those t_{i-1}, t_i with $t_i = \bar{t}_{i-1}\zeta^{j_1+u}$. So if B is any one of the copies of $\mathbb{B}^n(\mathbb{R})$, and $G = (A_f/B_f)_H$ is its stabilizer, then it remains to prove that $G \setminus B \to A_f \setminus Y_f$ is an isometry. Surjectivity follows from the transitivity of A_f on the $\mathbb{B}^n(\mathbb{R})'s$. It is a piecewise isometry so we only need to prove injectivity. This will follow from an elementary lemma.

Lemma 4.38. *Let X be a set on which a group G acts, let Y and I be sets, and let $\{\phi_i : Y \hookrightarrow X\}_{i \in I}$ be a set of embeddings. Write $Y_i = \phi_i(Y)$ and suppose that $X = \cup_i Y_i$. Fix $0 \in I$. Let $H \subset G$ be the stabilizer of Y_0. Suppose that for all $y \in X$, the stabilizer of y in G acts transitively on the sets Y_i containing y. Then $H \setminus Y_0 \to G \setminus X$ is injective.*

Proof. Let $x, y \in Y_0$ and $g \in G$ such that $g \cdot x = y$. Then $y = gx \in gY_0$. Since also $y \in Y_0$, there is an element $h \in \mathrm{Stab}_G(y)$ such that $hgY_0 = Y_0$ and $hg(x) = h(y) = y$. Let $f = hg$; then $f \in H$ and $f \cdot x = y$, which proves what we want. □

Now let us use the lemma: suppose that $y \in B_f \setminus Y_f$. We need to prove that $\mathrm{Stab}_{A_f/B_f}(y)$ acts transtivitely on the copies of $\mathbb{B}^n(\mathbb{R})$ containing y. There exists

$$j = (j_1, \ldots, j_a, j_{2a+1}, \ldots, j_k) \in (\mathbb{Z}/m)^{a+b}$$

such that $y = (t_1, \ldots, t_n)$ with $t_i = \bar{t}_{i-1}\zeta^{j_i}$ for $i \leq 2a$ even, $t_i = \bar{t}_{i-1}\zeta^{j_i}$ for $2a < i \leq k$, and $t_i \in \mathbb{R}$ for $i > k$. If all t_i are non-zero, then $y \in \cup_{j'} \mathbb{R}H^n_{\alpha_{j'}}$ is only contained in $\mathbb{R}H^n_{\alpha_j}$, so there is nothing to prove. Let us suppose that $t_1 = t_2 = 0$ and the other t_i are non-zero. Then y is contained in all the $\mathbb{R}H^n_{\alpha_{j'}}$ with $j'_i = j_i$ for $i \geq 2$; there are m of them. The stabilizer of y multiplies t_1 and t_2 by powers of ζ and leaves the other t_i invariant; it acts transitively on the $\mathbb{R}H^n_{\alpha_{j'}}$ containing y for if $t_2 = \bar{t}_1\zeta^{j'_1}$ then $\zeta^{(j''_1 - j'_1)} t_2 = \bar{t}_1 \zeta^{j''_1}$. The general case is similar. □

As indicated above, we can now prove Theorem 4.24.

Proof of Theorem 4.24. 1. The path metric on $P\Gamma \setminus Y$ is given by Proposition 4.30. Note that the map $\mathscr{P} : Y \to \mathbb{C}H^n$ is a local embedding by Lemma 4.27, which was used to define the metric on Y (Proposition 4.29). Thus, almost by definition, $\mathscr{P}$ is a local isometry. For each $f \in Y$ we can find a $P\Gamma_f$-invariant open neighborhood $U_f \subset Y_f \subset Y$ such that $P\Gamma_f \setminus U_f \subset P\Gamma \setminus Y$, with U_f mapping bijectively onto an open subset V_f in the closed subset $\mathscr{P}(Y_f) = \cup_i \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n$. By $P\Gamma$-equivariance of $\mathscr{P}$, the set V_f is $P\Gamma_f$-invariant, and we have $P\Gamma_f \setminus V_f \subset P\Gamma \setminus \mathbb{C}H^n$. Thus

$$\overline{\mathscr{P}} : P\Gamma \setminus Y \to P\Gamma \setminus \mathbb{C}H^n$$

is also a local isometry.

2. Note that the map $\mathscr{P} : Y \to \mathbb{C}H^n$ is proper because any compact set in $\mathbb{C}H^n$ meets only finitely many $\mathbb{R}H^{n'}_\alpha s$, $\alpha \in P\mathscr{A}$ (Lemma 4.26), and $\mathscr{P}$ carries each $H_\alpha = p\left(\mathbb{R}H^n_\alpha\right)$ homeomorphically onto $\mathbb{R}H^n_\alpha$. Since $P\Gamma \setminus \mathbb{C}H^n$ is complete, the space $P\Gamma \setminus Y$ is complete as well.

Finally, let $[f] \in P\Gamma \setminus Y$ be the image of $f \in Y$. Then $[f]$ has an open neighborhood isometric to the quotient of an open set W in $\mathbb{R}H^n$ by a finite group of isometries Γ_f. Indeed, take $Y_f \subset Y$ as in Equation (4.6), and $f \in U_f \subset Y_f$ as in Lemma 4.27.2. We let $A_f = P\Gamma_f$ be the stabilizer of f in $P\Gamma$ as before, and take an A_f-equivariant open neighborhood $V_f \subset U_f$ such that $A_f \setminus V_f \subset P\Gamma \setminus Y$. By Proposition 4.37.5, we know that $A_f \setminus Y_f$ is isometric to $\Gamma_f \setminus \mathbb{R}H^n$ for some finite group of isometries of $\mathbb{R}H^n$. This implies that $A_f \setminus V_f$ is isometric to some open set W' in $\Gamma_f \setminus \mathbb{R}H^n$. Take $W \subset \mathbb{R}H^n$ to be the preimage of W'.

Claim: For any path metric space X locally isometric to quotients of $\mathbb{R}H^n$ by finite groups of isometries, there is a unique real-hyperbolic orbifold structure on X whose path metric is the given one.

Proof of the Claim: If U and U' are connected open subsets of $\mathbb{R}H^n$ and Γ and Γ' finite groups of isometries of $\mathbb{R}H^n$ preserving U and U' respectively, then any isometry $\bar{\phi} : \Gamma \setminus U \to \Gamma' \setminus U'$ extends to an isometry $\phi : \mathbb{R}H^n \to \mathbb{R}H^n$ such that $\phi(U) = U'$ and $\phi\Gamma\phi^{-1} = \Gamma' \subset \mathrm{Isom}(\mathbb{R}H^n)$.

We conclude that $P\Gamma \setminus Y$ is naturally a real hyperbolic orbifold.

3. Let us show that

$$O := \coprod_{\alpha \in C\mathscr{A}} \left[P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})\right] \subset P\Gamma \setminus Y$$

as hyperbolic orbifolds. It suffices to show the following

Claim: For those $f = p(x_\alpha) \in Y$ that have no nodes, the stabilizer $A_f = P\Gamma_f \subset P\Gamma$ of $f \in Y$ and the stabilizer $P\Gamma_{\alpha,x} \subset P\Gamma_\alpha$ of x in $\mathbb{R}H^n_\alpha$ agree as subgroups of $P\Gamma$.

Proof of the Claim: To prove that $A_f = P\Gamma_{\alpha,x}$, we first observe that $p : \widetilde{Y} \to Y$ induces an isomorphism between $P\Gamma_{x_\alpha}$, the stabilizer of $x_\alpha \in \widetilde{Y}$ and $P\Gamma_f$, the stabilizer of $f = [x, \alpha] \in Y$. So it suffices to show that $P\Gamma_{x_\alpha} = P\Gamma_{\alpha,x}$. For this we use that the normalizer $N_{P\Gamma}(\alpha)$ and the stabilizer $P\Gamma_\alpha \subset P\Gamma$ of α in $P\Gamma$ are equal, which implies that $P\Gamma_{\alpha,x} = P\Gamma_{x_\alpha}$ because

$$\begin{aligned}\{g \in P\Gamma_\alpha : gx = x\} &= \{g \in N_{P\Gamma}(\alpha) : gx = x\} \\ &= \left\{g \in P\Gamma : g \cdot x_\alpha = (g(x), g\alpha g^{-1}) = x_\alpha\right\}.\end{aligned}$$

So the claim is proved. Part 3 of the theorem can be deduced from it as follows. Let $f = p(x_\alpha) \in Y$ have no nodes. We have $Y_f = \mathbb{R}H^n_\alpha$, hence

$$A_f \setminus \mathbb{R}H^n_\alpha = A_f \setminus Y_f = \Gamma_f \setminus \mathbb{R}H^n \quad \text{with} \quad \Gamma_f = A_f \setminus B_f = A_f.$$

By construction, an orbifold chart of the glued space $P\Gamma \setminus Y$ is given by

$$W \to A_f \setminus W \subset P\Gamma_\alpha \setminus \mathbb{R}H^n_\alpha \subset Y$$

for an invariant open subset W of $\mathbb{R}H^n_\alpha$ containing x. Because $A_f = P\Gamma_{\alpha,x}$ by the claim, this is also an orbifold chart for O at the point x_α.

4. The real-hyperbolic orbifold $P\Gamma \setminus Y$ is complete by part 1, so the uniformization of the connected components of $P\Gamma \setminus Y$ follows from the Ehresmann–Thurston uniformization theorem for (G, X)-orbifolds, see [Thu80, Proposition 13.3.2]. This concludes the proof of Theorem 4.24, and thereby also of Theorem 4.2. □

4.4 UNITARY SHIMURA VARIETIES

The goal of this section is to prove Proposition 4.45, which describes (in case the signature of h is hyperbolic for one place of K and definite for all others) our ball quotient $P\Gamma \setminus \mathbb{C}H^n$ in terms of moduli of abelian varieties with $\mathcal{O}_K$-action of hyperbolic signature, and Proposition 4.48, which interprets the divisor $P\Gamma \setminus \mathscr{H}$

as the locus of abelian varieties A that admit a homomorphism $\mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$. This has two applications:

1. Consider a relative uniform cyclic cover (see e.g. [AV04])

$$\mathscr{X} \to P \to S,$$

where $P = \mathbb{P}^1_S$ (resp. $\mathbb{P}^3_S$), the fibers of $\mathscr{X} \to S$ are curves (resp. threefolds with $\mathrm{H}^{0,3} = 0$) and the induced hermitian form on middle cohomology satisfies the above signature condition. Since the image $\mathfrak{I} \subset P\Gamma \setminus \mathbb{C}H^n$ of the period map $S(\mathbb{C}) \to P\Gamma \setminus \mathbb{C}H^n$ is contained in the locus of abelian varieties whose theta divisor is irreducible, one has $\mathfrak{I} \subset P\Gamma \setminus (\mathbb{C}H^n - \mathscr{H})$.

2. If the different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by some $\eta \in \mathcal{O}_K - \mathcal{O}_F$ with $\eta^2 \in \mathcal{O}_F$, then the hyperplanes in the arrangement $\mathscr{H} \subset \mathbb{C}H^n$ are orthogonal along their intersection (see Theorem 4.50).

4.4.1 *Alternating and hermitian forms on the lattice*

The goal of this subsection is to prove two lemmas. They will later be used to show that $P\Gamma \setminus \mathbb{C}H^n$ is a moduli space of abelian varieties, and to give a modular interpretation of the divisor $P\Gamma \setminus \mathscr{H} \subset P\Gamma \setminus \mathbb{C}H^n$.

We continue with the notation of Section 4.2.1. In particular, Λ is a free $\mathcal{O}_K$-module of rank $n+1$.

Lemma 4.39. *The assignment $T \mapsto \mathrm{Tr}_{K/\mathbb{Q}} \circ T$ defines a bijection between:*

1. *The set of skew-hermitian forms $T : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to K$.*
2. *The set of alternating forms $E : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to \mathbb{Q}$ such that $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$.*

Under this correspondence, $T(\Lambda, \Lambda) \subset \mathfrak{D}_K^{-1}$ if and only if $E(\Lambda, \Lambda) \subset \mathbb{Z}$.

Proof. Let

$$T \colon \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to K$$

be as in 1. Define $E_T = \mathrm{Tr}_{K/\mathbb{Q}} \circ T$. Since T is skew-hermitian, we have, for each $x, y \in \Lambda_\mathbb{Q}$, that

$$\mathrm{Tr}_{K/\mathbb{Q}} T(x,y) = -\mathrm{Tr}_{K/\mathbb{Q}} \overline{T(y,x)}.$$

Since $K/\mathbb{Q}$ is separable, for any $x \in K$, we have [Ste08, (7-1)]:

$$\mathrm{Tr}_{K/\mathbb{Q}}(x) = \sum_{1 \le i \le g} \left(\tau_i(x) + \tau_i\sigma(x)\right).$$

Thus, we have $\mathrm{Tr}_{K/\mathbb{Q}}(\sigma(x)) = \mathrm{Tr}_{K/\mathbb{Q}}(x)$, so that $E_T(x,y) = -E_T(y,x)$ for any $x, y \in \Lambda_{\mathbb{Q}}$. The property in 2 is easily checked.

Conversely, let $E : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ be as in 2. Choose a basis $\{b_1, \ldots, b_{n+1}\} \subset \Lambda$ for Λ over $\mathcal{O}_K$. Define Q to be the induced map $K^{n+1} \times K^{n+1} \to \mathbb{Q}$ and consider the map $K \to \mathbb{Q}$, $a \mapsto Q(a \cdot e_i, e_j)$. Since the trace pairing

$$K \times K \to \mathbb{Q}, \quad (x,y) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(xy)$$

is non-degenerate [Sta18, Tag 0BIE], there is a unique $t_{ij} \in K$ such that $Q(a \cdot e_i, e_j) = \mathrm{Tr}_{K/\mathbb{Q}}(a \cdot t_{ij})$ for every $a \in K$. This gives a matrix $(t_{ij})_{ij} \in M_{n+1}(K)$ such that $\sigma(t_{ij}) = -t_{ji}$, and the basis $\{b_i\}$ induces a skew-hermitian form $T_E : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to K$.

The last claim from the definition of $\mathfrak{D}_K^{-1} \subset K$ as the trace dual of $\mathcal{O}_K$, see [Ser79, Chapter III, §3]. $\square$

Examples 4.40. 1. Suppose $K = \mathbb{Q}(\sqrt{\Delta})$ is imaginary quadratic with discriminant Δ and non-trivial Galois automorphism $a \mapsto a^\sigma$. Let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be an alternating form with $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$. The form $T : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1} = (\sqrt{\Delta})^{-1}$ is defined as

$$T(x,y) = \frac{E(\sqrt{\Delta} \cdot x, y) + E(x,y)\sqrt{\Delta}}{2\sqrt{\Delta}}.$$

2. Let $K = \mathbb{Q}(\zeta)$ where $\zeta = \zeta_p = e^{2\pi i/p} \in \mathbb{C}$ for some prime number $p > 2$. Let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be an alternating form with $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$. Then $\mathfrak{D}_K = \left(p/(\zeta - \zeta^{-1})\right)$ and

$$T : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}, \quad T(x,y) = \frac{1}{p} \sum_{j=0}^{p-1} \zeta^j E\left(x, \zeta^j \cdot y\right).$$

Now consider a corresponding pair

$$(E\colon \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}, \quad T : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to K)$$

as in Lemma 4.39, and suppose that E is non-degenerate. Let $\varphi : K \to \mathbb{C}$ be an embedding. Define a skew-hermitian form T^φ as

$$
\begin{aligned}
&T^\varphi := \Lambda \otimes_{\mathcal{O}_K,\varphi} \mathbb{C} \times \Lambda \otimes_{\mathcal{O}_K,\varphi} \mathbb{C} \to \mathbb{C}, \\
&T^\varphi(\sum_i x_i \otimes \lambda_i, \sum_j y_j \otimes \mu_j) = \sum_{ij} \lambda_i \overline{\mu_j} \cdot \varphi\left(T(x_i, y_j)\right).
\end{aligned}
$$

On $\Lambda_{\mathbb{C}}$, we also have the skew-hermitian form $A(x,y) = E_{\mathbb{C}}(x, \bar{y})$. The composition

$$
(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\varphi \to \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \to \Lambda \otimes_{\mathcal{O}_K,\varphi} \mathbb{C}
$$

is an isomorphism. Define A^φ to be the restriction of A to the subspace $(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\varphi = \Lambda \otimes_{\mathcal{O}_K,\varphi} \mathbb{C} \subset \Lambda_{\mathbb{C}}$. Note that

$$
\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \cong \oplus_{\phi: K \to \mathbb{C}} (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\phi .
$$

For $x \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, let x^ϕ be the image of x under $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \to (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\phi$.

Lemma 4.41. *Let $\varphi\colon K \to \mathbb{C}$ be an embedding. We have an equality of skew-hermitian forms:*

$$
T^\varphi = A^\varphi \colon\ (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\varphi \times (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\varphi \to \mathbb{C}.
$$

More precisely, we have $A(x,y) = \sum_{\phi: K \to \mathbb{C}} T^\phi(x^\phi, y^\phi)$ for every $x, y \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$.

Proof. Write $V = \Lambda_{\mathbb{Q}}$. The lemma follows from the fact that the following diagram commutes:

$$
\begin{tikzcd}
V \times V \arrow[r, "T"] \arrow[d, hook] & K \arrow[r, "\mathrm{Tr}_{K/\mathbb{Q}}"] \arrow[d, hook] & \mathbb{Q} \arrow[dd, hook] \\
V \otimes_{\mathbb{Q}} \mathbb{C} \times V \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "T_{\mathbb{C}}"] \arrow[d, equal] \arrow[drr, "A(x{,}y)"] & K \otimes_{\mathbb{Q}} \mathbb{C} \arrow[d, equal] & \\
\oplus_\phi (V \otimes_{\mathbb{Q}} \mathbb{C})_\phi \times (V \otimes_{\mathbb{Q}} \mathbb{C})_\phi \arrow[r, "\oplus T^\phi"] & \oplus_\phi \mathbb{C}_\phi \arrow[r, "\Sigma"'] & \mathbb{C}.
\end{tikzcd}
$$

Here, ϕ ranges over the set of embeddings $K \to \mathbb{C}$, $\mathbb{C}_\phi$ is the K-module $\mathbb{C}$ where K acts via ϕ, and

$$
T_{\mathbb{C}} : V \otimes_{\mathbb{Q}} \mathbb{C} \times V \otimes_{\mathbb{Q}} \mathbb{C} \to K \otimes_{\mathbb{Q}} \mathbb{C}
$$

is the map that sends $(v \otimes \lambda, x \otimes \mu)$ to $\lambda \bar{\mu} T(v, w)$. ☐

4.4.2 *Moduli of abelian varieties acted upon by the ring of integers of a CM field*

Notation 4.42. In the rest of Section 4.4, we fix:

1. a non-degenerate hermitian form $\mathfrak{h} : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}$; and
2. an element $\xi \in \mathfrak{D}_K^{-1}$ such that $\sigma(\xi) = -\xi$ and $\Im(\tau_i(\xi)) < 0$ for $1 \leq i \leq g$ and write $\eta = \xi^{-1}$. Here, the embeddings $\tau_i \colon K \to \mathbb{C}$ are those introduced in (4.1).

These data define a skew-hermitian form

$$T \colon \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}, \quad T := \xi \cdot \mathfrak{h}.$$

The form T is in turn attached to a symplectic form (see Lemma 4.39)

$$E : \Lambda \times \Lambda \to \mathbb{Z} \quad \text{such that} \quad E(ax, y) = E(x, a^\sigma y) \quad \text{for all} \quad a \in \mathcal{O}_K, \; x, y \in \Lambda.$$

Write $V_i = \Lambda_\mathbb{Q} \otimes_{K,\tau_i} \mathbb{C}$ and define

$$\mathfrak{h}^{\tau_i} : V_i \times V_i \to \mathbb{C}$$

to be the hermitian form restricting to $\tau_i \circ \mathfrak{h}$ on Λ. Let (r_i, s_i) be the signature of the hermitian form $\mathfrak{h}^{\tau_i}$.

Let A be a complex abelian variety, ι a homomorphism $\mathcal{O}_K \to \mathrm{End}(A)$, and λ a polarization $A \to A^\vee$, satisfying the following (c.f. [KR14, Part I, §2.1]):

Conditions 4.43. *1. We have $\iota(a)^\dagger = i(a^\sigma)$ for the Rosati involution*

$$\dagger \colon \mathrm{End}(A)_\mathbb{Q} \to \mathrm{End}(A)_\mathbb{Q}, \qquad \textit{and}$$

2. $\mathrm{char}(t, \iota(a) | \mathrm{Lie}(A)) = \prod_{\nu=1}^{g} (t - a^{\tau_i})^{r_i} \cdot (t - a^{\tau_i \sigma})^{s_i} \; \in \; \mathbb{C}[t]$
(the characteristic polynomial of $\iota(a)$).

Note that $\dim A = g(n+1)$. Define $E_A : \mathrm{H}_1(A, \mathbb{Z}) \times \mathrm{H}_1(A, \mathbb{Z}) \to \mathbb{Z}$ to be the alternating form corresponding to λ. The condition on the Rosati involution implies that $E_A(\iota(a)x, y) = E_A(x, \iota(a^\sigma)y)$ for $x, y \in \mathrm{H}_1(A, \mathbb{Q})$. Define a hermitian form $\mathfrak{h}_A$ on the $\mathcal{O}_K$-module $\mathrm{H}_1(A, \mathbb{Z})$ as follows:

$$\mathfrak{h}_A = \eta T_A \colon \; \mathrm{H}_1(A, \mathbb{Z}) \times \mathrm{H}_1(A, \mathbb{Z}) \to \mathfrak{D}_K^{-1}.$$

Here, $T_A\colon \mathrm{H}_1(A,\mathbb{Z}) \times \mathrm{H}_1(A,\mathbb{Z}) \to \mathfrak{D}_K^{-1}$ is the skew-hermitian form attached to E_A via Lemma 4.39.

Definition 4.44.
1. Let $\widetilde{\mathrm{Sh}}_K(\mathfrak{h})$ be the set of isomorphism classes of four-tuples (A,i,λ,j), where (A,i,λ) is as above and satisfies Conditions 4.43, and where $j\colon \mathrm{H}_1(A,\mathbb{Z}) \to \Lambda$ is a symplectic isomorphism of $\mathcal{O}_K$-modules.
2. Let $\mathbb{D}(V_i)$ be the space of negative s_i-planes in the hermitian space $(V_i,\mathfrak{h}^{\tau_i})$.

We have the following proposition which is due to Shimura, see [Shi63, Theorem 2] or [Shi64, §1]. We give a different proof since it will imply Proposition 4.48 below, whereas we did not know how to deduce Proposition 4.48 from *loc.cit.* We remark that Shimura assumes Λ to be an R-module for any order $R \subset \mathcal{O}_K$; our proof carries over, but we do not need this generalization.

Proposition 4.45. *There is a canonical bijection*

$$\widetilde{\mathrm{Sh}}_K(\mathfrak{h}) \cong \mathbb{D}(V_1) \times \cdots \times \mathbb{D}(V_g).$$

Proof. Let (A,i,λ,j) be a representative of an isomorphism class in $\widetilde{\mathrm{Sh}}_K(\mathfrak{h})$. Let $\mathrm{H}_1(A,\mathbb{C}) = \mathrm{H}^{-1,0} \oplus \mathrm{H}^{0,-1}$ be the Hodge decomposition of A. For $1 \leq i \leq g$ there is a decomposition

$$\mathrm{H}_1(A,\mathbb{C})_{\tau_i} = \mathrm{H}^{-1,0}_{\tau_i} \oplus \mathrm{H}^{0,-1}_{\tau_i}, \tag{4.10}$$

with $\dim \mathrm{H}^{-1,0}_{\tau_i} = r_i$ and $\dim \mathrm{H}^{0,-1}_{\tau_i} = s_i$. The latter holds because

$$\overline{\mathrm{H}^{-1,0}_{\tau_i\sigma}} = \mathrm{H}^{0,-1}_{\tau_i}.$$

By Lemma 4.41, $\tau_i(\eta)E_{A,\mathbb{C}}(x,\bar{y})$ and $\mathfrak{h}^{\tau_i}_{A,\mathbb{C}}(x,y)$ agree as hermitian forms on the complex vector space $\mathrm{H}_1(A,\mathbb{Z}) \otimes_{\mathcal{O}_K,\tau_i} \mathbb{C}$. Since $\Im\tau_i(\eta) > 0$ for every i, the decomposition of $\mathrm{H}_1(A,\mathbb{C})_{\tau_i}$ in (4.10) is a decomposition into a positive definite r_i-dimensional subspace and a negative definite s_i-dimensional subspace. The isomorphism $j : \mathrm{H}_1(A,\mathbb{Q}) \to \Lambda_{\mathbb{Q}}$ induces an isometry $j_i : \mathrm{H}_1(A,\mathbb{C})_{\tau_i} \to V_i$ for every i, and so we obtain a negative s_i-plane $j(\mathrm{H}^{0,-1}_{\tau_i})$ in the hermitian space V_i for all i.

Reversing the argument shows that given a negative s_i-plane $X_i \subset V_i$ for every i, there is a canonical polarized abelian variety $A = \mathrm{H}^{-1,0}/\Lambda$, acted upon by $\mathcal{O}_K$ and inducing the planes $X_i \subset V_i$. $\square$

Definition 4.46. 1. Let $\mathrm{Sh}_K(\mathfrak{h})$ be the set of isomorphism classes of polarized $\mathcal{O}_K$-linear abelian varieties (A, i, λ), satisfying Conditions 4.43, such that $\mathrm{H}_1(A, \mathbb{Z})$ is isometric to Λ as hermitian $\mathcal{O}_K$-modules.

2. Let $\Gamma(\mathfrak{h}) = \mathrm{Aut}_{\mathcal{O}_K}(\Lambda, \mathfrak{h})$; this is the group of $\mathcal{O}_K$-linear automorphisms of Λ preserving our form $\mathfrak{h} : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}$.

The bijection in Proposition 4.45 being $\Gamma(\mathfrak{h})$-equivariant, we obtain the following:

Corollary 4.47. *There is a canonical bijection*

$$\mathrm{Sh}_K(\mathfrak{h}) \cong \Gamma(\mathfrak{h}) \setminus \mathbb{D}(V_1) \times \cdots \times \mathbb{D}(V_g).$$

□

4.4.3 *Abelian varieties with moduli in the hyperplane arrangement*

The set of embeddings Ψ defined in (4.1) defines a map $\Psi : \mathcal{O}_K \to \mathbb{C}^g$, giving a complex torus $\mathbb{C}^g/\Psi(\mathcal{O}_K)$. The map

$$Q : K \times K \to \mathbb{Q}, \quad Q(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(\xi x \bar{y})$$

is a non-degenerate $\mathbb{Q}$-bilinear form such that $Q(ax, y) = Q(x, a^\sigma y)$ for every $a, x, y \in K$. Moreover, $Q(\mathcal{O}_K, \mathcal{O}_K) \subset \mathbb{Z}$ because $\xi \in \mathfrak{D}_K^{-1}$. By [Mil20, Example 2.9 & Footnote 16], Q defines a Riemann form on the complex torus $\mathbb{C}^g/\Psi(\mathcal{O}_K)$.

As in Section 4.2.1, let $\mathbb{C}H^n$ be the set of negative lines in $\Lambda \otimes_{\mathcal{O}_K, \tau_1} \mathbb{C}$, and define $\mathscr{H} = \cup_{\mathfrak{h}(r,r)=1} \langle r_\mathbb{C} \rangle^\perp \subset \mathbb{C}H^n$.

Proposition 4.48. *Suppose that* $(r_1, s_1) = (n, 1)$ *and* $(r_i, s_i) = (n+1, 0)$ *for* $2 \leq i \leq g$. *Then under the bijection* $\widetilde{\mathrm{Sh}}_K(\mathfrak{h}) \cong \mathbb{C}H^n$ *of Proposition 4.45, the subset* $\mathscr{H} \subset \mathbb{C}H^n$ *corresponds to the isomorphism classes of those polarized marked* $\mathcal{O}_K$*-linear abelian varieties* A *that admit a* $\mathcal{O}_K$*-linear homomorphism* $\mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$ *of polarized abelian varieties.*

Proof. Consider an isomorphism class $[(A, i, \lambda, y)] \in \widetilde{\mathrm{Sh}}_K(\mathfrak{h})$ corresponding to a point $[x] \in \mathbb{C}H^n$. We may assume that $A = \mathrm{H}^{-1,0}/\Lambda$ with $\Lambda \otimes_\mathbb{Z} \mathbb{C} = \mathrm{H}^{-1,0} \oplus \mathrm{H}^{0,-1}$, and that $T_A = T$. Let

$$\phi : \mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$$

be a homomorphism as in the proposition. We obtain a homomorphism

$$\mathcal{O}_K \to \Psi(\mathcal{O}_K) \to \mathrm{H}_1(A, \mathbb{Z}) = \Lambda$$

which, for simplicity, we also denote by $\phi : \mathcal{O}_K \to \Lambda$. Let $r \in \Lambda$ be the image of $1 \in \mathcal{O}_K$. The fact that $Q = \phi^* E_A$ implies that $T_Q = \phi^* T_A = \phi^* T$. Therefore, we have

$$\eta^{-1} = T_Q(1,1) = T_A(\phi(1), \phi(1)) = T(\phi(1), \phi(1)) = T(r, r),$$

so that $\mathfrak{h}(r,r) = \eta \cdot T(r,r) = 1$. We claim that $\mathfrak{h}(x, r_\tau) = 0$, where the element $r_\tau \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau$ is the image of $r \in \Lambda$. To see this, write

$$\Psi(\mathcal{O}_K) = L, \quad L \otimes \mathbb{C} = W^{-1,0} \oplus W^{0,-1},$$

and let $\alpha \in L$ correspond to $1 \in \mathcal{O}_K$. Notice that $(L \otimes_{\mathbb{Z}} \mathbb{C})_\tau = W_\tau^{-1,0}$. Consequently, since the composition

$$W_\tau^{-1,0} = (L \otimes_{\mathbb{Z}} \mathbb{C})_\tau \to (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau = \mathrm{H}_\tau^{-1,0} \oplus \mathrm{H}_\tau^{0,-1}$$

factors through the inclusion of $\mathrm{H}_\tau^{-1,0}$ into $(L \otimes_{\mathbb{Z}} \mathbb{C})_\tau$, we see that

$$r_\tau = r_\tau^{-1,0} \in \mathrm{H}_\tau^{-1,0} = \left(\mathrm{H}_\tau^{0,-1}\right)^\perp = \langle x \rangle^\perp,$$

and the claim follows.

Conversely, let $[x] \in \langle r_{\mathbb{C}} \rangle^\perp \subset \mathscr{H}$ with $r \in \Lambda$ such that $\mathfrak{h}(r,r) = 1$ and consider the marked abelian variety $A = \mathrm{H}^{-1,0}/\Lambda$ corresponding to $[x]$. Define a homomorphism $\phi : \mathcal{O}_K \to \Lambda$ by $\phi(1) = r$. Then ϕ can be shown to be a morphism of Hodge structures using the fact that its $\mathbb{C}$-linear extension preserves the eigenspace decompositions. We obtain an $\mathcal{O}_K$-linear homorphism $\phi : \mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$. The fact that $\mathfrak{h}(r,r) = 1$ implies that ϕ preserves the polarizations on both sides. $\square$

Observe that if the different $\mathfrak{D}_K \subset \mathcal{O}_K$ is a principal ideal $(\eta) \subset \mathcal{O}_K$, then we have

$$\begin{aligned} \{x \in K : \mathrm{Tr}_{K/\mathbb{Q}}\left(x\eta^{-1}\mathcal{O}_K\right) \subset \mathbb{Z}\} &= \{x \in K : x \cdot \eta^{-1}\mathcal{O}_K \subset \eta^{-1}\mathcal{O}_K\} \\ &= \{x \in K : x\mathcal{O}_K \subset \mathcal{O}_K\} = \mathcal{O}_K. \end{aligned}$$

Thus, $Q : \Psi(\mathcal{O}_K) \times \Psi(\mathcal{O}_K) \to \mathbb{Z}$ defines a *principal* polarization on the torus $\mathbb{C}^g/\Psi(\mathcal{O}_K)$ in this case. In fact, for $\beta \in K$, the rational Riemann form

$$\Psi(K) \times \Psi(K) \to \mathbb{Q}, \quad (\Psi(x), \Psi(y)) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(\beta^{-1} x \bar{y})$$

defines a principal polarization on $\mathbb{C}^g/\Psi(\mathcal{O}_K)$ if and only if (i) we have that β generates the different ideal $\mathfrak{D}_K$, (ii) we have that $\sigma(\beta) = -\beta$, and (iii) we have that $\Im(\varphi(\beta)) > 0$ for every $\varphi \in \Psi$. This follows from the above; see also [Wam99].

Consider the following:

Conditions 4.49. 1. *The CM type (K, Ψ) is primitive.*

2. *We have $\mathfrak{D}_K = (\eta)$ for some $\eta \in \mathcal{O}_K$ such that $\sigma(\eta) = -\eta$.*

3. *The signature of $\mathfrak{h}^{\tau_i}$ is $(n, 1)$ for $i = 1$ and $(n+1, 0)$ for $i \neq 1$.*

Theorem 4.50. *Suppose that Conditions 4.49 hold. Let $r_1, r_2 \in \Lambda$ satisfy $H_{r_1} \cap H_{r_2} \neq \emptyset$ and $H_{r_1} \neq H_{r_2} \subset \mathbb{C}H^n$ for $H_{r_i} = \langle r_{i,\mathbb{C}} \rangle^{\perp} \subset \mathbb{C}H^n$. Then $\mathfrak{h}(r_1, r_2) = 0$.*

Proof. Let $[x] \in H_r \cap H_t \subset \mathbb{C}H^n(V)$, and let A be an abelian variety whose isomorphism class gives $[x]$. Define B to be the principally polarized abelian variety $\mathbb{C}^g/\Psi(\mathcal{O}_K)$. By Proposition 4.48, the roots r and t induce $\mathcal{O}_K$-linear embeddings

$$\phi_1 : B \hookrightarrow A \quad \text{and} \quad \phi_2 : B \hookrightarrow A$$

of polarized abelian varieties. By Lemma 4.51 below, the ϕ_i induce decompositions

$$A \cong B \times C_1 \quad \text{and} \quad A \cong B \times C_2$$

as polarized abelian varieties. Note that B is non-decomposable as an abelian variety because $\mathrm{End}(B) \otimes_{\mathbb{Z}} \mathbb{Q} = K$ is a field (here we use that the CM type (K, Ψ) is primitive). By [Deb96], the decomposition of (A, λ) into non-decomposable polarized abelian subvarieties is unique, in the strong sense that if (A_i, λ_i), $i \in \{1, \ldots, r\}$ and (B_j, μ_j), $j \in \{1, \ldots, m\}$ are polarized abelian subvarieties such that the natural homomorphisms $\prod_i (A_i, \lambda_i) \to (A, \lambda)$ and $\prod_j (B_j, \lambda_j) \to (A, \lambda)$ are isomorphisms, then $r = m$ and there exists a permutation σ on $\{1, \ldots, r\}$ such that B_j and $A_{\sigma(j)}$ are *equal* as polarized abelian subvarieties of (A, λ), for every $j \in \{1, \ldots, r\}$. Consequently, for the two abelian subvarieties

$$B_i = \phi_i(B) \subset A, \quad \text{we have either that} \quad B_1 = B_2 \subset A \quad \text{or that} \quad B_1 \cap B_2 = \{0\}.$$

Suppose first that $B_1 = B_2$. Then

$$\mathcal{O}_K \cdot r = \phi_1(\mathcal{O}_K) = \phi_2(\mathcal{O}_K) = \mathcal{O}_K \cdot t \subset \Lambda.$$

Therefore, $r = \lambda t$ for some $\lambda \in \mathcal{O}_K^*$; but then $H_r = H_t$ which is absurd. Thus, we must have

$$A \cong B_1 \times B_2 \times C$$

as polarized abelian varieties, for some polarized abelian subvariety C of A. This implies that

$$\mathrm{H}^{-1,0} = \mathrm{Lie}(A) \cong \mathrm{Lie}(B_1) \times \mathrm{Lie}(B_2) \times \mathrm{Lie}(C),$$

which is orthogonal for the positive definite hermitian form $iE_{\mathbb{C}}(x, \bar{y})$ on $\mathrm{H}^{-1,0}$.

Observe that $r_\tau = r_\tau^{-1,0} \in \mathrm{H}_\tau^{-1,0}$ and $t_\tau = t_\tau^{-1,0} \in \mathrm{H}_\tau^{-1,0}$: see the proof of Proposition 4.48. By Lemma 4.41, we have

$$\begin{aligned}\mathfrak{h}(r,t) = \mathfrak{h}^\tau(r_\tau, t_\tau) &= \tau(\eta) \cdot T_{\mathbb{C}}^\tau(r_\tau, t_\tau) \\ &= \tau(\eta) \cdot E_{\mathbb{C}}(r_\tau, \bar{t_\tau}) = \tau(\eta) \cdot E_{\mathbb{C}}(r_\tau^{-1,0}, \overline{t_\tau^{-1,0}}).\end{aligned}$$

Since $r_\tau^{-1,0} \in \mathrm{Lie}(B_1)$ and $t_\tau^{-1,0} \in \mathrm{Lie}(B_2)$, we have $iE_{\mathbb{C}}(r_\tau^{-1,0}, \overline{t_\tau^{-1,0}}) = 0$. □

Lemma 4.51. *Let A be an abelian variety over a field k, with polarization $\lambda : A \to \widehat{A}$. Let $B \subset A$ be an abelian subvariety such that the polarization $\mu = \lambda|_B$ is principal. There is a polarized abelian subvariety $Z \subset A$ such that $A \cong B \times Z$ as polarized abelian varieties.*

Proof. Let $W = \mathrm{Ker}(A \xrightarrow{\lambda} \widehat{A} \to \widehat{B})$. Let $Z = W^0_{\mathrm{red}}$. Then Z is an abelian subvariety of dimension $\dim(A) - \dim(B)$ of A. The kernel of the natural homomorphism $B \times Z \to A$ is contained in $(B \cap Z) \times (B \cap Z)$. However, $B \cap Z \subset B \cap W = (0)$ because $\mu : B \to \widehat{B}$ is an isomorphism. Thus, $B \times Z \to A$ is an isomorphism. □

Finally, we remark that the condition on the different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ in Theorem 4.50 (see Conditions 4.49) is satisfied in two interesting cases:

Lemma 4.52. *Suppose that $K/\mathbb{Q}$ is an imaginary quadratic extension, or that $K = \mathbb{Q}(\zeta_n)$ is a cyclotomic field for some integer $n \geq 3$. Then Condition 4.49.2 is satisfied. That is, we have $\mathfrak{D}_K = (\eta) \subset \mathcal{O}_K$ for some element $\eta \in \mathcal{O}_K$ such that $\sigma(\eta) = -\eta$.*

Proof. If $K/\mathbb{Q}$ is imaginary quadratic with discriminant Δ, then $\mathfrak{D}_K = (\sqrt{\Delta})$ and the assertion is immediate. Let $n \geq 3$ be an integer, and consider the fields

$$K = \mathbb{Q}(\zeta_n) \ \supset F = \mathbb{Q}(\alpha), \quad \text{with} \quad \alpha = \zeta_n + \zeta_n^{-1}.$$

Since $\mathcal{O}_K = \mathbb{Z}[\zeta_n]$ by [Neu99, I, Proposition 10.2], we have $\mathcal{O}_K = \mathcal{O}_F[\zeta_n]$. Notice that $f(x) = x^2 - \alpha x + 1 \in \mathcal{O}_F[x]$ is the minimal polynomial of ζ_n over F. We have $f'(\zeta_n) = 2\zeta_n - \alpha\zeta_n = \zeta_n - \zeta_n^{-1}$. Therefore,

$$\mathfrak{D}_{K/F} = \left(f'(\zeta_n)\right) = \left(\zeta_n - \zeta_n^{-1}\right), \quad \text{see [Neu99, III, Proposition 2.4].}$$

By [Lia76], we know that $\mathcal{O}_F = \mathbb{Z}[\alpha]$. Moreover, if $g(x) \in \mathbb{Z}[x]$ is the minimal polynomial of α over $\mathbb{Q}$, then $\mathfrak{D}_{F/\mathbb{Q}} = (g'(\alpha))$. By [Neu99, III, Proposition 2.2], we have that $\mathfrak{D}_{K/\mathbb{Q}} = \mathfrak{D}_{K/F}\mathfrak{D}_{F/\mathbb{Q}}$. Combining all this yields

$$\mathfrak{D}_{K/\mathbb{Q}} = \mathfrak{D}_{K/F}\mathfrak{D}_{F/\mathbb{Q}} = \left(\zeta_n - \zeta_n^{-1}\right) \cdot \left(g'(\alpha)\right) = \left((\zeta_n - \zeta_n^{-1})g'(\alpha)\right).$$

□

Remark 4.53. It would be more natural to attach an orthogonal hyperplane arrangement $\mathcal{H} \subset \mathbb{C}H^n$ to every primitive CM field K and integral hermitian form $\mathfrak{h}$ of hyperbolic signature, such that $\mathcal{H} = \mathscr{H} = \cup_{\mathfrak{h}(r,r)=1}\langle r_{\mathbb{C}}\rangle^{\perp}$ if $\mathfrak{D}_K = (\eta)$ for some $\eta \in \mathcal{O}_K$ such that $\sigma(\eta) = -\eta$. This turns out to be possible. The idea is as follows.

Consider our CM field K. Choose $\beta \in \mathcal{O}_K - \mathcal{O}_F$ such that $\beta^2 \in \mathcal{O}_F$; then choose a CM type $\Psi = \{\tau_i : K \to \mathbb{C}\}_{1 \leq i \leq g}$ such that $\Im(\tau_i(\beta)) > 0$ for all i. Let $\mathfrak{h}$ be a non-degenerate hermitian form $\Lambda \times \Lambda \to \mathfrak{D}_K^{-1}$ such that $\operatorname{sign}(\mathfrak{h}^{\tau}) = (n,1)$ and $\operatorname{sign}(\mathfrak{h}^{\tau_i}) = (n+1,0)$ for $i \neq 1$. Let $\mathscr{S}$ be the set of fractional ideals $\mathfrak{a} \subset K$ for which there exist an element $b \in \mathcal{O}_F$ such that $\mathfrak{D}_K\mathfrak{a}\bar{\mathfrak{a}} = (b\beta)$. By [Wam99, Theorem 4], $\mathscr{S}$ is not empty. For $\mathfrak{a} \in \mathscr{S}$, define $\eta = b\beta \in \mathcal{O}_K$ and consider the complex torus $B = \mathbb{C}^g/\Psi(\mathfrak{a})$. It is equipped with the Riemann form $Q : \Psi(\mathfrak{a}) \times \Psi(\mathfrak{a}) \to \mathbb{Z}$, $(x,y) \mapsto \operatorname{Tr}_{K/\mathbb{Q}}(\eta^{-1}x\bar{y})$, and Q defines a principal polarization on B [Wam99, Theorem 3]. Let $\mathscr{R}$ be the set of embeddings $\phi : \mathfrak{a} \to \Lambda$, $\mathfrak{a} \in \mathscr{S}$, such that $\mathfrak{h}(\phi(x),\phi(y)) = x\bar{y}$ for all $x, y \in \mathfrak{a}$. For $\phi \in \mathscr{R}$, one obtains a hyperplane $H_\phi = \{x \in \mathbb{C}H^n : \mathfrak{h}^{\tau}(x,\phi(\mathfrak{a})) = 0\} \subset \mathbb{C}H^n$. The sought-for hyperplane arrangement $\mathcal{H} \subset \mathbb{C}H^n$ is defined as $\mathcal{H} = \cup_{\phi \in \mathscr{R}} H_\phi$. Indeed, if the CM type (K, Ψ) is primitive, then $\mathcal{H}$ is an orthogonal arrangement by arguments similar to those used to prove Proposition 4.48 and Theorem 4.50.

5

THE MODULI SPACE OF REAL BINARY QUINTICS

5.1 INTRODUCTION

In the previous Chapter 4 we saw that to any hermitian $\mathcal{O}_K$-lattice (Λ, h) over the ring of integers $\mathcal{O}_K$ of a CM field K, one can attach a certain path metric space $P\Gamma \setminus Y$ in a canonical way. Under favourable circumstances, this glued space is a complete hyperbolic orbifold, thus a disjoint union of real ball quotients. In this Chapter 5, we work out the structure of the glued space in a specific example.

Indeed, describing moduli of real binary quintics was the main motivation behind setting up the glueing procedure in Chapter 4. With that theory in place, we can prove that the moduli space of stable real binary quintics is isomorphic to the quotient of the hyperbolic plane by a non-arithmetic triangle group.

Let us explain this result in more detail. Let $X \cong \mathbb{A}^6_{\mathbb{R}}$ be the affine space parametrizing homogeneous degree 5 polynomials $F \in \mathbb{R}[x, y]$. Let the varieties

$$X_0 \subset X, \quad X_s \subset X$$

parametrize polynomials with distinct roots, respectively polynomials with roots of multiplicity at most two (i.e. stable in the sense of geometric invariant theory). The goal of Chapter 5 is to study the moduli spaces

$$\mathcal{M}_s(\mathbb{R}) := \mathrm{GL}_2(\mathbb{R}) \setminus X_s(\mathbb{R}) \qquad \text{and}$$
$$\mathcal{M}_0(\mathbb{R}) := \mathrm{GL}_2(\mathbb{R}) \setminus X_0(\mathbb{R}),$$

of stable and smooth *real binary quintics*. If $P_s \subset (\mathbb{P}^1_{\mathbb{R}})^5$ is the variety that parametrizes ordered five-tuples $(x_1, \ldots, x_5)$ such that no three x_i coincide (c.f. [MS72]), and $P_0 \subset P_s$ the subvariety of five-tuples whose coordinates are pairwise distinct, then

$$\begin{aligned} \mathcal{M}_s(\mathbb{R}) &\cong \mathrm{PGL}_2(\mathbb{R}) \setminus (P_s/\mathfrak{S}_5)(\mathbb{R}), \qquad \text{and} \\ \mathcal{M}_0(\mathbb{R}) &\cong \mathrm{PGL}_2(\mathbb{R}) \setminus (P_0/\mathfrak{S}_5)(\mathbb{R}). \end{aligned}$$

In other words, $\mathcal{M}_0(\mathbb{R})$ is the space of subsets $S \subset \mathbb{P}^1(\mathbb{C})$ of cardinality $|S| = 5$ stable by complex conjugation modulo real projective transformations, and in $\mathcal{M}_s(\mathbb{R})$ one or two pairs of points are allowed to collapse. For $i = 0, 1, 2$, we define $\mathscr{M}_i$ to be the connected component of $\mathcal{M}_0(\mathbb{R})$ parametrizing five-tuples in $\mathbb{P}^1(\mathbb{C})$ with $2i$ complex and $5 - 2i$ real points.

There is a natural occult period map that defines an isomorphism

$$\mathcal{M}_s(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) \setminus X_s(\mathbb{C}) \xrightarrow{\sim} P\Gamma \setminus \mathbb{C}H^2$$

for a certain arithmetic ball quotient $P\Gamma \setminus \mathbb{C}H^2$ (see Theorem 5.14, which depends on [Shi64, DM86]). Moreover, one can prove that strictly stable quintics correspond to points in a hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^2$ (Proposition 5.15). Investigating the equivariance of the period map with respect to a suitable set of anti-holomorphic involutions $\alpha_i : \mathbb{C}H^2 \to \mathbb{C}H^2$, we obtain the following real analogue:

Theorem 5.1. *For each $i \in \{0, 1, 2\}$, the period map induces an isomorphism of real analytic orbifolds*

$$\mathscr{M}_i \cong P\Gamma_i \setminus \left(\mathbb{R}H^2 - \mathscr{H}_i\right). \tag{5.1}$$

Here $\mathbb{R}H^2$ is the real hyperbolic plane, $\mathscr{H}_i$ a union of geodesic subspaces in $\mathbb{R}H^2$ and $P\Gamma_i$ an arithmetic lattice in $\mathrm{PO}(2,1)$. Moreover, the $P\Gamma_i$ are projective orthogonal groups attached to explicit quadratic forms over $\mathbb{Z}[\zeta_5 + \zeta_5^{-1}]$, see (5.32).

In particular, Theorem 5.1 endows each component $\mathscr{M}_i$ with a hyperbolic metric. Since one can deform the topological type of a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-stable five-element subset of $\mathbb{P}^1(\mathbb{C})$ by allowing two points to collide, the compactification $\mathcal{M}_s(\mathbb{R}) \supset \mathcal{M}_0(\mathbb{R})$ is connected. One may wonder whether the metrics on the components $\mathscr{M}_i$ extend to a metric on the whole of $|\mathcal{M}_s(\mathbb{R})|$. If so, what the resulting space looks like at the boundary? The answer to this question is the main result of this chapter:

Theorem 5.2. *There exists a complete hyperbolic metric on $|\mathcal{M}_s(\mathbb{R})|$ that restricts to the metrics on $\mathscr{M}_i$ induced by Theorem 5.1. If $\overline{\mathscr{M}}_{\mathbb{R}}$ denotes the resulting metric space, and*

$$P\Gamma_{3,5,10} = \langle \alpha_1, \alpha_2, \alpha_3 | \alpha_i^2 = (\alpha_1\alpha_2)^3 = (\alpha_1\alpha_3)^5 = (\alpha_2\alpha_3)^{10} = 1 \rangle, \tag{5.2}$$

then there exist open embeddings $P\Gamma_i \setminus \left(\mathbb{R}H^2 - \mathscr{H}_i\right) \subset P\Gamma_{3,5,10} \setminus \mathbb{R}H^2$ and an isometry

$$\overline{\mathscr{M}}_{\mathbb{R}} \cong P\Gamma_{3,5,10} \setminus \mathbb{R}H^2 \tag{5.3}$$

extending the orbifold isomorphisms (5.1) in Theorem 5.1. In particular, $\overline{\mathscr{M}}_{\mathbb{R}}$ is isometric to the hyperbolic triangle $\Delta_{3,5,10}$ of angles $\pi/3, \pi/5, \pi/10$, see Figure 5.1 on the next page.

Remark 5.3. The lattice $P\Gamma_{3,5,10} \subset \mathrm{PO}(2,1)$ is *non-arithmetic*, as follows from [Tak77]. The isometry (5.3) in Theorem 5.2 seems to provide $\overline{\mathscr{M}}_{\mathbb{R}}$ with a hyperbolic orbifold structure. The proof of Theorem 5.2 actually goes in the other direction: we use the theory developed in the previous Chapter 4 to show that the pieces on the right hand side of (5.1) glue into the hyperbolic quotient on the right hand side of (5.3), and afterwards, we prove that the period maps (5.1) glue to form (5.3).

By Theorem 5.2, the topological space $|\mathcal{M}_s(\mathbb{R})|$ underlies two orbifold structures: the one of $\mathcal{M}_s(\mathbb{R})$ and the one of $\overline{\mathscr{M}}_{\mathbb{R}}$. These structures only differ at one point of $|\mathcal{M}_s(\mathbb{R})|$, which is the point $(\infty, i, i, -i, -i)$ in Figure 5.1, see Proposition 5.32.

As explained in Section 1.2.2, equivariant (occult) period maps are often used in real algebraic geometry to uniformize components of a real moduli space of smooth varieties. Think of real moduli of abelian varieties [GH81], algebraic curves [SS89], K3 surfaces [Nik80] and quartic curves [HR18]. Only recently, Allcock, Carlson and Toledo have shown that for moduli of cubic surfaces [ACT10] and binary sextics [ACT06, ACT07], the real ball quotient components can be glued along the hyperplane arrangement to form a larger ball quotient that parametrizes moduli of stable varieties. Binary quintics provide the first new example of this phenomenon.

Remark 5.4. Our glueing construction relies on Condition 4.1, saying that $\mathscr{H} \subset \mathbb{C}H^n$ is an *orthogonal arrangement* in the sense of [ACT02b]. Such arrangements are interesting in their own right. Indeed, if $n > 1$ then the orbifold fundamental $\pi_1^{\mathrm{orb}}\left(P\Gamma \setminus (\mathbb{C}H^n - \mathscr{H})\right)$ is not a lattice in any Lie group with finitely many connected components [*loc.cit.*, Theorem 1.2]. In particular, neither $\pi_1\left(P_0/\mathfrak{S}_5\right)$ nor $\pi_1^{\mathrm{orb}}\left(\mathscr{M}_{\mathbb{C}}\right)$ is a lattice in any Lie group with finitely many connected components.

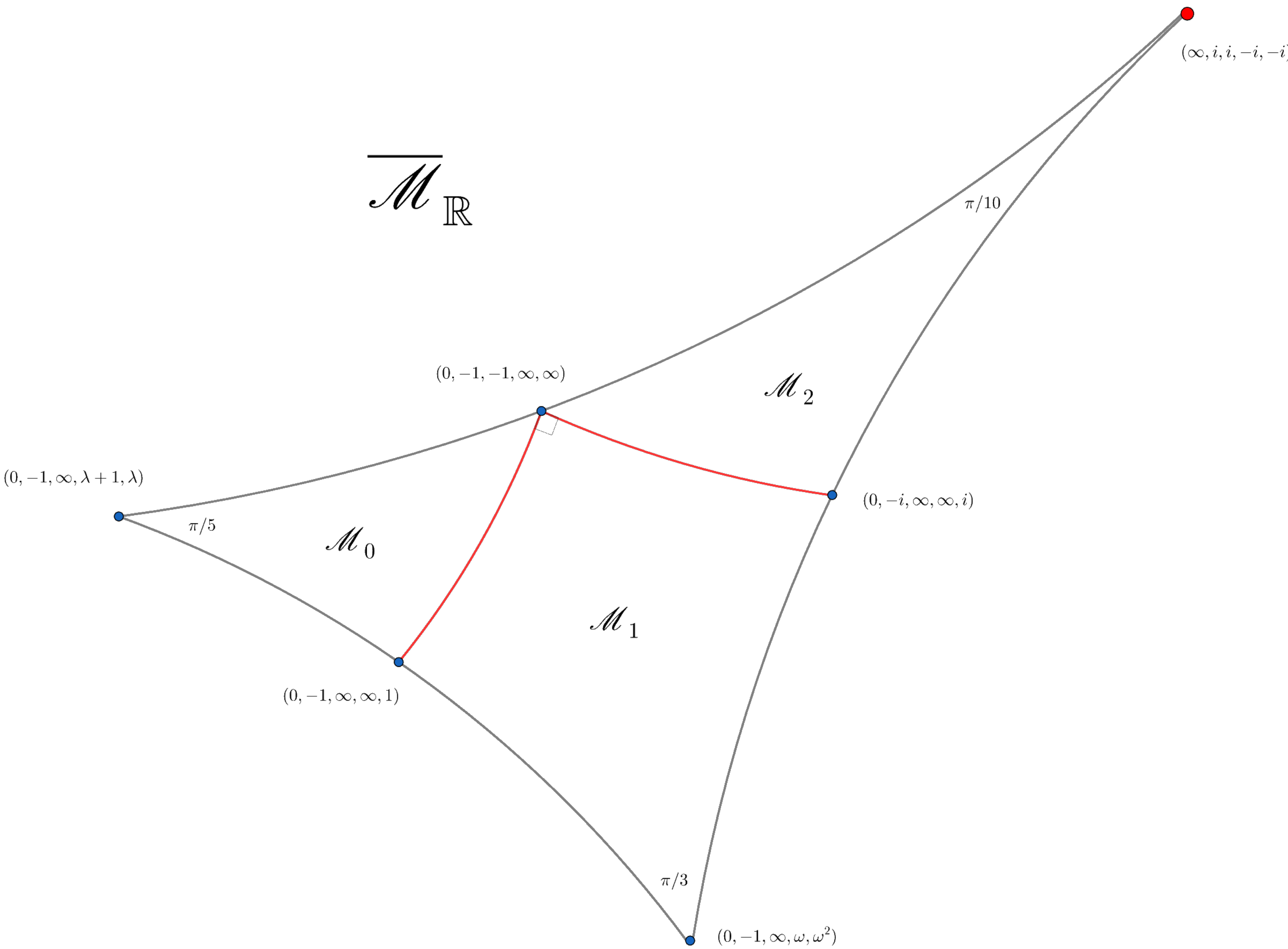

Figure 5.1: $\overline{\mathscr{M}}_{\mathbb{R}}$ as the hyperbolic triangle $\Delta_{3,5,10} \subset \mathbb{R}H^2$. Here $\lambda = \zeta_5 + \zeta_5^{-1}$ and $\omega = \zeta_3$.

5.2 MODULI OF COMPLEX BINARY QUINTICS

Recall from Section 5.1 that $X \cong \mathbb{A}^6_{\mathbb{R}}$ is the real affine space of homogeneous degree 5 polynomials $F \in \mathbb{R}[x, y]$, X_0 the subvariety of polynomials with distinct roots, and $X_s \subset X$ the subvariety of polynomials with roots of multiplicity at most two, i.e. non-zero polynomials whose class in the associated projective space is stable in the sense of geometric invariant theory [MFK94] for the action of $\mathrm{SL}_{2,\mathbb{R}}$ on it.

Notation 5.5. In Chapter 5, the CM field K is the cyclotomic field $\mathbb{Q}(\zeta)$, with $\zeta = \zeta_5 = e^{2\pi i/5} \in \mathbb{C}$. Recall that the ring of integers $\mathcal{O}_K$ of K is the ring $\mathbb{Z}[\zeta]$ [Neu99, Chapter I, Proposition (10.2)]. We let $\mu_K \subset \mathcal{O}_K^*$ denote the group of finite units as before; in this case μ_K is generated by $-\zeta$, and is therefore of order ten.

The goal of this Section 5.2 is to prove that there exists a hermitian $\mathcal{O}_K$-lattice Λ of rank three, such that, if $\Gamma = \mathrm{Aut}(\Lambda)$, $\mathbb{G}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/D$, and $\mathscr{H} \subset \mathbb{C}H^n$ is the hyperplane arrangement defined by the norm one vectors in Λ, then there is an isomorphism of analytic spaces $\mathcal{M}_s(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_s(\mathbb{C}) \cong P\Gamma \setminus \mathbb{C}H^2$ restricting to an orbifold isomorphism $\mathcal{M}_0(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_0(\mathbb{C}) \cong P\Gamma \setminus \left(\mathbb{C}H^2 - \mathscr{H}\right)$.

5.2.1 *The Jacobian of a cyclic quintic cover of the projective line*

We begin with the following:

Lemma 5.6. *Let $Z \subset \mathbb{P}^1_{\mathbb{C}}$ be a smooth quintic hypersurface. Let $\mathbb{P}^2_{\mathbb{C}} \supset C \to \mathbb{P}^1_{\mathbb{C}}$ be the quintic cover of $\mathbb{P}^1$ ramified along Z. Then C has the following refined Hodge numbers:*

$$h^{1,0}(C)_\zeta = 3, \quad h^{1,0}(C)_{\zeta^2} = 2, \quad h^{1,0}(C)_{\zeta^3} = 1, \quad h^{1,0}(C)_{\zeta^4} = 0 \tag{5.4}$$

$$h^{0,1}(C)_\zeta = 0, \quad \mathrm{H}^{0,1}(C)_{\zeta^2} = 1, \quad h^{0,1}(C)_{\zeta^3} = 2, \quad h^{0,1}(C)_{\zeta^4} = 3. \tag{5.5}$$

Proof. This follows from the Hurwitz-Chevalley-Weil formula, see [MO13, Proposition 5.9]. Alternatively, see [CT99, Section 5]. □

Fix a point $F_0 \in X_0(\mathbb{C})$ and let

$$C = \{z^5 = F_0(x, y)\} \subset \mathbb{P}^2_{\mathbb{C}} \tag{5.6}$$

be the corresponding cyclic cover of $\mathbb{P}^1_{\mathbb{C}}$. Let

$$\left(A = J(C) = \mathrm{Pic}^0(C), \quad \lambda\colon A \to \widehat{A}, \quad \iota\colon \mathcal{O}_K = \mathbb{Z}[\zeta] \to \mathrm{End}(A)\right)$$

be the Jacobian of C, viewed as a principally polarized abelian variety of dimension six equipped with an $\mathcal{O}_K$-action compatible with the polarization, see (4.43).

Write $\Lambda = \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$. We have $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{H}^{-1,0} \oplus \mathrm{H}^{0,-1}$, the Hodge decomposition of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. Define a CM-type $\Psi \subset \mathrm{Hom}(K, \mathbb{C})$ as follows:

$$\tau_i : K \to \mathbb{C}, \quad \tau_1(\zeta) = \zeta^3, \quad \tau_2(\zeta) = \zeta^4; \qquad \Psi = \{\tau_1, \tau_2\}. \tag{5.7}$$

Since $\mathrm{H}^{-1,0} = \mathrm{Lie}(A) = \mathrm{H}^1(C, \mathcal{O}_C) = \mathrm{H}^{0,1}(C)$, Lemma 5.6 implies that

$$\dim_{\mathbb{C}} \mathrm{H}^{-1,0}_{\tau_1} = 2, \quad \dim_{\mathbb{C}} \mathrm{H}^{-1,0}_{\tau_1\sigma} = 1, \quad \dim_{\mathbb{C}} \mathrm{H}^{-1,0}_{\tau_2} = 3, \quad \dim_{\mathbb{C}} \mathrm{H}^{-1,0}_{\tau_2\sigma} = 0. \tag{5.8}$$

Define $\eta = 5/(\zeta - \zeta^{-1})$. Then $\mathfrak{D}_K = (\eta)$ (see Lemma 4.52). Let

$$E : \Lambda \times \Lambda \to \mathbb{Z}$$

be the alternating form corresponding to the polarization λ of the abelian variety A. For $a \in \mathcal{O}_K$ and $x, y \in \Lambda$, we have $E(\iota(a)x, y) = E(x, \iota(a^\sigma)y)$. Define

$$T\colon \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}, \quad T(x,y) = \frac{1}{5}\sum_{j=0}^{4} \zeta^j E\left(x, \iota(\zeta)^j y\right).$$

By Example 4.40.2, this is the skew-hermitian form corresponding to E via Lemma 4.39. We obtain a hermitian form on the free $\mathcal{O}_K$-module Λ as follows:

$$\mathfrak{h} : \Lambda \times \Lambda \to \mathcal{O}_K, \quad \mathfrak{h}(x,y) = \eta T(x,y) = (\zeta - \zeta^{-1})^{-1} \sum_{j=0}^{4} \zeta^j E\left(x, \iota(\zeta)^j y\right). \tag{5.9}$$

By Lemma 4.39, the hermitian lattice $(\Lambda, \mathfrak{h})$ is unimodular, because (Λ, E) is unimodular. For each embedding $\varphi : K \to \mathbb{C}$, the restriction of the hermitian form

$\varphi(\eta) \cdot E_{\mathbb{C}}(x, \bar{y})$ on $\Lambda_{\mathbb{C}}$ to $(\Lambda_{\mathbb{C}})_{\varphi} \subset \Lambda_{\mathbb{C}}$ coincides with $\mathfrak{h}^{\varphi}$ by Lemma 4.41. Since $\Im(\tau_i(\zeta - \zeta^{-1})) < 0$ for $i = 1, 2$, the signature of $\mathfrak{h}^{\tau_i}$ on $V_i = \Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C}$ is

$$\operatorname{sign}(\mathfrak{h}^{\tau_i}) = \begin{cases} (\mathrm{h}_{\tau_1}^{-1,0}, h_{\tau_1}^{0,-1}) = (2,1) & \text{for } i = 1, \quad \text{and} \\ (\mathrm{h}_{\tau_2}^{-1,0}, h_{\tau_2}^{0,-1}) = (3,0) & \text{for } i = 2. \end{cases} \tag{5.10}$$

5.2.2 *The monodromy representation*

Consider the real algebraic variety X_0 introduced in Section 5.1. Let $D \subset \mathrm{GL}_2(\mathbb{C})$ be the subgroup $D = \{\zeta^i \cdot I_2\} \subset \mathrm{GL}_2(\mathbb{C})$ of scalar matrices $\zeta^i \cdot I_2$, where $I_2 \in \mathrm{GL}_2(\mathbb{C})$ is the identity matrix of rank two, and define

$$\mathbb{G}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/D. \tag{5.11}$$

The group $\mathbb{G}(\mathbb{C})$ acts from the left on $X_0(\mathbb{C})$ in the following way: if $F(x,y) \in \mathbb{C}[x,y]$ is a binary quintic, we may view F as a function $\mathbb{C}^2 \to \mathbb{C}$, and define $g \cdot F = F(g^{-1})$ for $g \in \mathbb{G}(\mathbb{C})$. This gives a canonical isomorphism of complex analytic orbifolds

$$\mathcal{M}_0(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_0(\mathbb{C}),$$

where $\mathcal{M}_0$ is the moduli stack of smooth binary quintics.

Consider two families

$$\pi : \mathscr{C} \to X_0 \quad \text{and} \quad \phi : J \to X_0,$$

defined as follows. We define π as the universal family of cyclic covers $C \to \mathbb{P}^1$ ramified along a smooth binary quintic $\{F = 0\} \subset \mathbb{P}^1$. We let ϕ be the relative Jacobian of π. By Section 5.2.1, ϕ is an abelian scheme of relative dimension six over X_0, with $\mathcal{O}_K$-action of signature $\{(2,1), (3,0)\}$ with respect to $\Psi = \{\tau_1, \tau_2\}$.

Let $\mathbb{V} = R^1\pi_*\mathbb{Z}$ be the local system of hermitian $\mathcal{O}_K$-modules underlying the abelian scheme J/X_0. Attached to $\mathbb{V}$, we have a representation

$$\rho' : \pi_1(X_0(\mathbb{C}), F_0) \to \Gamma, \quad \Gamma = \mathrm{Aut}_{\mathcal{O}_K}(\Lambda, \mathfrak{h}),$$

whose composition with the quotient map $\Gamma \to P\Gamma = \Gamma/\mu_K$ defines a homomorphism

$$\rho : \pi_1(X_0(\mathbb{C}), F_0) \to P\Gamma. \tag{5.12}$$

We shall see that ρ is surjective, see Corollary 5.8 below.

5.2.3 *Marked binary quintics*

For $F \in X_0(\mathbb{C})$, define Z_F as the hypersurface

$$Z_F = \{F = 0\} \subset \mathbb{P}^1_{\mathbb{C}}.$$

A *marking* of F is a ring isomorphism $m : \mathrm{H}^0(Z_F(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^5$. To give a marking is to give a labelling of the points $p \in Z_F(\mathbb{C})$. Let $\mathcal{N}_0$ be the space of marked binary quintics (F, m); this is a manifold, equipped with a holomorphic map

$$\mathcal{N}_0 \to X_0(\mathbb{C}). \tag{5.13}$$

Let

$$\psi : \mathcal{Z} \to X_0(\mathbb{C})$$

be the universal complex binary quintic, and consider the local system $H = \psi_* \mathbb{Z}$ of stalk $H_F = \mathrm{H}^0(Z_F(\mathbb{C}), \mathbb{Z})$ for $F \in X_0(\mathbb{C})$. Then H corresponds to a monodromy representation

$$\tau : \pi_1(X_0(\mathbb{C}), F_0) \to \mathfrak{S}_5. \tag{5.14}$$

It can be shown that τ is surjective using the results of [Bea86a]. This implies that (5.13) is covering space, i.e. that $\mathcal{N}_0$ is connected.

If we choose a marking $m_0 : \mathrm{H}^0(Z_{F_0}(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^5$ lying over our base point $F_0 \in X_0(\mathbb{C})$, we obtain an embedding $\pi_1(\mathcal{N}_0, m_0) \hookrightarrow \pi_1(X_0(\mathbb{C}), F_0)$, whose composition with ρ in (5.12) defines a homomorphism

$$\mu : \pi_1(\mathcal{N}_0, m_0) \to P\Gamma. \tag{5.15}$$

Define $\theta = \zeta - \zeta^{-1}$ and consider the 3-dimensional $\mathbb{F}_5$ vector space $\Lambda/\theta\Lambda$ and the quadratic space

$$W := (\Lambda/\theta\Lambda, q),$$

where q is the quadratic form obtained by reducing $\mathfrak{h}$ modulo $\theta\Lambda$. Define two groups Γ_θ and $P\Gamma_\theta$ as follows:

$$\Gamma_\theta = \mathrm{Ker}\left(\Gamma \to \mathrm{Aut}(W)\right), \quad P\Gamma_\theta = \mathrm{Ker}\left(P\Gamma \to P\mathrm{Aut}(W)\right) \subset \mathrm{PU}(2,1).$$

Remark that the composition $\mathcal{N}_0 \to X_0(\mathbb{C}) \to X_s(\mathbb{C})$ admits an essentially unique *completion* $\mathcal{N}_s \to X_s(\mathbb{C})$, see [Fox57] or [DM86, §8]. Here $\mathcal{N}_s$ a manifold and $\mathcal{M}_s \to X_s(\mathbb{C})$ is a ramified covering space.

Proposition 5.7. *The image of μ in (5.15) is the group $P\Gamma_\theta$, and the induced homomorphism $\pi_1(X_0(\mathbb{C}), F_0)/\pi_1(\mathcal{N}_0, m_0) = \mathfrak{S}_5 \to P\Gamma/P\Gamma_\theta$ is an isomorphism. In other words, we obtain the following commutative diagram with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(\mathcal{N}_0, m_0) \arrow[r] \arrow[d, two heads, "\mu"] & \pi_1(X_0(\mathbb{C}), F_0) \arrow[r, "\tau"] \arrow[d, "\rho"] & \mathfrak{S}_5 \arrow[r] \arrow[d, "\gamma"', "\wr"] & 0 \\ 0 \arrow[r] & P\Gamma_\theta \arrow[r] & P\Gamma \arrow[r] & P\mathrm{Aut}(W) \arrow[r] & 0. \end{tikzcd} \tag{5.16}$$

Proof. Consider the quotient

$$Q = \mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_0 = \mathrm{PGL}_2(\mathbb{C}) \setminus P_0(\mathbb{C}),$$

where $P_0 \subset (\mathbb{P}^1_{\mathbb{R}})^5$ is the subvariety of distinct five-tupes, see Section 5. Let $0 \in Q$ be the image of $m_0 \in \mathcal{N}_0$. In [DM86], Deligne and Mostow define a hermitian space bundle $B_Q \to Q$ over Q whose fiber over $0 \in Q$ is $\mathbb{C}H^2$. Consequently, writing $V_1 = \Lambda \otimes_{\mathcal{O}_K, \tau_1} \mathbb{C}$, this gives a monodromy representation

$$\pi_1(Q, 0) \to \mathrm{PU}(V_1, \mathfrak{h}^{\tau_1}) \cong \mathrm{PU}(2,1)$$

whose image we denote by Γ_{DM}. Kondō has shown that in fact, $\Gamma_{\mathrm{DM}} = P\Gamma_\theta$ [Kon07, Theorem 7.1]. Since $\mathcal{N}_0 \to Q$ is a covering space (the action of $\mathbb{G}(\mathbb{C})$ on $\mathcal{N}_0$ being free) we have an embedding $\pi_1(\mathcal{N}_0, m_0) \hookrightarrow \pi_1(Q, 0)$ whose composition with $\pi_1(Q, 0) \to \mathrm{PU}(2,1)$ is the map $\mu : \pi_1(\mathcal{N}_0, m_0) \to P\Gamma \subset \mathrm{PU}(2,1)$.

To prove that the image of μ is $P\Gamma_\theta$, it suffices to give a section of the map $\mathcal{N}_0 \to Q$. Indeed, such a section induces a retraction of $\pi_1(\mathcal{N}_0, m_0) \hookrightarrow \pi_1(Q, 0)$, so that the images of these two groups in $\mathrm{PU}(2,1)$ are the same.

To define such a section, observe that if $\Delta \subset \mathbb{P}^1(\mathbb{C})^5$ is the union of all hyperplanes $\{x_i = x_j\} \subset \mathbb{P}^1(\mathbb{C})^5$ for $i \neq j$, then

$$\begin{aligned} Q = \mathrm{PGL}_2(\mathbb{C}) \setminus P_0(\mathbb{C}) &= \mathrm{PGL}_2(\mathbb{C}) \setminus \left(\mathbb{P}^1(\mathbb{C})^5 - \Delta\right) \\ &\cong \{(x_4, x_5) \in \mathbb{C}^2 : x_i \neq 0, 1 \text{ and } x_1 \neq x_2\}. \end{aligned}$$

The section $Q \to \mathcal{N}_0$ may then be defined by sending (x_4, x_5) to the binary quintic

$$F(x,y) = x(x-y)y(x - x_4 \cdot y)(x - x_5 \cdot y) \in X_0(\mathbb{C}),$$

marked by the labelling of its roots $\{[0:1], [1:1], [1:0], [x_4:1], [x_5:1]\}$.

It remains to prove that the homomorphism $\gamma : \mathfrak{S}_5 \to P\Gamma/P\Gamma_\theta$ appearing on the right in (5.16) is an isomorphism. We use Theorem 5.34, proven by Shimura in [Shi64], which says that

$$(\Lambda, \mathfrak{h}) \cong \left(\mathcal{O}_K^3, \mathrm{diag}\left(1, 1, \frac{1-\sqrt{5}}{2}\right)\right).$$

It follows that

$$P\Gamma/P\Gamma_\theta = P\mathrm{Aut}(W) \cong \mathrm{PO}_3(\mathbb{F}_5) \cong \mathfrak{S}_5.$$

Next, consider the manifold $\mathcal{N}_s$. Remark that $\mathfrak{S}_5$ embeds into $\mathrm{Aut}(\mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s)$. Moreover, there is a natural isomorpism

$$p \colon \mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s \cong P\Gamma_\theta \setminus \mathbb{C}H^2, \qquad \text{see [DM86, Kon07]}.$$

(See also (5.18).) The two compositions

$$\mathfrak{S}_5 \subset \mathrm{Aut}(\mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s) \cong \mathrm{Aut}(P\Gamma_\theta \setminus \mathbb{C}H^2) \text{ and } \mathfrak{S}_5 \to P\Gamma/P\Gamma_\theta \subset \mathrm{Aut}(P\Gamma_\theta \setminus \mathbb{C}H^2)$$

agree, because of the equivariance of p with respect to γ. Thus, γ is injective. □

Corollary 5.8. *The monodromy representation ρ in (5.12) is surjective.* □

5.2.4 *Framed binary quintics*

By a *framing* of a point $F \in X_0(\mathbb{C})$ we mean a projective equivalence class $[f]$, where

$$f \colon \mathbb{V}_F = \mathrm{H}^1(C_F(\mathbb{C}), \mathbb{Z}) \to \Lambda$$

is an $\mathcal{O}_K$-linear isometry: two such isometries are in the same class if and only if they differ by an element in μ_K. Let $\mathcal{F}_0$ be the collection of all framings of all points $x \in X_0(\mathbb{C})$. The set $\mathcal{F}_0$ is naturally a complex manifold, by arguments similar to those in [ACT02a]. Note that Corollary 5.8 implies that $\mathcal{F}_0$ is connected, hence

$$\mathcal{F}_0 \to X_0(\mathbb{C}) \tag{5.17}$$

is a covering, with Galois group $P\Gamma$.

Lemma 5.9. *The spaces $P\Gamma_\theta \setminus \mathcal{F}_0$ and $\mathcal{N}_0$ are isomorphic as covering spaces of $X_0(\mathbb{C})$. In particular, there is a covering map $\mathcal{F}_0 \to \mathcal{N}_0$ with Galois group $P\Gamma_\theta$.*

Proof. We have $P\Gamma / P\Gamma_\theta \cong \mathfrak{S}_5$ as quotients of $P\Gamma$, see Proposition 5.7. □

Lemma 5.10. $\Delta := X_s(\mathbb{C}) - X_0(\mathbb{C})$ *is an irreducible normal crossings divisor of* $X_s(\mathbb{C})$.

Proof. The proof is similar to the proof of Proposition 6.7 in [Bea09]. □

Lemma 5.11. *The local monodromy transformations of $\mathcal{F}_0 \to X_0(\mathbb{C})$ around every $x \in \Delta$ are of finite order. More precisely, if $x \in \Delta$ lies on the intersection of k local components of Δ, then the local monodromy group around x is isomorphic to $(\mathbb{Z}/10)^k$.*

Proof. See [DM86, Proposition 9.2] or [CT99, Proposition 6.1] for the generic case, i.e. when a quintic $Z = \{F = 0\} \subset \mathbb{P}^1_{\mathbb{C}}$ aquires only one node. In this case, the local equation of the singularity is $x^2 = 0$, hence the curve C_F acquires a singularity of the form $y^5 + x^2 = 0$. If the quintic acquires two nodes, then C_F acquires two such singularities; the vanishing cohomology splits as an orthogonal direct sum, hence the local monodromy transformations commute. □

In the following corollary, we let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ denote the open unit disc, and $D^* = D - \{0\}$ the punctured open unit disc.

Corollary 5.12. *There is an essentially unique branched cover $\pi : \mathcal{F}_s \to X_s(\mathbb{C})$, with $\mathcal{F}_s$ a complex manifold, such that for any $x \in \Delta$, any open $x \in U \subset X_s(\mathbb{C})$ with*

$U \cong D^k \times D^{6-k}$ and $U \cap X_0(\mathbb{C}) \cong (D^*)^k \times D^{6-k}$, and any connected component U' of $\pi^{-1}(U) \subset \mathcal{F}_s$, there is an isomorphism $U' \cong D^k \times D^{6-k}$ such that the composition

$$D^k \times D^{6-k} \cong U' \to U \cong D^6 \text{ is the map } (z_1, \dots, z_6) \mapsto (z_1^{r_1}, \dots, z_k^{r_k}, z_{k+1}, \dots, z_6).$$

Proof. See [Bea09, Lemma 7.2]. See also [Fox57] and [DM86, Section 8]. □

The group $\mathbb{G}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/D$ (see (5.11)) acts on $\mathcal{F}_0$ over its action on X_0. Explicitly, if $g \in \mathbb{G}(\mathbb{C})$ and if $([\phi], \phi : \mathbb{V}_F \cong \Lambda)$ is a framing of $F \in X_0(\mathbb{C})$, then

$$\left([\phi \circ g^*], \phi \circ g^* : \mathbb{V}_{g \cdot F} \to \Lambda\right)$$

is a framing of $g \cdot F \in X_0(\mathbb{C})$. This is a left action. The group $P\Gamma$ also acts on $\mathcal{F}_0$ from the left, and the actions of $P\Gamma$ and $\mathbb{G}(\mathbb{C})$ on $\mathcal{F}_0$ commute. By functoriality of the Fox completion, the action of $\mathbb{G}(\mathbb{C})$ on $\mathcal{F}_0$ extends to an action of $\mathbb{G}(\mathbb{C})$ on $\mathcal{F}_s$.

Lemma 5.13. *The group $\mathbb{G}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/D$ acts freely on $\mathcal{F}_s$.*

Proof. The functoriality of the Fox completion gives an action of $\mathbb{G}(\mathbb{C})$ on $\mathcal{N}_s$ such that, by Lemma 5.9, there is a $\mathbb{G}(\mathbb{C})$-equivariant commutative diagram

$$\begin{tikzcd} P\Gamma_\theta \setminus \mathcal{F}_s \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{N}_s \arrow[dl] \\ & X_s(\mathbb{C}). & \end{tikzcd}$$

In particular, it suffices to show that $\mathbb{G}(\mathbb{C})$ acts freely on $\mathcal{N}_s$. Note that $\mathcal{N}_0$ admits a natural $\mathbb{G}_m$-covering map $\mathcal{N}_0 \to P_0(\mathbb{C})$ where $P_0(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C})^5$ is the space of distinct ordered five-tuples in $\mathbb{P}^1(\mathbb{C})$ introduced in Section 5.1. Consequently, there is a $\mathbb{G}_m$-quotient map $\mathcal{N}_s \to P_s(\mathbb{C})$, where $P_s(\mathbb{C})$ is the space of stable ordered five-tuples, and this map is equivariant for the homomorphism $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$.

Let $g \in \mathrm{GL}_2(\mathbb{C})$ and $x \in \mathcal{N}_s$ such that $gx = x$. It is clear that $\mathrm{PGL}_2(\mathbb{C})$ acts freely on $P_s(\mathbb{C})$. Therefore, $g = \lambda \in \mathbb{C}^*$. Let $F \in X_s(\mathbb{C})$ be the image of $x \in \mathcal{N}_s$; then

$$gF(x, y) = F(g^{-1}(x, y)) = F(\lambda^{-1}x, \lambda^{-1}y) = \lambda^{-5}F(x, y).$$

The equality $gF = F$ implies that $\lambda^5 = 1 \in \mathbb{C}$, from which we conclude that $\lambda \in \langle \zeta \rangle$. Therefore, $[g] = [\mathrm{id}] \in \mathbb{G}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/D$. □

5.2.5 *Complex uniformization*

Consider the hermitian space $V_1 = \Lambda \otimes_{\mathcal{O}_K,\tau_1} \mathbb{C}$; define $\mathbb{C}H^2$ to be the space of negative lines in V_1. Using Proposition 4.45 we see that the abelian scheme $J \to X_0$ induces a $\mathbb{G}(\mathbb{C})$-equivariant morphism $\mathcal{P} : \mathcal{F}_0 \to \mathbb{C}H^2$. Explicitly, if $(F,[f]) \in \mathcal{F}_0$ is the framing $[f : \mathrm{H}^1(C_F(\mathbb{C}),\mathbb{Z}) \xrightarrow{\sim} \Lambda]$ of the binary quintic $F \in X_0(\mathbb{C})$, and A_F is the Jacobian of the curve C_F, then

$$\mathcal{P} : \mathcal{F}_0 \to \mathbb{C}H^2, \quad \mathcal{P}\left(F,[f]\right) = f\left(H^{0,-1}(A_F)_{\tau_1}\right) = f\left(H^{1,0}(C_F)_{\zeta^3}\right) \in \mathbb{C}H^2. \tag{5.18}$$

The map $\mathcal{P}$ is holomorphic, and descends to a morphism of complex analytic spaces

$$\mathcal{M}_0(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_0(\mathbb{C}) \to P\Gamma \setminus \mathbb{C}H^2. \tag{5.19}$$

By Riemann extension, (5.18) extends to a $\mathbb{G}(\mathbb{C})$-equivariant holomorphic map

$$\overline{\mathcal{P}} : \mathcal{F}_s \to \mathbb{C}H^2. \tag{5.20}$$

Theorem 5.14 (Deligne–Mostow). *The period map (5.20) induces an isomorphism of complex manifolds*

$$\mathcal{M}_s^f(\mathbb{C}) := \mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_s \cong \mathbb{C}H^2. \tag{5.21}$$

Taking $P\Gamma$-quotients gives an isomorphism of complex analytic spaces

$$\mathcal{M}_s(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_s(\mathbb{C}) \cong P\Gamma \setminus \mathbb{C}H^2. \tag{5.22}$$

Proof. In [DM86], Deligne and Mostow define $\widetilde{Q} \to Q$ to be the covering space corresponding to the monodromy representation $\pi_1(Q,0) \to \mathrm{PU}(2,1)$; since the image of this homomomorphism is $P\Gamma_\theta$ (see the proof of Proposition 5.7), it follows that $\mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_0 \cong \widetilde{Q}$ as covering spaces of Q. Consequently, if $\widetilde{Q}_{\mathrm{st}}$ is the Fox completion of the spread

$$\widetilde{Q} \to Q \to Q_{\mathrm{st}} := \mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s = \mathrm{PGL}_2(\mathbb{C}) \setminus P_s(\mathbb{C}),$$

then there is an isomorphism $\mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_s \cong \widetilde{Q}_{\mathrm{st}}$ of branched covering spaces of Q_{st}. We obtain commutative diagrams, where the lower right morphism uses (5.16):

$$\begin{tikzcd}
\mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_s \arrow[r, "\sim"] \arrow[d] & \widetilde{Q}_{\mathrm{st}} \arrow[r] \arrow[d] & \mathbb{C}H^2 \arrow[d] \\
\mathbb{G}(\mathbb{C}) \setminus \mathcal{N}_s \arrow[r, "\sim"] \arrow[d] & Q_{\mathrm{st}} \arrow[r] \arrow[d] & P\Gamma_\theta \setminus \mathbb{C}H^2 \arrow[d] \\
\mathbb{G}(\mathbb{C}) \setminus X_s(\mathbb{C}) \arrow[r, "\sim"] & Q_{\mathrm{st}}/\mathfrak{S}_5 \arrow[r] & P\Gamma \setminus \mathbb{C}H^2.
\end{tikzcd}$$

The map $\widetilde{Q}_{\mathrm{st}} \to \mathbb{C}H^2$ is an isomorphism by [DM86, (3.11)]. Therefore, we are done if the composition $\mathbb{G}(\mathbb{C}) \setminus \mathcal{F}_0 \to \widetilde{Q} \to \mathbb{C}H^2$ agrees with the period map $\mathcal{P}$ of equation (5.18). This follows from [DM86, (2.23) and (12.9)]. □

Proposition 5.15. *The isomorphism (5.22) induces an isomorphism of complex analytic spaces*

$$\mathcal{M}_0(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_0(\mathbb{C}) \cong P\Gamma \setminus \left(\mathbb{C}H^2 - \mathscr{H}\right). \tag{5.23}$$

Proof. We have $\overline{\mathcal{P}}(\mathcal{F}_0) \subset \mathbb{C}H^2 - \mathscr{H}$ by Proposition 4.48, because the Jacobian of a smooth curve cannot contain an abelian subvariety whose induced polarization is principal. Therefore, we have $\overline{\mathcal{P}}^{-1}(\mathscr{H}) \subset \mathcal{F}_s - \mathcal{F}_0$. Since $\mathcal{F}_s$ is irreducible (it is smooth by Corollary 5.12 and connected by Corollary 5.8), the analytic space $\overline{\mathcal{P}}^{-1}(\mathscr{H})$ is a divisor. Since $\mathcal{F}_s - \mathcal{F}_0$ is also a divisor by Corollary 5.12, we have

$$\overline{\mathcal{P}}^{-1}(\mathscr{H}) = \mathcal{F}_s - \mathcal{F}_0$$

and we are done.

Alternatively, let $H_{0,5}$ be the moduli space of degree 5 covers of $\mathbb{P}^1$ ramified along five distinct marked points [HM98, §2.G]. The period map

$$H_{0,5}(\mathbb{C}) \to P\Gamma \setminus \mathbb{C}H^2,$$

that sends the moduli point of a curve $C \to \mathbb{P}^1$ to the moduli point of the $\mathbb{Z}[\zeta]$-linear Jacobian $J(C)$, extends to the stable compactification $\overline{H}_{0,5}(\mathbb{C}) \supset H_{0,5}(\mathbb{C})$ because the curves in the limit are of compact type. Since the divisor $\mathscr{H} \subset \mathbb{C}H^2$ parametrizes abelian varieties that are products of lower dimensional ones by Proposition 4.48, the image of the boundary is exactly $P\Gamma \setminus \mathscr{H}$. □

5.3 MODULI OF REAL BINARY QUINTICS

Having understood the period map for complex binary quintics, we turn to the period map of real binary quintics in this Section 5.3.

5.3.1 *The period map for stable real binary quintics*

Define κ as the anti-holomorphic involution

$$\kappa\colon X_0(\mathbb{C}) \to X_0(\mathbb{C}), \quad F(x,y) = \sum_{i+j=5} a_{ij}x^i y^j \mapsto \overline{F(x,y)} = \sum_{i+j=5} \overline{a_{ij}} x^i y^j.$$

Let $\mathscr{A}$ be the set of anti-unitary involutions $\alpha : \Lambda \to \Lambda$, and $P\mathscr{A} = \mu_K \setminus \mathscr{A}$, see Section 4.4. For each $\alpha \in P\mathscr{A}$, there is a natural anti-holomorphic involution $\alpha : \mathcal{F}_0 \to \mathcal{F}_0$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}_0 \arrow[r, "\alpha"] \arrow[d] & \mathcal{F}_0 \arrow[d] \\ X_0(\mathbb{C}) \arrow[r, "\kappa"] & X_0(\mathbb{C}). \end{tikzcd}$$

It is defined as follows. Consider a framed binary quintic $(F,[f]) \in \mathcal{F}_0$, where $f : \mathbb{V}_F \to \Lambda$ is an $\mathcal{O}_K$-linear isometry. Let $C_F \to \mathbb{P}^1_{\mathbb{C}}$ be the quintic cover defined by a smooth binary quintic $F \in X_0(\mathbb{C})$. Complex conjugation $\mathrm{conj}\colon \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ induces an anti-holomorphic map

$$\sigma_F : C_F(\mathbb{C}) \to C_{\kappa(F)}(\mathbb{C}).$$

Consider the pull-back $\sigma_F^* : \mathbb{V}_{\kappa(F)} \to \mathbb{V}_F$ of σ. The composition

$$\mathbb{V}_{\kappa(F)} \xrightarrow{\sigma_F^*} \mathbb{V}_F \xrightarrow{f} \Lambda \xrightarrow{\alpha} \Lambda$$

induces a framing of $\kappa(F) \in X_0(\mathbb{C})$, and we define

$$\alpha(F,[f]) := (\kappa(F), [\alpha \circ f \circ \sigma_F^*]) \in \mathcal{F}_0.$$

Although we have chosen a representative $\alpha \in \mathscr{A}$ of the class $\alpha \in P\mathscr{A}$, the element $\alpha(F, [f]) \in \mathcal{F}_0$ does not depend on this choice.

Consider the covering map $\mathcal{F}_0 \to X_0(\mathbb{C})$ introduced in (5.3), and define

$$\mathcal{F}_0(\mathbb{R}) = \bigsqcup_{\alpha \in P\mathscr{A}} \mathcal{F}_0^\alpha \subset \mathcal{F}_0 \tag{5.24}$$

as the preimage of $X_0(\mathbb{R})$ in the space $\mathcal{F}_0$. To see why the union on the left hand side of (5.24) is disjoint, observe that

$$\mathcal{F}_0^\alpha = \left\{ (F, [f]) \in \mathcal{F}_0 : \kappa(F) = F \text{ and } [f \circ \sigma_F^* \circ f^{-1}] = \alpha \right\}.$$

Thus, for $\alpha, \beta \in P\mathscr{A}$ and $(F, [f]) \in \mathcal{F}_0^\alpha \cap \mathcal{F}_0^\beta$, we have $\alpha = [f \circ \sigma \circ f^{-1}] = \beta$.

Lemma 5.16. *The anti-holomorphic involution $\alpha \colon \mathcal{F}_0 \to \mathcal{F}_0$ defined by $\alpha \in P\mathscr{A}$ makes the period map $\mathcal{P} \colon \mathcal{F}_0 \to \mathbb{C}H^2$ equivariant for the $\mathbb{G}(\mathbb{C})$-actions on both sides.*

Proof. Indeed, if $\mathrm{conj} \colon \mathbb{C} \to \mathbb{C}$ is complex conjugation, then for any $F \in X_0(\mathbb{C})$, the induced map

$$\sigma_F^* \otimes \mathrm{conj} \colon \mathbb{V}_{\kappa(F)} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{V}_F \otimes_{\mathbb{Z}} \mathbb{C}$$

is anti-linear, preserves the Hodge decompositions [Sil89, Chapter I, Lemma (2.4)] as well as the eigenspace decompositions. $\square$

We obtain a *real period map*

$$\mathcal{P}_{\mathbb{R}} \colon \mathcal{F}_0(\mathbb{R}) = \textstyle\coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_0^\alpha \longrightarrow \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^2_\alpha = \widetilde{Y}. \tag{5.25}$$

Define $\mathbb{G}(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$. The map (5.25) is constant on $\mathbb{G}(\mathbb{R})$-orbits, since the same is true for the complex period map $\mathcal{P} \colon \mathcal{F}_0 \to \mathbb{C}H^2$.

By abuse of notation, we for $\alpha \in P\mathscr{A}$, we write $\mathbb{R}H^2_\alpha - \mathscr{H} = \mathbb{R}H^2_\alpha - \left(\mathscr{H} \cap \mathbb{R}H^2_\alpha\right)$.

Proposition 5.17. *The real period map (5.25) descends to a $P\Gamma$-equivariant diffeomorphism*

$$\mathcal{M}_0(\mathbb{R})^f := \mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_0(\mathbb{R}) \cong \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^2_\alpha - \mathscr{H}. \tag{5.26}$$

By $P\Gamma$-equivariance, the map (5.26) induces an isomorphism of real-analytic orbifolds

$$\mathcal{P}_{\mathbb{R}} \colon \mathcal{M}_0(\mathbb{R}) = \mathbb{G}(\mathbb{R}) \setminus X_0(\mathbb{R}) \cong \coprod_{\alpha \in C\mathscr{A}} P\Gamma_\alpha \setminus \left(\mathbb{R}H^2_\alpha - \mathscr{H}\right). \tag{5.27}$$

Proof. This follows from [ACT10, *proof of Theorem 3.3*]. It is crucial that the actions of G and $P\Gamma$ on $\mathcal{F}_0$ commute and are free, which is the case, see Corollary 5.13. □

5.3.2 *The period map for smooth real binary quintics*

Our next goal will be to prove the real analogue of the isomorphisms (5.21) and (5.21) in Theorem 5.14. We need a lemma, a definition, and then two more lemmas.

Consider the CM-type $\Psi = \{\tau_1, \tau_2\}$ defined in (5.7), the hermitian $\mathcal{O}_K$-lattice $(\Lambda, \mathfrak{h})$ defined in (5.9), and the sets (c.f. Definition 4.11):

$$\mathcal{H} = \left\{H_r \subset \mathbb{C}H^2 \mid r \in \mathscr{R}\right\}, \qquad \text{and} \qquad \mathscr{H} = \cup_{H \in \mathcal{H}} H \subset \mathbb{C}H^2.$$

Here, $\mathscr{R} \subset \Lambda$ is the set of short roots (see Section 4.2.1).

Lemma 5.18. *The hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^2$ satisfies Condition 4.9, that is: any two distinct $H_1, H_2 \in \mathcal{H}$ either meet orthogonally, or not at all.*

Proof. Condition 4.49.1 holds because K does not contain proper CM-subfields. By Lemma 4.52, we have that Condition 4.49.2 is satisfied. By equation (5.10), Condition 4.49.3 holds. By Theorem 4.50, we obtain the desired result. □

Definition 5.19.

1. For $k = 1, 2$, define $\Delta_k \subset \Delta = X_s(\mathbb{C}) - X_0(\mathbb{C})$ to be the locus of stable binary quintics with exactly k nodes. Define $\widetilde{\Delta} = \mathcal{F}_s - \mathcal{F}_0$, and let $\widetilde{\Delta}_k \subset \widetilde{\Delta}$ be the inverse image of Δ_k in $\widetilde{\Delta}$ under the map $\widetilde{\Delta} \to \Delta$.
2. For $k = 1, 2$, define $\mathscr{H}_k \subset \mathscr{H}$ as the set $\mathscr{H}_k = \left\{x \in \mathbb{C}H^2 \mid |\mathcal{H}(x)| = k\right\}$. Thus, this is the locus of points in $\mathscr{H}$ where exactly k hyperplanes meet.

Lemma 5.20.

1. *The period map $\overline{\mathcal{P}}$ of (5.20) satisfies $\overline{\mathcal{P}}(\widetilde{\Delta}_k) \subset \mathscr{H}_k$.*
2. *If $f \in \widetilde{\Delta}_k$, $x = \overline{\mathcal{P}(f)} \in \mathbb{C}H^2$, and $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$ for $r_i \in \mathscr{R}$, then $\overline{\mathcal{P}}$ induces a group isomorphism $P\Gamma_f \cong G(x)$.* □

The naturality of the Fox completion implies that for $\alpha \in P\mathscr{A}$, the anti-holomorphic involution $\alpha : \mathcal{F}_0 \to \mathcal{F}_0$ extends to an anti-holomorphic involution $\alpha : \mathcal{F}_s \to \mathcal{F}_s$.

Lemma 5.21. *For every $\alpha \in P\mathscr{A}$, the restriction of $\overline{\mathcal{P}} : \mathcal{F}_s \to \mathbb{C}H^2$ to $\mathcal{F}_s^\alpha$ defines a diffeomorphism $\mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_s^\alpha \cong \mathbb{R}H^2_\alpha$.*

Proof. See [ACT10, Lemma 11.3]. It is essential that G acts freely on $\mathcal{F}_s$, which holds by Corollary 5.13. □

We arrive at the main theorem of Section 5.3. Define

$$\mathcal{F}_s(\mathbb{R}) = \bigcup_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha = \pi^{-1}\left(X_s(\mathbb{R})\right).$$

This is not a manifold because of the ramification of $\pi : \mathcal{F}_s \to X_s(\mathbb{C})$, but a union of embedded submanifolds.

Theorem 5.22. *There is a smooth map*

$$\overline{\mathcal{P}}_\mathbb{R} : \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha \to \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^2_\alpha = \widetilde{Y} \tag{5.28}$$

that extends the real period map (5.25). The map (5.28) induces the following commutative diagram of topological spaces, in which $\mathscr{P}_\mathbb{R}$ and $\mathfrak{P}_\mathbb{R}$ are homeomorphisms:

$$\begin{tikzcd}
 & \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha \arrow[r, "\overline{\mathcal{P}}_\mathbb{R}"] \arrow[d] & \widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^2_\alpha \arrow[d] \\
 & \mathcal{F}_s(\mathbb{R}) \arrow[r, "\overline{\mathcal{P}}_\mathbb{R}"] \arrow[d] & Y \arrow[d, equal] \\
\mathcal{M}_s(\mathbb{R})^f \arrow[r, equal] \arrow[d] & \mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R}) \arrow[r, "\mathscr{P}_\mathbb{R}", "\sim"'] \arrow[d] & Y \arrow[d] \\
\mathcal{M}_s(\mathbb{R}) \arrow[r, equal] & \mathbb{G}(\mathbb{R}) \setminus X_s(\mathbb{R}) \arrow[r, "\mathfrak{P}_\mathbb{R}", "\sim"'] & P\Gamma \setminus Y.
\end{tikzcd}$$

Proof. The existence of $\overline{\mathcal{P}}_\mathbb{R}$ follows from the compatibility with the involutions $\alpha \in P\mathscr{A}$. We first show that the composition

$$\coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha \xrightarrow{\overline{\mathcal{P}}_\mathbb{R}} \widetilde{Y} \xrightarrow{p} Y$$

factors through $\mathcal{F}_s(\mathbb{R})$. Now f_α and $g_\beta \in \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha$ have the same image in $\mathcal{F}_s(\mathbb{R})$ if and only if $f = g \in \mathcal{F}_s^\alpha \cap \mathcal{F}_s^\beta$, in which case

$$x := \overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g) \in \mathbb{R}H^2_\alpha \cap \mathbb{R}H^2_\beta,$$

so we need to show is that $x_\alpha \sim x_\beta \in \widetilde{Y}$. For this, note that $\alpha\beta \in P\Gamma_f \cong (\mathbb{Z}/10)^k$, and $\overline{\mathcal{P}}$ induces an isomorphism $P\Gamma_f \cong G(x)$ by Lemma 5.20. Hence $\alpha\beta \in G(x)$ so that indeed, $x_\alpha \sim x_\beta$.

Let us prove the $\mathbb{G}(\mathbb{R})$-equivariance of $\overline{\mathcal{P}}_\mathbb{R}$. Suppose that

$$f \in \mathcal{F}_s^\alpha, g \in \mathcal{F}_s^\beta \quad | \quad a \cdot f = g \in \mathcal{F}_s(\mathbb{R}) \quad \text{for some} \quad a \in \mathbb{G}(\mathbb{R}).$$

Then $x := \overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g) \in \mathbb{C}H^2$, so we need to show that $\alpha\beta \in G(x)$. The actions of $\mathbb{G}(\mathbb{C})$ and $P\Gamma$ on $\mathbb{C}H^2$ commute, and the same holds for the actions of $\mathbb{G}(\mathbb{R})$ and $P\Gamma'$ on $\mathcal{F}_s^\mathbb{R}$. It follows that

$$\alpha(g) = \alpha(a \cdot f) = a \cdot \alpha(f) = a \cdot f = g,$$

hence $g \in \mathcal{F}_s^\alpha \cap \mathcal{F}_s^\beta$. This implies in turn that $\alpha\beta(g) = g$, hence $\alpha\beta \in P\Gamma_g \cong G(x)$, so that indeed, $x_\alpha \sim x_\beta$.

To prove that $\mathscr{P}_\mathbb{R}$ is injective, let again $f_\alpha, g_\beta \in \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha$ and suppose that they have the same image in Y. This implies that

$$x := \overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g) \in \mathbb{R}H^2_\alpha \cap \mathbb{R}H^2_\beta,$$

and that $\beta = \phi \circ \alpha$ for some $\phi \in G(x)$. We have $\phi \in G(x) \cong P\Gamma_f$ (by Lemma 5.20) hence

$$\beta(f) = \phi\left(\alpha(f)\right) = \phi(f) = f. \tag{5.29}$$

Therefore $f, g \in \mathcal{F}_s^\beta$; since $\overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g)$, it follows from Lemma 5.21 that there exists $a \in \mathbb{G}(\mathbb{R})$ such that $a \cdot f = g$. This proves injectivity of $\mathscr{P}_\mathbb{R}$, as desired.

The surjectivity of $\mathscr{P}_\mathbb{R} : \mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R}) \to Y$ is straightforward, using the surjectivity of $\overline{\mathcal{P}}_\mathbb{R}$, which follows from Lemma 5.21.

Finally, we claim that $\mathscr{P}_\mathbb{R}$ is open. Let $U \subset \mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_s^\mathbb{R}$, and write $U = \mathscr{P}_\mathbb{R}^{-1} \mathscr{P}_\mathbb{R}(U)$. Let V be the preimage of U in $\coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha$. Then

$$V = \overline{\mathcal{P}}_\mathbb{R}^{-1}\left(p^{-1}\left(\mathscr{P}_\mathbb{R}(U)\right)\right)$$

and hence

$$\overline{\mathcal{P}}_{\mathbb{R}}(V) = p^{-1}\left(\mathscr{P}_{\mathbb{R}}(U)\right).$$

The map $\overline{\mathcal{P}}_{\mathbb{R}}$ is open, being the coproduct of the maps $\mathcal{F}_s^\alpha \to \mathbb{R}H^2_\alpha$, which are open since they have surjective differential at each point. Thus $\mathscr{P}_{\mathbb{R}}(U)$ is open in Y. □

Corollary 5.23. *There is a lattice $P\Gamma_{\mathbb{R}} \subset \mathrm{PO}(2,1)$, an inclusion of orbifolds*

$$\coprod_{\alpha \in C\mathscr{A}} P\Gamma_\alpha \setminus \left(\mathbb{R}H^2_\alpha - \mathscr{H}\right) \hookrightarrow P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^2, \tag{5.30}$$

and a homeomorphism

$$\mathcal{M}_s(\mathbb{R}) = \mathbb{G}(\mathbb{R}) \setminus X_s(\mathbb{R}) \cong P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^2 \tag{5.31}$$

such that (5.31) restricts to (5.27) with respect to (5.30).

Proof. This follows directly from Theorems 4.24 and 5.22. □

Remark 5.24. The proof of Theorem 5.22 also shows that $\mathcal{M}_s(\mathbb{R})$ is homeomorphic to the glued space $P\Gamma \setminus Y$ (see Definition 4.22) if $\mathcal{M}_s$ is the stack of cubic surfaces or of binary sextics. This strategy to uniformize the real moduli space differs from the one used in [ACT10, ACT06, ACT07], since we first glue together the real ball quotients, and then prove that our real moduli space is homeomorphic to the result.

5.3.3 *Automorphism groups of stable real binary quintics*

Before we can finish the proof of Theorem 5.2, we need to understand the orbifold structure of $\mathcal{M}_s(\mathbb{R})$, and how this structure differs from the orbifold structure of the glued space $P\Gamma \setminus Y$. In the current Section 5.3.3 we start by analyzing the orbifold structure of $\mathcal{M}_s(\mathbb{R})$, by listing its stabilizer groups. There is a canonical orbifold isomorphism $\mathcal{M}_s(\mathbb{R}) = \mathbb{G}(\mathbb{R}) \setminus X_s(\mathbb{R}) = (P_s/\mathfrak{S}_5)(\mathbb{R})$. Therefore, to list the automorphism groups of binary quintics is to list the elements $x = [\alpha_1, \ldots, \alpha_5] \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ whose stabilizer $\mathrm{PGL}_2(\mathbb{R})_x$ is non-trivial, and calculate $\mathrm{PGL}_2(\mathbb{R})_x$ in these cases. This will be our next goal.

Proposition 5.25. *All stabilizer groups $\mathrm{PGL}_2(\mathbb{R})_x \subset \mathrm{PGL}_2(\mathbb{R})$ for points $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ are among $\mathbb{Z}/2, D_3, D_5$. For $n \in \{3,5\}$, there is a unique $\mathrm{PGL}_2(\mathbb{R})$-orbit in $(P_s/\mathfrak{S}_5)(\mathbb{R})$ of points x with stabilizer D_n.*

Proof. We have an injection $(P_s/\mathfrak{S}_5)(\mathbb{R}) \hookrightarrow P_s/\mathfrak{S}_5$ which is equivariant for the embedding $\mathrm{PGL}_2(\mathbb{R}) \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$. In particular, $\mathrm{PGL}_2(\mathbb{R})_x \subset \mathrm{PGL}_2(\mathbb{C})_x$ for every $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$. The groups $\mathrm{PGL}_2(\mathbb{C})_x$ for equivalence classes of distinct points $x \in P_0/\mathfrak{S}_5$ are calculated in [WX17, Theorem 22], and such a group is isomorphic to $\mathbb{Z}/2, D_3, \mathbb{Z}/4$ or D_5. None of these have subgroups isomorphic to $D_2 = \mathbb{Z}/2 \rtimes \mathbb{Z}/2$ or $D_4 = \mathbb{Z}/2 \rtimes \mathbb{Z}/4$. Define an involution

$$\nu := (z \mapsto 1/z) \in \mathrm{PGL}_2(\mathbb{R}).$$

The proof of Proposition 5.25 will follow from the following Lemmas 5.26, 5.27, 5.28 and 5.29.

Lemma 5.26. *Let $\tau \in \mathrm{PGL}_2(\mathbb{R})$. Consider a subset $S = \{x, y, z\} \subset \mathbb{P}^1(\mathbb{C})$ stabilized by complex conjugation, such that $\tau(x) = x$, $\tau(y) = z$ and $\tau(z) = y$. There is a transformation $g \in \mathrm{PGL}_2(\mathbb{R})$ that maps S to either $\{-1, 0, \infty\}$ or $\{-1, i, -i\}$, and that satisfies $g\tau g^{-1} = \nu = (z \mapsto 1/z) \in \mathrm{PGL}_2(\mathbb{R})$. In particular, $\tau^2 = \mathrm{id}$.*

Proof. Indeed, two transformations $g, h \in \mathrm{PGL}_2(\mathbb{C})$ that satisfy $g(x_i) = h(x_i)$ for three different points $x_1, x_2, x_3 \in \mathbb{P}^1(\mathbb{C})$ are necessarily equal. $\square$

Lemma 5.27. *There is no $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ stabilized by some $\phi \in \mathrm{PGL}_2(\mathbb{R})$ of order 4.*

Proof. By [Bea10, Theorem 4.2], all subgroups $G \subset \mathrm{PGL}_2(\mathbb{R})$ that are isomorphic to $\mathbb{Z}/4$ are conjugate to each other. Since the transformation $I : z \mapsto (z-1)/(z+1)$ is of order 4, it gives a representative $G_I = \langle I \rangle$ of this conjugacy class. Hence, assuming there exists x and ϕ as in the lemma, possibly after replacing x by gx for some $g \in \mathrm{PGL}_2(\mathbb{R})$, we may and do assume that $\phi = I$. On the other hand, it is easily shown that I cannot fix any $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$. $\square$

Define

$$\rho \in \mathrm{PGL}_2(\mathbb{R}), \quad \rho(z) = \frac{-1}{z+1}.$$

Lemma 5.28. *Let $x = (x_1, \ldots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$. Suppose $\phi(x) = x$ for an element $\phi \in \mathrm{PGL}_2(\mathbb{R})$ of order 3. There is a transformation $g \in \mathrm{PGL}_2(\mathbb{R})$ mapping x to $z = (-1, \infty, 0, \omega, \omega^2)$ with ω a primitive third root of unity, and the stabilizer of x to the subgroup of $\mathrm{PGL}_2(\mathbb{R})$ generated by ρ and ν. In particular, $\mathrm{PGL}_2(\mathbb{R})_x$ is isomorphic to D_3.*

Proof. It follows from Lemma 5.26 that there must be three elements x_1, x_2, x_3 which form an orbit under ϕ. Since complex conjugation preserves this orbit, one element in it is real; since g is defined over $\mathbb{R}$, they are all real. Let $g \in \mathrm{PGL}_2(\mathbb{R})$ such that $g(x_1) = -1$, $g(x_2) = \infty$ and $g(x_3) = 0$. Define $\kappa = g\phi g^{-1}$. Then $\kappa^3 = \mathrm{id}$, and κ preserves $\{-1, \infty, 0\}$ and sends -1 to ∞ and ∞ to 0. Consequently, $\kappa(0) = -1$, and it follows that $\kappa = \rho$. Hence x is equivalent to an element of the form $z = (-1, \infty, 0, \alpha, \beta)$. Moreover, $\beta = \bar{\alpha}$ and $\alpha^2 + \alpha + 1 = 0$. $\square$

Recall that $\zeta_5 = e^{2i\pi/5} \in \mathbb{P}^1(\mathbb{C})$ and define

$$\lambda = \zeta_5 + \zeta_5^{-1} \in \mathbb{R}, \qquad \gamma(z) = \frac{(\lambda+1)z - 1}{z+1} \in \mathrm{PGL}_2(\mathbb{R}).$$

Lemma 5.29. *Let $x = (x_1, \ldots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$. Suppose x is stabilized by a subgroup of $\mathrm{PGL}_2(\mathbb{R})$ of order 5. There is a transformation $g \in \mathrm{PGL}_2(\mathbb{R})$ mapping x to $z = (0, -1, \infty, \lambda+1, \lambda)$ and identifying the stabilizer of x with the subgroup of $\mathrm{PGL}_2(\mathbb{R})$ generated by γ and ν. In particular, the stabilizer $\mathrm{PGL}_2(\mathbb{R})_x$ of x is isomorphic to D_5.*

Proof. Let $\phi \in \mathrm{PGL}_2(\mathbb{R})_x$ be an element of order 5. Using Lemma 5.26 one shows that x must be smooth, i.e. all x_i are distinct, and $x_i = \phi^{i-1}(x_1)$. Since there is one real x_i and ϕ is defined over $\mathbb{R}$, all x_i are real. Now note that $z = (0, -1, \infty, \lambda+1, \lambda)$ is the orbit of 0 under $\gamma : z \mapsto ((\lambda+1)z - 1)/(z+1)$. The reflection $\nu : z \mapsto 1/z$ preserves z as well: if $\zeta = \zeta_5$ then $\lambda = \zeta + \zeta^{-1}$ hence $\lambda + 1 = -(\zeta^2 + \zeta^{-2}) = -\lambda^2 + 2$, so that $\lambda(\lambda+1) = 1$. So we have $\mathrm{PGL}_2(\mathbb{R})_z \cong D_5$. By [WX17, Theorem 22], the point z with its stabilizer $\mathrm{PGL}_2(\mathbb{R})_z$ must be equivalent under $\mathrm{PGL}_2(\mathbb{C})$ to the point $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ with its stabilizer $\langle x \mapsto \zeta x, x \mapsto 1/x \rangle$. Thus, there exists $g \in \mathrm{PGL}_2(\mathbb{C})$ such that $g(x_1) = 0$, $g(x_2) = -1$, $g(x_3) = \infty$, $g(x_4) = \lambda + 1$ and $g(x_5) = \lambda$, and such that $g\mathrm{PGL}_2(\mathbb{R})_x g^{-1} = \mathrm{PGL}_2(\mathbb{R})_z$. Since all x_i and $z_i \in z$ are real, we see that $\bar{g}(x_i) = z_i$ for every i, hence g and $\bar{g}$ coincide on more than 2 points, hence $g = \bar{g} \in \mathrm{PGL}_2(\mathbb{R})$. $\square$

Proposition 5.25 follows. $\square$

5.3.4 *Binary quintics with automorphism group of order two*

The goal of Section 5.3.4 is to prove that there are no cone points in the orbifold $\mathrm{PGL}_2(\mathbb{R}) \setminus (P_s/\mathfrak{S}_5)(\mathbb{R})$, i.e. orbifold points whose stabilizer group is $\mathbb{Z}/n$ for some n acting on the orbifold chart by rotations. By Proposition 5.25, this fact will follow from the following:

Proposition 5.30. *Let $x = (x_1, \ldots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ such that $\mathrm{PGL}_2(\mathbb{R})_x = \langle \tau \rangle$ has order two. There is a $\mathrm{PGL}_2(\mathbb{R})_x$-stable open neighborhood $U \subset (P_s/\mathfrak{S}_5)(\mathbb{R})$ of x such that $\mathrm{PGL}_2(\mathbb{R})_x \setminus U \to \mathcal{M}_s(\mathbb{R})$ is injective, and a homeomorphism $\phi : (U, x) \to (B, 0)$ for $0 \in B \subset \mathbb{R}^2$ an open ball, such that ϕ identifies $\mathrm{PGL}_2(\mathbb{R})_x$ with $\mathbb{Z}/2$ acting on B by reflections in a line through 0.*

Proof. Using Lemma 5.26, one checks that the only possibilities for the element $x = (x_1, \ldots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ are $(-1, 0, \infty, \beta, \beta^{-1})$, $(-1, i, -i, \beta, \beta^{-1})$, $(-1, -1, \beta, 0, \infty)$, $(-1, -1, \beta, i, -i)$, $(0, 0, \infty, \infty, -1)$ and $(-1, i, i, -i, -i)$. □

5.3.5 *Comparing the orbifold structures*

Consider the moduli space $\mathcal{M}_s(\mathbb{R})$ of real stable binary quintics.

Definition 5.31. Let $\overline{\mathscr{M}}_{\mathbb{R}}$ be the hyperbolic orbifold with $|\mathcal{M}_s(\mathbb{R})|$ as underlying space, whose orbifold structure is induced by the homeomorphism of Corollary 5.23 and the natural orbifold structure of $P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^2$.

There are two orbifold structures on the space $|\mathcal{M}_s(\mathbb{R})|$: the natural orbifold structure of $\mathcal{M}_s(\mathbb{R})$, see Proposition 2.12 (i.e. the orbifold structure of the quotient $\mathbb{G}(\mathbb{R}) \setminus X_s(\mathbb{R})$), and the orbifold structure $\overline{\mathscr{M}}_{\mathbb{R}}$ introduced in Definition 5.31.

Proposition 5.32.

1. *The orbifold structures of $\mathcal{M}_s(\mathbb{R})$ and $\overline{\mathscr{M}}_{\mathbb{R}}$ differ only at the moduli point attached to the five-tuple $(\infty, i, i, -i, -i)$. The stabilizer group of $\mathcal{M}_s(\mathbb{R})$ at that moduli point is $\mathbb{Z}/2$, whereas the stabilizer group of $\overline{\mathscr{M}}_{\mathbb{R}}$ at that point is the dihedral group D_{10} of order twenty.*
2. *The orbifold $\overline{\mathscr{M}}_{\mathbb{R}}$ has no cone points and three corner reflectors, whose orders are $\pi/3, \pi/5$ and $\pi/10$.*

Proof. The statements can be deduced from Proposition 4.37. The notation of that proposition was as follows: for $f \in Y \cong \mathbb{G}(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R})$ (see Theorem 5.22) the group $A_f \subset P\Gamma$ is the stabilizer of $f \in K$. Moreover, if $\tilde{f} \in \mathcal{F}_s(\mathbb{R})$ represents f and if $F = [\tilde{f}] \in X_s(\mathbb{R})$ has $k = 2a + b$ nodes, then the image $x \in \mathbb{C}H^2$ has $k = 2a + b$ nodes in the sense of Definition 4.11. If F has no nodes ($k = 0$), then $G(x)$ is trivial by Proposition 4.37.1 and $G_F = A_f = \Gamma_f$. If F has only real nodes, then $B_f = G(x)$ hence $G_F = A_f/G(x) = A_f/B_f = \Gamma_f$. Now suppose that $a = 1$ and $b = 0$: the equation F defines a pair of complex conjugate nodes. In other words, the zero set of

F defines a 5-tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_5) \in \mathbb{P}^1(\mathbb{C})$, well-defined up to the $\mathrm{PGL}_2(\mathbb{R}) \times \mathfrak{S}_5$ action on $\mathbb{P}^1$, where $\alpha_1 \in \mathbb{P}^1(\mathbb{R})$ and $\alpha_3 = \bar{\alpha}_2 = \alpha_5 = \bar{\alpha}_4 \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. So we may write $\underline{\alpha} = (\rho, \alpha, \bar{\alpha}, \alpha, \bar{\alpha})$ with $\rho \in \mathbb{P}^1(\mathbb{R})$ and $\alpha \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. Then there is a unique $T \in \mathrm{PGL}_2(\mathbb{R})$ such that $T(\rho) = \infty$ and $T(\alpha) = i$. But this gives $T(x) = (\infty, i, -i, i, -i)$ hence F is unique up to isomorphism. As for the stabilizer $G_F = A_f / G(x)$, we have $G(x) \cong (\mathbb{Z}/10)^2$. Since there are no real nodes, B_f is trivial. By Proposition 4.37.3, K_f is the union of ten copies of $\mathbb{B}^2(\mathbb{R})$ meeting along a common point $\mathbb{B}^0(\mathbb{R})$. In fact, in the local coordinates (t_1, t_2) around f, the $\alpha_j : \mathbb{B}^2(\mathbb{C}) \to \mathbb{B}^2(\mathbb{C})$ are defined by $(t_1, t_2) \mapsto (\bar{t}_2 \zeta^j, \bar{t}_1 \zeta^j)$, for $j \in \mathbb{Z}/10$, and so the fixed points sets are given by the equations $\mathbb{R}H^2_j = \{t_2 = \bar{t}_1 \zeta^j\} \subset \mathbb{B}^2(\mathbb{C})$, $j \in \mathbb{Z}/10$. Notice that the subgroup $E \subset G(x)$ that stabilizes $\mathbb{R}H^2_j$ is the cyclic group of order 10 generated by the transformations $(t_1, t_2) \mapsto (\zeta t_1, \zeta^{-1} t_2)$. There is only one non-trivial transformation $T \in \mathrm{PGL}_2(\mathbb{R})$ that fixes ∞ and sends the subset $\{i, -i\} \subset \mathbb{P}^1(\mathbb{C})$ to itself, and T is of order 2. Hence $G_F = \mathbb{Z}/2$ so that we have an exact sequence $0 \to \mathbb{Z}/10 \to \Gamma_f \to \mathbb{Z}/2 \to 0$ and this splits since G_F is a subgroup of Γ_f. We are done by Propositions 5.25 and 5.30. $\square$

5.3.6 *The real moduli space as a hyperbolic triangle*

The goal of Section 5.3.6 is to show that $\overline{\mathscr{M}}_{\mathbb{R}}$ (see Definition 5.31) is isomorphic, as hyperbolic orbifolds, to the triangle $\Delta_{3,5,10}$ in the real hyperbolic plane $\mathbb{R}H^2$ with angles $\pi/3, \pi/5$ and $\pi/10$. The results in the above Sections 5.3.3, 5.3.4 and 5.3.5 give the orbifold singularities of $\overline{\mathscr{M}}_{\mathbb{R}}$ together with their stabilizer groups. In order to complete determine the hyperbolic orbifold structure of $\overline{\mathscr{M}}_{\mathbb{R}}$, however, we also need to know the underlying topological space $|\mathcal{M}_s(\mathbb{R})|$ of $\overline{\mathscr{M}}_{\mathbb{R}}$. The first observation is that $|\mathcal{M}_s(\mathbb{R})|$ is compact. Indeed, it is classical that the topological space $\mathcal{M}_s(\mathbb{C}) = \mathbb{G}(\mathbb{C}) \setminus X_s(\mathbb{C})$, parametrizing complex stable binary quintics, is compact. This follows from the fact that it is homeomorphic to $\overline{M}_{0,5}(\mathbb{C})/\mathfrak{S}_5$, and the stack of stable five-pointed curves $\overline{M}_{0,5}$ is proper [Knu83], or from the fact that it is homeomorphic to a compact ball quotient [Shi64]. Moreover, the map $\mathcal{M}_s(\mathbb{R}) \to \mathcal{M}_s(\mathbb{C})$ is proper, which proves the compactness of $\mathcal{M}_s(\mathbb{R})$.

The second observation is that $\mathcal{M}_s(\mathbb{R})$ is connected, since $X_s(\mathbb{R})$ is obtained from the euclidean space $\{F \in \mathbb{R}[x, y] : F \text{ homogeneous } \mid \deg(F) = 5\}$ by removing a subspace of codimension at least two. We can prove more:

Lemma 5.33. *The moduli space $\mathcal{M}_s(\mathbb{R})$ of real stable binary quintics is simply connected.*

Proof. The idea is to show that the following holds:

1. For each $i \in \{0,1,2\}$, the embedding $\mathscr{M}_i \hookrightarrow \overline{\mathscr{M}}_i \subset \mathcal{M}_s(\mathbb{R})$ of the connected component $\mathscr{M}_i$ of $\mathcal{M}_0(\mathbb{R})$ into its closure in $\mathcal{M}_s(\mathbb{R})$ is homeomorphic to the embedding $D \hookrightarrow \overline{D}$ of the open unit disc into the closed unit disc in $\mathbb{R}^2$.
2. We have $\mathcal{M}_s(\mathbb{R}) = \overline{\mathscr{M}}_0 \cup \overline{\mathscr{M}}_1 \cup \overline{\mathscr{M}}_2$, and this glueing corresponds up to homeomorphism to the glueing of three closed discs $\overline{D}_i \subset \mathbb{R}^2$ as in Figure 5.1.

To do this, one considers the moduli spaces of real smooth (resp. stable) genus zero curves with five real marked points [Knu83], as well as twists of this space. Define two anti-holomorphic involutions $\sigma_i : \mathbb{P}^1(\mathbb{C})^5 \to \mathbb{P}^1(\mathbb{C})^5$ by $\sigma_1(x_1, x_2, x_3, x_4, x_5) = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_5, \bar{x}_4)$, and $\sigma(x_1, x_2, x_3, x_4, x_5) = (\bar{x}_1, \bar{x}_3, \bar{x}_2, \bar{x}_5, \bar{x}_4)$. Then define

$$P_0^1(\mathbb{R}) = P_0(\mathbb{C})^{\sigma_1}, \quad P_s^1(\mathbb{R}) = P_1(\mathbb{C})^{\sigma_1}, \quad P_0^2(\mathbb{R}) = P_0(\mathbb{C})^{\sigma_2}, \quad P_s^2(\mathbb{R}) = P_1(\mathbb{C})^{\sigma_2}.$$

It is clear that $\mathscr{M}_0 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0(\mathbb{R}) / \mathfrak{S}_5$. Similarly, we have:

$$\mathscr{M}_1 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0^1(\mathbb{R}) / \mathfrak{S}_3 \times \mathfrak{S}_2 \quad \text{and} \quad \mathscr{M}_2 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0^2(\mathbb{R}) / \mathfrak{S}_2 \times \mathfrak{S}_2.$$

Moreover, we have $\overline{\mathscr{M}}_0 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s(\mathbb{R}) / \mathfrak{S}_5$. We define

$$\overline{\mathscr{M}}_1 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s^1(\mathbb{R}) / \mathfrak{S}_3 \times \mathfrak{S}_2, \quad \text{and} \quad \overline{\mathscr{M}}_2 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s^2(\mathbb{R}) / \mathfrak{S}_2 \times \mathfrak{S}_2.$$

Each $\overline{\mathscr{M}}_i$ is simply connected. Moreover, the natural maps $\overline{\mathscr{M}}_i \to \mathcal{M}_s(\mathbb{R})$ are closed embeddings of topological spaces, and one can check that the images glue to form $\mathcal{M}_s(\mathbb{R})$ in the prescribed way. We leave the details to the reader. □

Proof of Theorem 5.2. To any closed 2-dimensional orbifold O one can associate a set of natural numbers $S_O = \{n_1, \dots, n_k; m_1, \dots, m_l\}$ by letting k be the number of cone points of X_O, l the number of corner reflectors, n_i the order of the i-th cone point and $2m_j$ the order of the j-th corner reflector. A closed 2-dimensional orbifold O is then determined, up to orbifold-structure preserving homeomorphism, by its underlying space X_O and the set S_O [Thu80]. By Lemma 5.33, $\overline{\mathscr{M}}_{\mathbb{R}}$ is simply connected. By Proposition 5.32, $\overline{\mathscr{M}}_{\mathbb{R}}$ has no cone points and three corner reflectors whose orders are $\pi/3, \pi/5$ and $\pi/10$. This implies $\overline{\mathscr{M}}_{\mathbb{R}}$ and $\Delta_{3,5,10}$ are isomorphic as topological orbifolds. Consequently, the orbifold fundamental group of $\overline{\mathscr{M}}_{\mathbb{R}}$ is abstractly isomorphic to the group $P\Gamma_{3,5,10}$ defined in (5.2).

Let $\phi : P\Gamma_{3,5,10} \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$ be *any* embedding such that $X := \phi\,(P\Gamma_{3,5,10}) \setminus \mathbb{R}H^2$ is a hyperbolic orbifold; we claim that there is a fundamental domain Δ of X isometric to $\Delta_{3,5,10}$. Consider the generator $a \in P\Gamma_{3,5,10}$. Since $\phi(a)^2 = 1$, there exists a geodesic $L_1 \subset \mathbb{R}H^2$ such that $\phi(a) \in \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}(\mathbb{R}H^2)$ is the reflection across L_1. Next, consider the generator $b \in P\Gamma_{3,5,10}$. There exists a geodesic $L_2 \subset \mathbb{R}H^2$ such that $\phi(b)$ is the reflection across L_2. One easily shows that $L_2 \cap L_1 \neq \emptyset$. Let $x \in L_1 \cap L_2$. Then $\phi(a)\phi(b)$ is an element of order three that fixes x, hence $\phi(a)\phi(b)$ is a rotation around x. Therefore, one of the angles between L_1 and L_2 must be $\pi/3$. Finally, we know that $\phi(c)$ is an element of order 2 in $\mathrm{PSL}_2(\mathbb{R})$, hence a reflection across a line L_3. By the previous arguments, $L_3 \cap L_2 \neq \emptyset$ and $L_3 \cap L_1 \neq \emptyset$. It also follows that $x \in L_3 \cap L_2 \cap L_1 = \emptyset$. Consequently, the three geodesics $L_i \subset \mathbb{R}H^2$ enclose a hyperbolic triangle; the orders of $\phi(a)\phi(b)$, $\phi(a)\phi(c)$ and $\phi(b)\phi(c)$ imply that the three interior angles of the triangle are $\pi/3$, $\pi/5$ and $\pi/10$. $\square$

5.4 THE MONODROMY GROUPS

In this section, we describe the monodromy group $P\Gamma$, as well as the groups $P\Gamma_\alpha$ appearing in Proposition 5.17. As for the lattice $(\Lambda, \mathfrak{h})$ (see (5.9)), we have:

Theorem 5.34 (Shimura). *There is an isomorphism of hermitian $\mathcal{O}_K$-lattices*

$$(\Lambda, \mathfrak{h}) \cong \left(\mathcal{O}_K^3, \mathrm{diag}\left(1, 1, \frac{1-\sqrt{5}}{2}\right)\right).$$

Proof. See [Shi64, Section 6] as well as item (5) in the table on page 1. $\square$

Let us write $\Lambda = \mathcal{O}_K^3$ and $\mathfrak{h} = \mathrm{diag}(1, 1, \frac{1-\sqrt{5}}{2})$ in the remaining part of Section 5.4. Write $\alpha = \zeta_5 + \zeta_5^{-1} = \frac{\sqrt{5}-1}{2}$. Recall that $\theta = \zeta_5 - \zeta_5^{-1}$ and observe that $|\theta|^2 = \frac{\sqrt{5}+5}{2}$. Define three quadratic forms q_0, q_1 and q_2 on $\mathbb{Z}[\alpha]^3$ as follows:

$$\begin{aligned} q_0(x_0, x_1, x_2) &= x_0^2 + x_1^2 - \alpha x_2^2, \\ q_1(x_0, x_1, x_2) &= |\theta|^2 x_0^2 + x_1^2 - \alpha x_2^2, \\ q_2(x_0, x_1, x_2) &= |\theta|^2 x_0^2 + |\theta|^2 x_1^2 - \alpha x_2^2. \end{aligned} \tag{5.32}$$

We consider $\mathbb{Z}[\alpha]$ as a subring of $\mathbb{R}$ via the standard embedding.

Theorem 5.35. *Consider the quadratic forms q_j defined in (5.32). There is a union of geodesic subspaces $\mathscr{H}_j \subset \mathbb{R}H^2$ ($j \in \{0,1,2\}$) and an isomorphism of hyperbolic orbifolds*

$$\mathcal{M}_0(\mathbb{R}) \cong \coprod_{j=0}^{2} \mathrm{PO}(q_j, \mathbb{Z}[\alpha]) \setminus \left(\mathbb{R}H^2 - \mathscr{H}_j\right). \tag{5.33}$$

Proof. Recall that $\theta = \zeta_5 - \zeta_5^{-1}$; we consider the $\mathbb{F}_5$-vector space W equipped with the quadratic form $q = \mathfrak{h} \mod \theta$. Define three anti-isometric involutions as follows:

$$\begin{aligned} \alpha_0 :&(x_0, x_1, x_2) \mapsto (\ \ \bar{x}_0, \ \ \ \bar{x}_1, \bar{x}_2) \\ \alpha_1 :&(x_0, x_1, x_2) \mapsto (-\bar{x}_0, \ \ \ \bar{x}_1, \bar{x}_2) \\ \alpha_2 :&(x_0, x_1, x_2) \mapsto (-\bar{x}_0, -\bar{x}_1, \bar{x}_2). \end{aligned} \tag{5.34}$$

For isometries $\alpha : W \to W$, the dimension and determinant of the fixed space $(W^\alpha, q|_{W^\alpha})$ are conjugacy-invariant. Using this, one easily shows that an anti-unitary involution of Λ is Γ-conjugate to exactly one of the $\pm\alpha_j$, hence $C\mathscr{A}$ has cardinality 3 and is represented by $\alpha_0, \alpha_1, \alpha_2$ of (5.34). By Proposition 5.17, we obtain $\mathcal{M}_0(\mathbb{R}) \cong \coprod_{j=0}^{2} P\Gamma_{\alpha_j} \setminus (\mathbb{R}H^2_{\alpha_j} - \mathscr{H})$ where each hyperbolic quotient $P\Gamma_{\alpha_j} \setminus (\mathbb{R}H^2_{\alpha_j} - \mathscr{H})$ is connected. Next, consider the fixed lattices

$$\begin{aligned} \Lambda_0 &:= \Lambda^{\alpha_0} = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \\ \Lambda_1 &:= \Lambda^{\alpha_1} = \theta\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \\ \Lambda_2 &:= \Lambda^{\alpha_2} = \theta\mathbb{Z}[\alpha] \oplus \theta\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha]. \end{aligned} \tag{5.35}$$

One easily shows that $P\Gamma_{\alpha_j} = N_{P\Gamma}(\alpha_j)$ for the normalizer $N_{P\Gamma}(\alpha_j)$ of α_j in $P\Gamma$. Moreover, if h_j denotes the restriction of $\mathfrak{h}$ to Λ^{α_j}, then there is a natural embedding

$$\iota : N_{P\Gamma}(\alpha_j) \hookrightarrow \mathrm{PO}(\Lambda_j, h_j, \mathbb{Z}[\alpha]). \tag{5.36}$$

We claim that ι is actually an isomorphism. Indeed, this follows from the fact that the natural homomorphism $\pi : N_\Gamma(\alpha_j) \to O(\Lambda_j, h_j)$ is surjective, where $N_\Gamma(\alpha_j) = \{g \in \Gamma : g \circ \alpha_j = \alpha_j \circ g\}$ is the normalizer of α_j in Γ. The surjectivity of π follows in turn from the equality

$$\Lambda = \mathcal{O}_K \cdot \Lambda_j + \mathcal{O}_K \cdot \theta\Lambda_j^\vee \subset K^3$$

which follows from (5.35). Since $\mathrm{PO}(\Lambda_j, h_j, \mathbb{Z}[\alpha]) = \mathrm{PO}(q_j, \mathbb{Z}[\alpha])$, we are done. $\square$

Part II

Real algebraic cycles

6

INTEGRAL FOURIER TRANSFORMS

This chapter is based on joint work with Thorsten Beckmann.

6.1 Introduction

In the second part of this thesis, we focus on algebraic cycles on complex and real abelian varieties. A central role in this study – which takes up Chapters 6, 7 and 8 – is played by a certain correspondence. It has since long been known that for an abelian variety A over a field k, with dual abelian variety $\widehat{A}$, the Fourier transform

$$\mathcal{F}_A \colon \mathrm{CH}(A)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}(\widehat{A})_{\mathbb{Q}} \tag{6.1}$$

provides a powerful tool to study the $\mathbb{Q}$-linear algebraic cycles of A. It is used to study the rational Chow ring of A, as well as the cycle class map to rational cohomology. Recently, Moonen and Polishchuk [MP10] initiated the study of the integrality aspects of the Fourier transform (6.1). Indeed, it is natural to ask:

Question 6.1. How does $\mathcal{F}_A$ interact with integral algebraic cycles?

The goal of the current Chapter 6 is to work on Question 6.1, building on the results of Moonen–Polishchuk. The applications of *loc.cit.* primarily concern the structure of the integral Chow rings themselves. We continue with their study, but we also address the compatibility of Fourier transforms with integral cycle class maps. Since Question 6.1 was phrased somewhat imprecisely, let us explain in some detail the steps that we take during our search for an answer:

Chapter 6: We further develop the theory of *integral Fourier transforms*, on Chow rings as well as on cohomology. The main result of this chapter (Theorem 6.9) will provide necessary and sufficient conditions for the Fourier transform (6.1) to lift to a motivic homomorphism between integral Chow groups.

Chapter 7: We apply the theory of Chapter 6 to complex abelian varieties. The main outcome of this project is Theorem 7.1, which says that on a principally polarized complex abelian variety A whose minimal cohomology class is algebraic, all integral Hodge classes of degree $2\dim(A) - 2$ are algebraic.

Chapter 8: We apply the theory of Chapter 6 (which is developed for abelian varieties over an arbitrary field k) to the case of abelian varieties over $k = \mathbb{R}$. The principal outcome is that modulo torsion, every real abelian threefold satisfies the real integral Hodge conjecture (Theorem 8.3). Other applications of integral Fourier transforms include a detailed analysis of the Hochschild-Serre filtration on equivariant cohomology of a real abelian variety (Theorem 8.8).

Having lifted a veil of what to do with integral Fourier transforms, let us now make Question 6.1 more precise. Let g be a positive integer and let A be an abelian variety of dimension g over a field k. *Fourier transforms* are correspondences between the derived categories, rational Chow groups and cohomology of A and $\widehat{A}$ attached to the Poincaré bundle $\mathcal{P}_A$ on $A \times \widehat{A}$ [Muk81, Bea82, Huy06]. On the level of cohomology, the Fourier transform preserves integral ℓ-adic étale cohomology when $k = k_s$ (separable closure), and integral Betti cohomology when $k = \mathbb{C}$. It is thus natural to ask whether the Fourier transform on rational Chow groups preserves integral cycles modulo torsion or, more generally, lifts to a homomorphism between integral Chow groups. This question was raised by Moonen–Polishchuk [MP10] and Totaro [Tot21]. More precisely, Moonen and Polishchuk gave a counterexample for abelian varieties over non-closed fields and asked about the case of algebraically closed fields.

The goal of Chapter 6 is to investigate this question further.

6.2 INTEGRAL FOURIER TRANSFORMS

The main result of Section 6 gives necessary and sufficient conditions for the Fourier transform on rational Chow groups or cohomology to lift to a motivic homomorphism between integral Chow groups. To get there, we need a precise definition of "integral Fourier transform", which we introduce in this Section 6.2.

6.2.1 *Notation and conventions*

We let k be a field with separable closure k_s and ℓ a prime number different from the characteristic of k. For a smooth projective variety X over k, we let $\mathrm{CH}(X)$ be the Chow group of X and define $\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{CH}(X) \otimes \mathbb{Q}$, $\mathrm{CH}(X)_{\mathbb{Q}_\ell} = \mathrm{CH}(X) \otimes \mathbb{Q}_\ell$ and $\mathrm{CH}(X)_{\mathbb{Z}_\ell} = \mathrm{CH}(X) \otimes \mathbb{Z}_\ell$. We let $\mathrm{H}^i_{\text{ét}}(X_{k_s}, \mathbb{Z}_\ell(a))$ be the i-th degree étale cohomology group with coeffients in $\mathbb{Z}_\ell(a)$, $a \in \mathbb{Z}$.

Often, A will denote an abelian variety of dimension g over k, with dual abelian variety $\widehat{A}$ and (normalized) Poincaré bundle on $\mathcal{P}_A$. The abelian group $\mathrm{CH}(A)$ will in that case be equipped with two ring structures: the usual intersection product $\cdot$ as well as the Pontryagin product $\star$. Recall that the latter is defined as follows:

$$\star\colon \mathrm{CH}(A) \times \mathrm{CH}(A) \to \mathrm{CH}(A), \quad x \star y = m_*(\pi_1^*(x) \cdot \pi_2^*(y)).$$

Here, as well as in the rest of the paper, π_i denotes the projection onto the i-th factor, $\Delta\colon A \to A \times A$ the diagonal morphism, and $m\colon A \times A \to A$ the group law morphism of A.

For any abelian group M and any element $x \in M$, we denote by $x_{\mathbb{Q}} \in M \otimes_{\mathbb{Z}} \mathbb{Q}$ the image of x in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ under the canonical homomorphism $M \to M \otimes_{\mathbb{Z}} \mathbb{Q}$.

6.2.2 *Integral Fourier transforms and integral Hodge classes*

For abelian varieties A over $k = k_s$, it is unknown whether the Fourier transform $\mathcal{F}_A\colon \mathrm{CH}(A)_{\mathbb{Q}} \to \mathrm{CH}(\widehat{A})_{\mathbb{Q}}$ preserves the subgroups of integral cycles modulo torsion. A sufficient condition for this to hold is that $\mathcal{F}_A$ lifts to a homomorphism $\mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$. In this section we outline a second consequence of such a lift $\mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ when A is defined over the complex numbers: the existence of an integral lift of $\mathcal{F}_A$ implies the integral Hodge conjecture for one-cycles on $\widehat{A}$.

Let A be an abelian variety over k. The Fourier transform on the level of Chow groups is the group homomorphism

$$\mathcal{F}_A\colon \mathrm{CH}(A)_{\mathbb{Q}} \to \mathrm{CH}(\widehat{A})_{\mathbb{Q}}$$

induced by the correspondence $\mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$, where $\mathrm{ch}(\mathcal{P}_A)$ is the Chern character of $\mathcal{P}_A$. Similarly, one defines the Fourier transform on the level of étale cohomology:

$$\mathscr{F}_A \colon \mathrm{H}^{\bullet}_{\text{ét}}(A_{k_s}, \mathbb{Q}_\ell(\bullet)) \to \mathrm{H}^{\bullet}_{\text{ét}}(\widehat{A}_{k_s}, \mathbb{Q}_\ell(\bullet)).$$

In fact, $\mathscr{F}_A$ preserves the integral cohomology classes and induces, for each integer j with $0 \leq j \leq 2g$, an isomorphism [Bea82, Proposition 1], [Tot21, page 18]:

$$\mathscr{F}_A \colon \mathrm{H}^{j}_{\text{ét}}(A_{k_s}, \mathbb{Z}_\ell(a)) \to \mathrm{H}^{2g-j}_{\text{ét}}(\widehat{A}_{k_s}, \mathbb{Z}_\ell(a+g-j)).$$

Similarly, if $k = \mathbb{C}$, then $\mathrm{ch}(\mathcal{P}_A)$ induces, for each integer i with $0 \leq i \leq 2g$, an isomorphism of Hodge structures

$$\mathscr{F}_A \colon \mathrm{H}^{i}(A, \mathbb{Z}) \to \mathrm{H}^{2g-i}(\widehat{A}, \mathbb{Z})(g-i). \tag{6.2}$$

In [MP10], Moonen and Polishchuk consider an isomorphism $\phi \colon A \xrightarrow{\sim} \widehat{A}$, a positive integer d, and define the notion of motivic integral Fourier transform of (A, ϕ) up to factor d. The definition goes as follows. Let $\mathcal{M}(k)$ be the category of effective Chow motives over k with respect to ungraded correspondences, and let $h(A)$ be the motive of A. Then a morphism

$$\mathcal{F} \colon h(A) \to h(A)$$

in $\mathcal{M}(k)$ is a *motivic integral Fourier transform of* (A, ϕ) *up to factor* d if the following three conditions are satisfied: (i) the induced morphism $h(A)_{\mathbb{Q}} \to h(A)_{\mathbb{Q}}$ is the composition of the usual Fourier transform with the isomorphism $\phi^* \colon h(\widehat{A})_{\mathbb{Q}} \xrightarrow{\sim} h(A)_{\mathbb{Q}}$, (ii) one has $d \cdot \mathcal{F} \circ \mathcal{F} = d \cdot (-1)^g \cdot [-1]_*$ as morphisms from $h(A)$ to $h(A)$, and (iii) $d \cdot \mathcal{F} \circ m_* = d \cdot \Delta^* \circ \mathcal{F} \otimes \mathcal{F} \colon h(A) \otimes h(A) \to h(A)$.

For our purposes, we consider similar homomorphisms $\mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$. To make their existence easier to verify (c.f. Theorem 6.9) we relax the above conditions:

Definition 6.2. Let $A_{/k}$ be an abelian variety and let $\mathcal{F}\colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ be a group homomorphism. We call $\mathcal{F}$ a *weak integral Fourier transform* if the following diagram commutes:

$$\begin{tikzcd} \mathrm{CH}(A) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathrm{CH}(\widehat{A}) \arrow[d] \\ \mathrm{CH}(A)_{\mathbb{Q}} \arrow[r, "\mathcal{F}_A"] & \mathrm{CH}(\widehat{A})_{\mathbb{Q}}. \end{tikzcd} \tag{6.3}$$

We call a weak integral Fourier transform $\mathcal{F}$ *motivic* if it is induced by a cycle Γ in $\mathrm{CH}(A \times \widehat{A})$ that satisfies $\Gamma_{\mathbb{Q}} = \mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$. A group homomorphism $\mathcal{F}\colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ is an *integral Fourier transform up to homology* if the following diagram commutes:

$$\begin{tikzcd} \mathrm{CH}(A) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathrm{CH}(\widehat{A}) \arrow[d] \\ \oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(A_{k_s}, \mathbb{Z}_\ell(r)) \arrow[r, "\mathscr{F}_A"] & \oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(\widehat{A}_{k_s}, \mathbb{Z}_\ell(r)). \end{tikzcd} \tag{6.4}$$

Similarly, a $\mathbb{Z}_\ell$-module homomorphism $\mathcal{F}_\ell\colon \mathrm{CH}(A)_{\mathbb{Z}_\ell} \to \mathrm{CH}(\widehat{A})_{\mathbb{Z}_\ell}$ is called an *ℓ-adic integral Fourier transform up to homology* if $\mathcal{F}_\ell$ is compatible with $\mathscr{F}_A$ and the ℓ-adic cycle class maps. Finally, an integral Fourier transform up to homology $\mathcal{F}$ (resp. an ℓ-adic integral Fourier transform up to homology $\mathcal{F}_\ell$) is called *motivic* if it is induced by a cycle $\Gamma \in \mathrm{CH}(A \times \widehat{A})$ (resp. $\Gamma_\ell \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Z}_\ell}$) such that $cl(\Gamma)$ (resp. $cl(\Gamma_\ell)$) equals $\mathrm{ch}(\mathcal{P}_A) \in \oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}((A \times \widehat{A})_{k_s}, \mathbb{Z}_\ell(r))$.

Remark 6.3. If $\mathcal{F}\colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ is a weak integral Fourier transform, then $\mathcal{F}$ is an integral Fourier transform up to homology, the $\mathbb{Z}_\ell$-module $\oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(\widehat{A}_{k_s}, \mathbb{Z}_\ell(r))$ being torsion-free. If $k = \mathbb{C}$, then $\mathcal{F}\colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ is an integral Fourier transform up to homology if and only if $\mathcal{F}$ is compatible with the Fourier transform $\mathscr{F}_A\colon \mathrm{H}^\bullet(A, \mathbb{Z}) \to \mathrm{H}^\bullet(\widehat{A}, \mathbb{Z})$ on Betti cohomology.

The relation between integral Fourier transforms and Hodge classes is as follows:

Lemma 6.4. *Let A be a complex abelian variety and let $\mathcal{F}\colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ be an integral Fourier transform up to homology.*

1. *For each $i \in \mathbb{Z}_{\geq 0}$, the integral Hodge conjecture for degree $2i$ classes on A implies the integral Hodge conjecture for degree $2g - 2i$ classes on $\widehat{A}$.*

2. *If $\mathcal{F}$ is motivic, then $\mathscr{F}_A$ induces a group isomorphism $\mathrm{Z}^{2i}(A) \xrightarrow{\sim} \mathrm{Z}^{2g-2i}(\widehat{A})$ and, therefore, the integral Hodge conjectures for degree $2i$ classes on A and degree $2g-2i$ classes on $\widehat{A}$ are equivalent for all i.*

Proof. We can extend Diagram (6.4) to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{CH}^i(A) \arrow[r] \arrow[d, "cl^i"] & \mathrm{CH}(A) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathrm{CH}(\widehat{A}) \arrow[r] \arrow[d] & \mathrm{CH}_i(\widehat{A}) \arrow[d, "cl_i"] \\
\mathrm{H}^{2i}(A,\mathbb{Z}) \arrow[r] & \mathrm{H}^\bullet(A,\mathbb{Z}) \arrow[r, "\mathscr{F}_A"] & \mathrm{H}^\bullet(\widehat{A},\mathbb{Z}) \arrow[r] & \mathrm{H}^{2g-2i}(\widehat{A},\mathbb{Z}).
\end{tikzcd}$$

The composition $\mathrm{H}^{2i}(A,\mathbb{Z}) \to \mathrm{H}^{2g-2i}(\widehat{A},\mathbb{Z})$ appearing on the bottom line agrees up to a suitable Tate twist with the map $\mathscr{F}_A$ of equation (6.2). Therefore, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{CH}^i(A) \arrow[r] \arrow[d, "cl^i"] & \mathrm{CH}_i(\widehat{A}) \arrow[d, "cl_i"] \\
\mathrm{Hdg}^{2i}(A,\mathbb{Z}) \arrow[r, "\sim"] & \mathrm{Hdg}^{2g-2i}(\widehat{A},\mathbb{Z}).
\end{tikzcd} \tag{6.5}$$

Thus the surjectivity of cl^i implies the surjectivity of cl_i. Moreover, if $\mathcal{F}$ is motivic, then replacing A by $\widehat{A}$ and $\widehat{A}$ by $\widehat{\widehat{A}}$ in the argument above shows that the images of cl^i and cl_i are identified under the isomorphism $\mathscr{F}_A\colon \mathrm{Hdg}^{2i}(A,\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hdg}^{2g-2i}(\widehat{A},\mathbb{Z})$ in diagram (6.5). □

6.3 RATIONAL FOURIER TRANSFORMS

The above suggests that to prove Theorem 7.1, one would need to show that for a complex abelian variety of dimension g whose minimal Poincaré class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A},\mathbb{Z})$ is algebraic, all classes of the form $c_1(\mathcal{P}_A)^i/i! \in \mathrm{H}^{2i}(A \times \widehat{A},\mathbb{Z})$ are algebraic. With this goal in mind we shall study Fourier transforms on rational Chow groups in Section 6.3, and investigate how these relate to $\mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$. In turns out that the cycles $c_1(\mathcal{P}_A)^i/i!$ in $\mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$ satisfy several relations that are very similar to those proved by Beauville for the cycles $\theta^i/i! \in \mathrm{CH}(A)_{\mathbb{Q}}$ in case A is principally polarized, see [Bea82]. Since we will need these results in any characteristic in order to prove Theorem 7.6, we will work over our general field k, see Section 6.2.1.

Let A be an abelian variety over k. Define cycles

$$\begin{aligned} \ell &= c_1(\mathcal{P}_A) \in \mathrm{CH}^1(A \times \widehat{A})_{\mathbb{Q}}, \\ \widehat{\ell} &= c_1(\mathcal{P}_{\widehat{A}}) \in \mathrm{CH}^1(\widehat{A} \times A)_{\mathbb{Q}}, \\ \mathscr{R}_A &= c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{CH}_1(A \times \widehat{A})_{\mathbb{Q}}, \quad \text{and} \\ \mathscr{R}_{\widehat{A}} &= c_1(\mathcal{P}_{\widehat{A}})^{2g-1}/(2g-1)! \in \mathrm{CH}_1(\widehat{A} \times A)_{\mathbb{Q}}. \end{aligned}$$

For $a \in \mathrm{CH}(A)_{\mathbb{Q}}$, define $\mathrm{E}(a) \in \mathrm{CH}(A)_{\mathbb{Q}}$ as the $\star$-exponential of a:

$$\mathrm{E}(a) := \sum_{n \geq 0} \frac{a^{\star n}}{n!} \in \mathrm{CH}(A)_{\mathbb{Q}}.$$

The key to Theorem 7.1 will be the following:

Lemma 6.5. *We have* $\mathrm{ch}(\mathcal{P}_A) = e^{\ell} = (-1)^g \cdot \mathrm{E}((-1)^g \cdot \mathscr{R}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$.

Proof. The most important ingredient in the proof is the following:

Claim $(*)$: Consider the Fourier transform $\mathcal{F}_{A \times \widehat{A}} \colon \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}} \to \mathrm{CH}(\widehat{A} \times A)_{\mathbb{Q}}$. One has

$$\mathcal{F}_{A \times \widehat{A}}(e^{\ell}) = (-1)^g \cdot e^{-\widehat{\ell}} \in \mathrm{CH}(\widehat{A} \times A)_{\mathbb{Q}}.$$

To prove Claim $(*)$, we lift the desired equality in the rational Chow group of $\widehat{A} \times A$ to an isomorphism in the derived category $\mathrm{D}^b(\widehat{A} \times A)$ of $\widehat{A} \times A$. For $X = A \times \widehat{A}$ the Poincaré line bundle $\mathcal{P}_X$ on $X \times \widehat{X} \cong A \times \widehat{A} \times \widehat{A} \times A$ is isomorphic to $\pi_{13}^* \mathcal{P}_A \otimes \pi_{24}^* \mathcal{P}_{\widehat{A}}$. Consider

$$\Phi_{\mathcal{P}_X}(\mathcal{P}_A) \cong \pi_{34,*}\left(\pi_{13}^* \mathcal{P}_A \otimes \pi_{24}^* \mathcal{P}_{\widehat{A}} \otimes \pi_{12}^* \mathcal{P}_A\right) \in \mathrm{D}^b(\widehat{A} \times A) \tag{6.6}$$

whose Chern character is exactly $\mathcal{F}_X(e^{\ell})$. Applying the pushforward along the permutation map

$$(123) \colon A \times \widehat{A} \times \widehat{A} \times A \cong \widehat{A} \times A \times \widehat{A} \times A$$

the object (6.6) becomes $\pi_{14,*}\left(\pi_{12}^* \mathcal{P}_{\widehat{A}} \otimes \pi_{23}^* \mathcal{P}_A \otimes \pi_{34}^* \mathcal{P}_{\widehat{A}}\right)$ which is isomorphic to the Fourier–Mukai kernel of the composition

$$\Phi_{\mathcal{P}_{\widehat{A}}} \circ \Phi_{\mathcal{P}_A} \circ \Phi_{\mathcal{P}_{\widehat{A}}}.$$

Since $\Phi_{\mathcal{P}_A} \circ \Phi_{\mathcal{P}_{\widehat{A}}}$ is isomorphic to $[-1_{\widehat{A}}]^* \circ [-g]$ by [Muk81, Theorem 2.2], we have

$$\Phi_{\mathcal{P}_{\widehat{A}}} \circ \Phi_{\mathcal{P}_A} \circ \Phi_{\mathcal{P}_{\widehat{A}}} \cong \Phi_{\mathcal{P}_{\widehat{A}}} \circ [-1_{\widehat{A}}]^* \circ [-g].$$

This is the Fourier–Mukai transform with kernel $\mathcal{E} = \mathcal{P}_{\widehat{A}}^{\vee}[-g] \in \mathrm{D}^b(\widehat{A} \times A)$. By uniqueness of the Fourier–Mukai kernel of an equivalence [Orl97, Theorem 2.2] and the fact that the Chern character of $\mathcal{E}$ equals $(-1)^g \cdot e^{-\widehat{\ell}} \in \mathrm{CH}(\widehat{A} \times A)_{\mathbb{Q}}$, this finishes the proof of Claim ($*$).

Next, we claim that $(-1)^g \cdot \mathcal{F}_{\widehat{A}\times A}(-\widehat{\ell}) = \mathscr{R}_A$. To see this, recall that for each integer i with $0 \leq i \leq g$, there is a canonical *Beauville decomposition*

$$\mathrm{CH}^i(A)_{\mathbb{Q}} = \oplus_{j=i-g}^{i} \mathrm{CH}^{i,j}(A)_{\mathbb{Q}}, \qquad \text{see [Bea86b].}$$

Since the Poincaré bundle $\mathcal{P}_A$ is symmetric, we have $\ell \in \mathrm{CH}^{1,0}(A \times \widehat{A})_{\mathbb{Q}}$ and hence $\ell^i \in \mathrm{CH}^{i,0}(A \times \widehat{A})_{\mathbb{Q}}$. In particular, we have $\mathscr{R}_A \in \mathrm{CH}^{2g-1,0}(A \times \widehat{A})_{\mathbb{Q}}$. The fact that $\mathcal{P}_A$ is symmetric also implies - via Claim ($*$) - that we have $\mathcal{F}_{\widehat{A}\times A}((-1)^g \cdot e^{-\widehat{\ell}}) = e^{\ell}$. Indeed,

$$\mathcal{F}_{\widehat{A}\times A} \circ \mathcal{F}_{A\times\widehat{A}} = [-1]^* \cdot (-1)^{2g} = [-1]^*,$$

see [DM91, Corollary 2.22]. Since $\mathcal{F}_{\widehat{A}\times A}$ identifies the group $\mathrm{CH}^{i,0}(\widehat{A} \times A)_{\mathbb{Q}}$ with the group $\mathrm{CH}^{g-i,0}(A \times \widehat{A})$ (see [DM91, Lemma 2.18]), we must indeed have

$$(-1)^g \cdot \mathcal{F}_{\widehat{A}\times A}(-\widehat{\ell}) = \mathcal{F}_{\widehat{A}\times A}((-1)^{g+1} \cdot \widehat{\ell}) = \frac{\ell^{2g-1}}{(2g-1)!} = \mathscr{R}_A. \tag{6.7}$$

For a g-dimensional abelian variety X and any $x, y \in \mathrm{CH}(X)_{\mathbb{Q}}$, one has $\mathcal{F}_X(x \cdot y) = (-1)^g \cdot \mathcal{F}_X(x) \star \mathcal{F}_X(y) \in \mathrm{CH}(\widehat{X})_{\mathbb{Q}}$, see [Bea82, Proposition 3]. This implies (see also [MP10, §3.7]) that if a is a cycle on X such that $\mathcal{F}_X(a) \in \mathrm{CH}_{>0}(\widehat{X})_{\mathbb{Q}}$, then $\mathcal{F}_X(e^a) = (-1)^g \cdot \mathrm{E}((-1)^g \cdot \mathcal{F}_X(a))$. This allows us to conclude that

$$\begin{aligned} e^{\ell} &= \mathcal{F}_{\widehat{A}\times A}((-1)^g \cdot e^{-\widehat{\ell}}) = (-1)^g \cdot \mathcal{F}_{\widehat{A}\times A}(e^{-\widehat{\ell}}) \\ &= (-1)^g \cdot \mathrm{E}(\mathcal{F}_{\widehat{A}\times A}(-\widehat{\ell})) = (-1)^g \cdot \mathrm{E}((-1)^g \cdot \mathscr{R}_A), \end{aligned}$$

which finishes the proof. $\square$

Next, assume that A is equipped with a *principal* polarization $\lambda\colon A \xrightarrow{\sim} \widehat{A}$, define $\ell = c_1(\mathcal{P}_A)$, and let

$$\Theta = \frac{1}{2} \cdot (\mathrm{id}, \lambda)^* \ell \in \mathrm{CH}^1(A)_{\mathbb{Q}} \tag{6.8}$$

be the symmetric ample class corresponding to the polarization. Here (id, λ) is the morphism $(\mathrm{id}, \lambda)\colon A \to A \times \widehat{A}$. One can understand the relation between

$$\Gamma_\Theta := \Theta^{g-1}/(g-1)! \in \mathrm{CH}_1(A)_{\mathbb{Q}}$$

and $\mathscr{R}_A = \ell^{2g-1}/(2g-1)! \in \mathrm{CH}_1(A \times \widehat{A})_{\mathbb{Q}}$ in the following way. Define

$$j_1\colon A \to A \times \widehat{A}, \quad x \mapsto (x, 0), \quad \text{and}$$
$$j_2\colon \widehat{A} \to A \times \widehat{A}, \quad y \mapsto (0, y).$$

Let $\widehat{\Theta} \in \mathrm{CH}^1(\widehat{A})_{\mathbb{Q}}$ be the dual of Θ, and define a one-cycle τ on $A \times \widehat{A}$ as follows:

$$\tau := j_{1,*}(\Gamma_\Theta) + j_{2,*}(\Gamma_{\widehat{\Theta}}) - (\mathrm{id}, \lambda)_*(\Gamma_\Theta) \in \mathrm{CH}_1(A \times \widehat{A})_{\mathbb{Q}}.$$

Lemma 6.6. *One has* $\tau = (-1)^{g+1} \cdot \mathscr{R}_A \in \mathrm{CH}_1(A \times \widehat{A})_{\mathbb{Q}}$.

Proof. Identify A and $\widehat{A}$ via λ. This gives $\ell = m^*(\Theta) - \pi_1^*(\Theta) - \pi_2^*(\Theta)$, and the Fourier transform becomes an endomorphism $\mathcal{F}_A\colon \mathrm{CH}(A)_{\mathbb{Q}} \to \mathrm{CH}(A)_{\mathbb{Q}}$.

We claim that

$$\tau = (-1)^g \cdot (\Delta_* \mathcal{F}_A(\Theta) - j_{1,*}\mathcal{F}_A(\Theta) - j_{2,*}\mathcal{F}_A(\Theta)).$$

For this, it suffices to show that $\mathcal{F}_A(\Theta) = (-1)^{g-1} \cdot \Theta^{g-1}/(g-1)! \in \mathrm{CH}_1(A)_{\mathbb{Q}}$. Now $\mathcal{F}_A(e^\Theta) = e^{-\Theta}$ by Lemma 6.7 below. Moreover, since Θ is symmetric, we have $\Theta \in \mathrm{CH}^{1,0}(A)_{\mathbb{Q}}$, hence $\Theta^i/i! \in \mathrm{CH}^{i,0}(A)_{\mathbb{Q}}$ for each $i \geq 0$. Therefore, $\mathcal{F}_A\left(\Theta^i/i!\right) \in \mathrm{CH}^{g-i,0}(A)_{\mathbb{Q}}$ by [DM91, Lemma 2.18]. This implies that, in fact,

$$\mathcal{F}_A\left(\Theta^i/i!\right) = (-1)^{g-i} \cdot \Theta^{g-i}/(g-i)! \in \mathrm{CH}^{g-i,0}(A)_{\mathbb{Q}}$$

for every i. In particular, the claim follows.

Next, recall that $\mathcal{F}_{A\times A}(\ell) = (-1)^{g+1} \cdot \mathscr{R}_A$, see Claim ($*$). So at this point, it suffices to prove the identity

$$\mathcal{F}_{A\times A}(\ell) = (-1)^g \cdot (\Delta_* \mathcal{F}_A(\Theta) - j_{1,*}\mathcal{F}_A(\Theta) - j_{2,*}\mathcal{F}_A(\Theta)).$$

To prove this, we use the following functoriality properties of the Fourier transform on the level of rational Chow groups. Let X and Y be abelian varieties and let $f\colon X \to Y$ be a homomorphism with dual homomorphism $\widehat{f}\colon \widehat{Y} \to \widehat{X}$. We then have the following equalities [MP10, (3.7.1)]:

$$(\widehat{f})^* \circ \mathcal{F}_X = \mathcal{F}_Y \circ f_*, \quad \mathcal{F}_X \circ f^* = (-1)^{\dim X - \dim Y} \cdot (\widehat{f})_* \circ \mathcal{F}_Y. \tag{6.9}$$

Since $\ell = m^*\Theta - \pi_1^*\Theta - \pi_2^*\Theta$, it follows from Equation (6.9) that

$$\begin{aligned}\mathcal{F}_{A\times A}(\ell) &= \mathcal{F}_{A\times A}\left(m^*\Theta\right) - \mathcal{F}_{A\times A}\left(\pi_1^*\Theta\right) - \mathcal{F}_{A\times A}\left(\pi_2^*\Theta\right) \\ &= (-1)^g \cdot \left(\Delta_*\mathcal{F}_A(\Theta) - j_{1,*}\mathcal{F}_A(\Theta) - j_{2,*}\mathcal{F}_A(\Theta)\right).\end{aligned}$$

□

Lemma 6.7 (Beauville)**.** *Let A be an abelian variety over k, principally polarized by $\lambda\colon A \xrightarrow{\sim} \widehat{A}$, and define $\Theta = \frac{1}{2}\cdot(\mathrm{id},\lambda)^* c_1(\mathcal{P}_A) \in \mathrm{CH}^1(A)_{\mathbb{Q}}$. Identify A and $\widehat{A}$ via λ. With respect to the Fourier transform*

$$\mathcal{F}_A\colon \mathrm{CH}(A)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}(A)_{\mathbb{Q}}, \quad \textit{one has} \quad \mathcal{F}_A(e^\Theta) = e^{-\Theta}.$$

Proof. Our proof follows the proof of [Bea82, Lemme 1], but has to be adapted, since Θ does not necessarily come from a symmetric ample line bundle on A. Since one still has $\ell = m^*\Theta - \pi_1^*\Theta - \pi_2^*\Theta$, the argument can be made to work: one has

$$\begin{aligned}\mathcal{F}_A(e^\Theta) &= \pi_{2,*}\left(e^\ell \cdot \pi_1^* e^\Theta\right) \\ &= \pi_{2,*}\left(e^{m^*\Theta - \pi_2^*\Theta}\right) = e^{-\Theta}\pi_{2,*}(m^*e^\Theta) \in \mathrm{CH}(A)_{\mathbb{Q}}.\end{aligned}$$

For codimension reasons, one has

$$\pi_{2,*}(m^*e^\Theta) = \pi_{2,*}m^*(\Theta^g/g!) = \deg(\Theta^g/g!) \in \mathrm{CH}^0(A)_{\mathbb{Q}} \cong \mathbb{Q}.$$

Pull back $\Theta^g/g!$ along $A_{k_s} \to A$ to see that

$$\deg(\Theta^g/g!) = 1 \in \mathrm{CH}^0(A)_{\mathbb{Q}} \cong \mathrm{CH}^0(A_{k_s})_{\mathbb{Q}},$$

since over k_s the cycle Θ becomes the cycle class attached to a symmetric ample line bundle. □

6.4 DIVIDED POWERS OF ALGEBRAIC CYCLES

It was asked by Bruno Kahn whether there exists a PD-structure on the Chow ring of an abelian variety over any field with respect to its usual (intersection) product. There are counterexamples over non-closed fields: see [Esn04], where Esnault constructs an abelian surface X and a line bundle $\mathcal{L}$ on X such that $c_1(\mathcal{L}) \cdot c_1(\mathcal{L})$ is not divisible by 2 in $\mathrm{CH}_0(X)$. However, the case of algebraically closed fields remains open [MP10, Section 3.2]. What we do know, is the following:

Theorem 6.8 (Moonen–Polishchuk)**.** *Let A be an abelian variety over k. The ring $(\mathrm{CH}(A), \star)$ admits a canonical PD-structure γ on the ideal $\mathrm{CH}_{>0}(A) \subset \mathrm{CH}(A)$. If $k = \bar{k}$, then γ extends to a PD-structure on the ideal generated by $\mathrm{CH}_{>0}(A)$ and the zero cycles of degree zero.*

In particular, for each element $x \in \mathrm{CH}_{>0}(A)$ and each $n \in \mathbb{Z}_{\geq 1}$, there is a canonical element $x^{[n]} \in \mathrm{CH}_{>0}(A)$ such that $n!x^{[n]} = x^{\star n}$, see [Sta18, Tag 07GM]. For $x \in \mathrm{CH}_{>0}(A)$, we may then define

$$\mathrm{E}(x) = \sum_{n \geq 0} x^{[n]} \in \mathrm{CH}(A)$$

as the $\star$-exponential of x in terms of its divided powers.

Together with the results of Section 6.3, Theorem 6.8 enables us to provide criteria for the existence of a motivic weak integral Fourier transform. Recall that for an abelian variety A over k, principally polarized by $\lambda \colon A \xrightarrow{\sim} \widehat{A}$, we defined $\Theta \in \mathrm{CH}^1(A)_{\mathbb{Q}}$ as the symmetric ample class attached to λ, see equation (6.8).

Theorem 6.9. *Let $A_{/k}$ be an abelian variety of dimension g. The following are equivalent:*

1. *The one-cycle $\mathscr{R}_A = c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$ lifts to $\mathrm{CH}_1(A \times \widehat{A})$.*
2. *The abelian variety A admits a motivic weak integral Fourier transform.*
3. *The abelian variety $A \times \widehat{A}$ admits a motivic weak integral Fourier transform.*

Moreover, if A carries a symmetric ample line bundle that induces a principal polarization $\lambda \colon A \xrightarrow{\sim} \widehat{A}$, then the above statements are equivalent to the following equivalent statements:

4. *The two-cycle $\mathscr{S}_A = c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$ lifts to $\mathrm{CH}_2(A \times \widehat{A})$.*
5. *The one-cycle $\Gamma_\Theta = \Theta^{g-1}/(g-1)! \in \mathrm{CH}(A)_{\mathbb{Q}}$ lifts to a one-cycle in $\mathrm{CH}(A)$.*

6. *The abelian variety A admits a weak integral Fourier transform.*

7. *The Fourier transform $\mathcal{F}_A$ satisfies* $\mathcal{F}_A\left(\mathrm{CH}(A)/(\mathrm{torsion})\right) \subset \mathrm{CH}(\widehat{A})/(\mathrm{torsion})$.

8. *There exists a PD-structure on the ideal* $\mathrm{CH}^{>0}(A)/(\mathrm{torsion}) \subset \mathrm{CH}(A)/(\mathrm{torsion})$.

Proof. Suppose that 1 holds, and let $\Gamma \in \mathrm{CH}_1(A \times \widehat{A})$ be a cycle such that $\Gamma_{\mathbb{Q}} = \mathscr{R}_A$. Then consider the cycle $(-1)^g \cdot \mathrm{E}((-1)^g \cdot \Gamma) \in \mathrm{CH}(A \times \widehat{A})$. By Lemma 6.5, we have

$$\begin{aligned}(-1)^g \cdot \mathrm{E}((-1)^g \cdot \Gamma)_{\mathbb{Q}} &= (-1)^g \cdot \mathrm{E}((-1)^g \cdot \Gamma_{\mathbb{Q}}) \\ &= (-1)^g \cdot \mathrm{E}((-1)^g \cdot \mathscr{R}_A) = \mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}.\end{aligned}$$

Thus 2 holds. We claim that 3 holds as well. Indeed, consider the line bundle $\mathcal{P}_{A\times\widehat{A}}$ on the abelian variety $X = A \times \widehat{A} \times \widehat{A} \times A$; one has that $\mathcal{P}_{A\times\widehat{A}} \cong \pi_{13}^*\mathcal{P}_A \otimes \pi_{24}^*\mathcal{P}_{\widehat{A}}$, which implies that we have the following equality in $\mathrm{CH}_1(X)_{\mathbb{Q}}$:

$$\begin{aligned}\mathscr{R}_{A\times\widehat{A}} &= \frac{\left(\pi_{13}^*c_1(\mathcal{P}_A) + \pi_{24}^*c_1(\mathcal{P}_{\widehat{A}})\right)^{4g-1}}{(4g-1)!} \\ &= \frac{\pi_{13}^*c_1(\mathcal{P}_A)^{2g-1} \cdot \pi_{24}^*c_1(\mathcal{P}_{\widehat{A}})^{2g} + \pi_{13}^*c_1(\mathcal{P}_A)^{2g} \cdot \pi_{24}^*c_1(\mathcal{P}_{\widehat{A}})^{2g-1}}{(2g)!(2g-1)!} \\ &= \frac{\pi_{13}^*c_1(\mathcal{P}_A)^{2g-1} \cdot \pi_{24}^*((2g)! \cdot [0]_{A\times\widehat{A}}) + \pi_{13}^*((2g)! \cdot [0]_{\widehat{A}\times A}) \cdot \pi_{24}^*c_1(\mathcal{P}_{\widehat{A}})^{2g-1}}{(2g)!(2g-1)!} \\ &= \pi_{13}^*\Big(\frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!}\Big) \cdot \pi_{24}^*([0]_{A\times\widehat{A}}) + \pi_{13}^*([0]_{\widehat{A}\times A}) \cdot \pi_{24}^*\Big(\frac{c_1(\mathcal{P}_{\widehat{A}})^{2g-1}}{(2g-1)!}\Big).\end{aligned} \tag{6.10}$$

We conclude that $\mathscr{R}_{A\times\widehat{A}}$ lifts to $\mathrm{CH}_1(X)$ which, by the implication [1 $\implies$ 2] (that has already been proved), implies that $A \times \widehat{A}$ admits a motivic weak integral Fourier transform. On the other hand, the implication [3 $\implies$ 1] follows from the fact that $(-1)^g \cdot \mathcal{F}_{\widehat{A}\times A}(-\widehat{\ell}) = \mathscr{R}_A$ (see Equation (6.7)) and the fact that an abelian variety admits a motivic weak integral Fourier transform if and only if its dual abelian variety does. Therefore, we have [1 $\iff$ 2 $\iff$ 3].

From now on, assume that A is principally polarized by $\lambda\colon A \xrightarrow{\sim} A$, where λ is the polarization attached to a symmetric ample line bundle $\mathcal{L}$ on A. Moreover, in what follows we shall identify $\widehat{A}$ and A via λ.

Suppose that 4 holds and let $S_A \in \mathrm{CH}_2(A \times A) = \mathrm{CH}^{2g-2}(A \times A)$ be such that $(S_A)_{\mathbb{Q}} = \mathscr{S}_A \in \mathrm{CH}_2(A \times A)_{\mathbb{Q}}$. Define $\mathrm{CH}^{1,0}(A) := \mathrm{Pic}^{\mathrm{sym}}(A)$ to be the group of isomorphism classes of symmetric line bundes on A. Then S_A induces a homomorphism $\mathcal{F}\colon \mathrm{CH}^{1,0}(A) \to \mathrm{CH}_1(A)$ defined as the composition

$$\begin{aligned}\mathrm{CH}^{1,0}(A) \xrightarrow{\pi_1^*} \mathrm{CH}^1(A\times A) \xrightarrow{\cdot S_A} &\mathrm{CH}^{2g-1}(A\times A)\\ &= \mathrm{CH}_1(A\times A) \xrightarrow{\pi_{2,*}} \mathrm{CH}_1(A).\end{aligned}$$

Since $\mathcal{F}_A\left(\mathrm{CH}^{1,0}(A)_{\mathbb{Q}}\right) \subset \mathrm{CH}_1(A)_{\mathbb{Q}}$ (see [DM91, Lemma 2.18]) we see that

$$\begin{tikzcd}
\mathrm{CH}^{1,0}(A) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathrm{CH}_1(A) \arrow[d] \\
\mathrm{CH}^{1,0}(A)_{\mathbb{Q}} \arrow[r, "\mathcal{F}_A"] & \mathrm{CH}_1(A)_{\mathbb{Q}}
\end{tikzcd} \tag{6.11}$$

commutes. Since the line bundle $\mathcal{L}$ is symmetric, we have

$$\begin{aligned}\Theta &= \frac{1}{2}\cdot(\mathrm{id},\lambda)^* c_1(\mathcal{P}_A) = \frac{1}{2}\cdot c_1\left((\mathrm{id},\lambda)^*\mathcal{P}_A\right)\\ &= \frac{1}{2}\cdot c_1(\mathcal{L}\otimes\mathcal{L}) = c_1(\mathcal{L}) \in \mathrm{CH}^1(A)_{\mathbb{Q}}.\end{aligned} \tag{6.12}$$

The class $c_1(\mathcal{L}) \in \mathrm{CH}^{1,0}(A)$ of the line bundle $\mathcal{L}$ thus lies above $\Theta \in \mathrm{CH}^1(A)_{\mathbb{Q}}$. Therefore, $\mathcal{F}(c_1(\mathcal{L})) \in \mathrm{CH}_1(A)$ lies above $\Gamma_\Theta = (-1)^{g-1}\mathcal{F}_A(\Theta)$ by the commutativity of (6.11), and 5 holds.

Suppose that 5 holds. Then 1 follows readily from Lemma 6.6. Moreover, if 2 holds, then $\mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A\times A)_{\mathbb{Q}}$ lifts to $\mathrm{CH}(A\times A)$, hence in particular 4 holds. Since we have already proved that 1 implies 2, we conclude that [4 $\implies$ 5 $\implies$ 1 $\implies$ 2 $\implies$ 4].

The implications [2 $\implies$ 6 $\implies$ 7] are trivial. Assume that 7 holds. By Equation (6.12), $\Theta \in \mathrm{CH}^1(A)_{\mathbb{Q}}$ lifts to $\mathrm{CH}^1(A)$, hence

$$\mathcal{F}_A(\Theta) = (-1)^{g-1}\cdot\Gamma_\Theta$$

lifts to $\mathrm{CH}_1(A)$, i.e. 5 holds.

Assume that 7 holds. The fact that $\mathcal{F}_A(\mathrm{CH}(A)/(\text{torsion})) \subset \mathrm{CH}(A)/(\text{torsion})$ implies that

$$\begin{aligned}\mathrm{CH}(A)/(\text{torsion}) &= \mathcal{F}_A(\mathcal{F}_A(\mathrm{CH}(A)/(\text{torsion}))) \\ &\subset \mathcal{F}_A(\mathrm{CH}(A)/(\text{torsion})) \subset \mathrm{CH}(A)/(\text{torsion}).\end{aligned}$$

Thus the restriction of the Fourier transform $\mathcal{F}_A$ to $\mathrm{CH}(A)/(\text{torsion})$ defines an isomorphism

$$\mathcal{F}_A \colon \mathrm{CH}(A)/(\text{torsion}) \xrightarrow{\sim} \mathrm{CH}(A)/(\text{torsion}).$$

If R is a ring and γ a PD-structure on an ideal $I \subset R$, then γ extends to a PD-structure on $I/(\text{torsion}) \subset R/(\text{torsion})$. Thus, the ideal $\mathrm{CH}_{>0}(A)/(\text{torsion})$ of $\mathrm{CH}(A)/(\text{torsion})$ admits a PD-structure for the Pontryagin product $\star$ by Theorem 6.8. Since $\mathcal{F}_A$ exchanges Pontryagin and intersection products up to sign [Bea82, Proposition 3(ii)], it follows that 8 holds. Since 8 trivially implies 5, we are done. □

Question 6.10 (Moonen–Polishchuk [MP10], Totaro [Tot21]). Let A be any principally polarized abelian variety over $k = \bar{k}$. Are the equivalent conditions in Theorem 6.9 satisfied for A?

Remark 6.11. For the Jacobian $A = J(C)$ of a hyperelliptic curve C, the answer to Question 6.10 is "yes" [MP10].

Similarly, there is a relation between integral Fourier transforms up to homology and the algebraicity of minimal cohomology classes induced by Poincaré line bundles and theta divisors.

Proposition 6.12. *Let A be an abelian variety of dimension g over k. The following are equivalent:*

1. *The class*

$$\rho_A := c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}_{\text{ét}}((A \times \widehat{A})_{k_s}, \mathbb{Z}_\ell(2g-1))$$

   is the class of a cycle in $\mathrm{CH}_1(A \times \widehat{A})$.

2. *The abelian variety A admits a motivic integral Fourier transform up to homology.*
3. *The abelian variety $A \times \widehat{A}$ admits a motivic integral Fourier transform up to homology.*

Moreover, if A carries an ample line bundle that induces a principal polarization $\lambda\colon A \xrightarrow{\sim} \widehat{A}$, then the above statements are equivalent to the following equivalent statements:

4. *The class*

$$\sigma_A := c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}_{\text{ét}}((A \times \widehat{A})_{k_s}, \mathbb{Z}_\ell(2g-2))$$

is the class of a cycle in $\mathrm{CH}_2(A \times \widehat{A})$.

5. *The class $\gamma_\theta = \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}_{\text{ét}}(A_{k_s}, \mathbb{Z}_\ell(g-1))$ lifts to a cycle in $\mathrm{CH}_1(A)$.*

6. *The abelian variety A admits an integral Fourier transform up to homology.*

Proof. The proof of Theorem 6.9 can easily be adapted to this situation. □

Proposition 6.13. 1. *If $k = \mathbb{C}$, then each of the statements 1 − 6 in Proposition 6.12 is equivalent to the same statement with étale cohomology replaced by Betti cohomology.*

2. *Proposition 6.12 remains valid if one replaces integral Chow groups in statements 1, 4 and 5 by their tensor product with $\mathbb{Z}_\ell$ and 'integral Fourier transform up to homology' by 'ℓ-adic integral Fourier transform up to homology' in statements 2, 3 and 6.*

Proof. 1. In this case $\mathbb{Z}_\ell(i) = \mathbb{Z}_\ell$ and the Artin comparison isomorphism

$$\mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^{2i}(A(\mathbb{C}), \mathbb{Z}_\ell)$$

[AGV71, III, Exposé XI] is compatible with the cycle class map. Since the map $\mathrm{H}^{2i}(A(\mathbb{C}), \mathbb{Z}) \to \mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell)$ is injective, a class $\beta \in \mathrm{H}^{2i}(A(\mathbb{C}), \mathbb{Z})$ is in the image of $cl\colon \mathrm{CH}^i(A) \to \mathrm{H}^{2i}(A(\mathbb{C}), \mathbb{Z})$ if and only if its image $\beta_\ell \in \mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell)$ is in the image of $cl\colon \mathrm{CH}^i(A) \to \mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell)$.

2. Indeed, for an abelian variety A over k, the PD-structure on $\mathrm{CH}_{>0}(A) \subset (\mathrm{CH}(A), \star)$ induces a PD-structure on $\mathrm{CH}_{>0}(A) \otimes \mathbb{Z}_\ell \subset (\mathrm{CH}(A)_{\mathbb{Z}_\ell}, \star)$ by [Sta18, Tag 07H1], because the ring map $(\mathrm{CH}(A), \star) \to (\mathrm{CH}(A)_{\mathbb{Z}_\ell}, \star)$ is flat. The latter follows from the flatness of $\mathbb{Z} \to \mathbb{Z}_\ell$.

□

7

ONE-CYCLES ON ABELIAN VARIETIES

This chapter is based on joint work with Thorsten Beckmann.

7.1 Introduction

In this chapter we provide applications of the results developed in the previous Chapter 6. These applications concern the cycle class map for curves on an abelian variety A. More precisely, we will consider the integral Hodge conjecture for one-cycles when A is defined over $\mathbb{C}$, and the integral Tate conjecture for one-cycles when A is defined over the separable closure of a finitely generated field.

To state the most important results of Chapter 7, let us recall how the complex cycle class map was defined (see also Section 1.2.3). Whenever $\iota\colon C \hookrightarrow A$ is a smooth curve, the image of the fundamental class under the pushforward map

$$\iota_*\colon \mathrm{H}_2(C,\mathbb{Z}) \to \mathrm{H}_2(A,\mathbb{Z}) \cong \mathrm{H}^{2g-2}(A,\mathbb{Z})$$

defines a cohomology class $[C] \in \mathrm{H}^{2g-2}(A,\mathbb{Z})$. This construction extends to one-cycles and factors modulo rational equivalence. The *cycle class map for curves on A* is the canonical homomorphism defined in this way:

$$cl\colon \mathrm{CH}_1(A) \to \mathrm{Hdg}^{2g-2}(A,\mathbb{Z}).$$

It extends to a natural graded ring homomorphism $cl\colon \mathrm{CH}(A) \to \mathrm{H}^\bullet(A,\mathbb{Z})$.

The liftability of the Fourier transform that we studied in Chapter 6 turns out to have important consequences for the image of the cycle class map. An element $\alpha \in \mathrm{H}^\bullet(A,\mathbb{Z})$ is called *algebraic* if it is in the image of cl, and that A satisfies the *integral Hodge conjecture for i-cycles* if all Hodge classes in $\mathrm{H}^{2g-2i}(A,\mathbb{Z})$ are algebraic.

Although the integral Hodge conjecture fails in general [AH62, tre92, Tot97], it is an open question for abelian varieties. The main result of Chapter 7 is as follows.

Theorem 7.1 (with T. Beckmann)**.** *Let A be a complex abelian variety of dimension g with Poincaré bundle $\mathcal{P}_A$. The following three statements are equivalent:*

1. *The cohomology class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic.*
2. *The Chern character $\mathrm{ch}(\mathcal{P}_A) = \exp(c_1(\mathcal{P}_A)) \in \mathrm{H}^\bullet(A \times \widehat{A}, \mathbb{Z})$ is algebraic.*
3. *The integral Hodge conjecture for one-cycles holds for $A \times \widehat{A}$.*

Any of these statements implies that

4. *The integral Hodge conjecture for one-cycles holds for A and $\widehat{A}$.*

Suppose that A is principally polarized by $\theta \in \mathrm{Hdg}^2(A, \mathbb{Z})$ and consider the following statements:

5. *The minimal cohomology class $\gamma_\theta := \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ is algebraic.*
6. *The cohomology class $c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}(A \times \widehat{A}, \mathbb{Z})$ is algebraic.*

Then statements $1-6$ are equivalent. If they hold, then the class $\theta^i/i! \in \mathrm{H}^{2i}(A, \mathbb{Z})$ is algebraic for every positive integer i.

Remark that Condition 5 is stable under products, so a product of principally polarized abelian varieties satisfies the integral Hodge conjecture for one-cycles if and only if each of the factors does. More importantly, if $J(C)$ is the Jacobian of a smooth projective curve C of genus g, then every integral Hodge class of degree $2g-2$ on $J(C)$ is a $\mathbb{Z}$-linear combination of curves classes:

Theorem 7.2. *Let $C_1, \ldots, C_n$ be smooth projective curves over $\mathbb{C}$. Then the integral Hodge conjecture for one-cycles holds for the product of Jacobians $J(C_1) \times \cdots \times J(C_n)$.*

See Remark 7.9.1 for another approach towards Theorem 7.2 in the case $n = 1$. A second consequence of Theorem 7.1 is that the integral Hodge conjecture for one-cycles on principally polarized complex abelian varieties is stable under specialization, see Corollary 7.10. An application of somewhat different nature is the following density result, proven in Section 7.2.2:

Theorem 7.3. *Let $\delta = (\delta_1, \ldots, \delta_g)$ be positive integers such that $\delta_i | \delta_{i+1}$ and let $\mathsf{A}_{g,\delta}(\mathbb{C})$ be the coarse moduli space of polarized abelian varieties over $\mathbb{C}$ with polarization type δ. There is a countable union $X \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ of closed algebraic subvarieties of dimension at least g, that satisfies the following property: X is dense in the analytic topology, and the integral Hodge conjecture for one-cycles holds for those polarized abelian varieties whose isomorphism class defines a point in X.*

Remark 7.4. The lower bound that we obtain on the dimension of the components of X actually depends on δ and is often greater than g. For instance, when $\delta = 1$ and $g \geq 2$, there is a set X as in the theorem, whose elements are prime-power isogenous to products of Jacobians of curves. Therefore, the components of X have dimension $3g - 3$ in this case, c.f. Remark 7.15.

One could compare Theorem 7.1 with the following statement, proven by Grabowski [Gra04]: if g is a positive integer such that the minimal class $\gamma_\theta = \theta^{g-1}/(g-1)!$ of every principally polarized abelian variety of dimension g is algebraic, then every abelian variety of dimension g satisfies the integral Hodge conjecture for one-cycles. In this way, he proved the integral Hodge conjecture for abelian threefolds, a result which has been extended to smooth projective threefolds X with $K_X = 0$ by Voisin and Totaro [Voi06, Tot21]. For abelian varieties of dimension greater than three, not many unconditional statements seem to have been known.

In Section 7.2.3, we consider an abelian variety $A_{/\mathbb{C}}$ of dimension g and ask: if $n \in \mathbb{Z}_{\geq 1}$ is such that $n \cdot c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})_{\mathrm{alg}}$, can we bound the order of $\mathrm{Z}^{2g-2}(A)$ in terms of g and n? For a smooth complex projective d-dimensional variety X, $\mathrm{Z}^{2d-2}(X)$ is called the degree $2d-2$ *Voisin group* of X [Per22], is a stably birational invariant [Voi07, Lemma 15], and related to the unramified cohomology groups by Colliot-Thélène–Voisin and Schreieder [CTV12, Sch20]. We prove that if $n \cdot c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is algebraic, then $\gcd(n^2, (2g-2)!) \cdot \mathrm{Z}^{2g-2}(A) = (0)$. In particular, $(2g-2)! \cdot \mathrm{Z}^{2g-2}(A) = (0)$ for any g-dimensional complex abelian variety A. Moreover, if A is principally polarized by $\theta \in \mathrm{NS}(A)$ and if $n \cdot \gamma_\theta \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ is algebraic, then $n \cdot c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is algebraic. Since it is well known that for Prym varieties, the Hodge class $2 \cdot \gamma_\theta$ is algebraic, these observations lead to the following result (see also Theorem 7.19).

Theorem 7.5. *Let A be a g-dimensional Prym variety over $\mathbb{C}$. Then $4 \cdot \mathrm{Z}^{2g-2}(A) = (0)$.*

Recall that in our study of the liftability of the Fourier transform, carried out in the previous Chapter 6, we considered abelian varieties defined over arbitrary fields. This generality allows us now to obtain the analogue of Theorem 7.1 over the separable closure k of a finitely generated field.

A smooth projective variety X of dimension d over k satisfies the *integral Tate conjecture for one-cycles over* k if, for every prime number ℓ different from $\mathrm{char}(k)$ and for some finitely generated field of definition $k_0 \subset k$ of X, the cycle class map

$$cl\colon \mathrm{CH}_1(X)_{\mathbb{Z}_\ell} = \mathrm{CH}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \bigcup_U \mathrm{H}^{2d-2}_{\text{ét}}(X, \mathbb{Z}_\ell(d-1))^U \tag{7.1}$$

is surjective, where U ranges over the open subgroups of $\mathrm{Gal}(k/k_0)$.

Theorem 7.6. *Let A be an abelian variety of dimension g over the separable closure k of a finitely generated field.*

- *The abelian variety A satisfies the integral Tate conjecture for one-cycles over k if the cohomology class*

$$c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}_{\text{ét}}(A \times \widehat{A}, \mathbb{Z}_\ell(2g-1))$$

is the class of a one-cycle with $\mathbb{Z}_\ell$-coefficients for every prime number $\ell < (2g-1)!$ unequal to $\mathrm{char}(k)$.

- *Suppose that A is principally polarized and let $\theta_\ell \in \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1))$ be the class of the polarization. The map (7.1) is surjective if*

$$\gamma_{\theta_\ell} := \theta_\ell^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}_{\text{ét}}(A, \mathbb{Z}_\ell(g-1))$$

is in its image. In particular, if $\ell > (g-1)!$ then this always holds. Thus A satisfies the integral Tate conjecture for one-cycles if and only if γ_{θ_ℓ} is in the image of (7.1) for every prime number $\ell < (g-1)!$ unequal to $\mathrm{char}(k)$.

Theorem 7.6 implies in particular that products of Jacobians of smooth projective curves over k satisfy the integral Tate conjecture for one-cycles over k. Moreover, for an abelian variety A_K over a number field $K \subset \mathbb{C}$, the integral Hodge conjecture for one-cycles on $A_{\mathbb{C}}$ is equivalent to the integral Tate conjecture for one-cycles on $A_{\bar{K}}$ (Corollary 7.21), which in turn implies the integral Tate conjecture for one-cycles on the geometric special fiber $A_{\bar{k}(\mathfrak{p})}$ of the Néron model of A_K over $\mathcal{O}_K$ for any prime $\mathfrak{p} \subset \mathcal{O}_K$ at which A_K has good reduction (Corollary 7.22). Finally, we obtain the analogue of Theorem 7.3 in positive characteristic as well. The definition for a smooth projective variety over the algebraic closure k of a finitely generated field to satisfy the *integral Tate conjecture for one-cycles over* k is analogous to the definition above (see e.g. [CP15]).

Theorem 7.7. *Let k be the algebraic closure of a finitely generated field of characteristic $p > 0$. Let A_g be the coarse moduli space over k of principally polarized abelian varieties of dimension g over k. Let $X \subset \mathsf{A}_g(k)$ be the subset of moduli points attached to principally polarized abelian varieties over k that satisfy the integral Tate conjecture for one-cycles over k. Then X is Zariski dense in A_g.*

7.2 THE INTEGRAL HODGE CONJECTURE

In this section we use the theory developed in Chapter 6 to prove Theorem 7.1. We also prove some applications of Theorem 7.1: the integral Hodge conjecture for one-cycles on products of Jacobians (Theorem 7.2), the fact that the integral Hodge conjecture for one-cycles on principally polarized complex abelian varieties is stable under specialization (Corollary 7.10) and density of polarized abelian varieties satisfying the integral Hodge conjecture for one-cycles (Theorem 7.3).

7.2.1 *Proof of the main theorem*

Let us prove Theorem 7.1.

Proof of Theorem 7.1. Suppose that 1 holds. Then 2 holds by Propositions 6.12 and 6.13.1. Suppose that 2 holds. Then 4 follows from Lemma 6.4. So we have [1 $\iff$ 2 $\implies$ 4]. If 1 holds, then $\rho_A = c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic, which implies that $\rho_{\widehat{A}} \in \mathrm{H}^{4g-2}(\widehat{A} \times A, \mathbb{Z})$ is algebraic. Therefore, $\rho_{A \times \widehat{A}} \in \mathrm{H}^{8g-2}(A \times \widehat{A} \times \widehat{A} \times A, \mathbb{Z})$ is algebraic by Equation (6.10). We then apply the implication [1 $\implies$ 4] to the abelian variety $A \times \widehat{A}$, which shows that 3 holds. Since [3 $\implies$ 1] is trivial, we have proven [1 $\iff$ 2 $\iff$ 3 $\implies$ 4].

Next, assume that A is principally polarized by $\theta \in \mathrm{NS}(A) \subset \mathrm{H}^2(A, \mathbb{Z})$. The directions [4 $\implies$ 5] and [2 $\implies$ 6] are trivial and [5 $\implies$ 1] follows from Propositions 6.12 and 6.13.1. We claim that 6 implies 4. Define

$$\sigma_A = c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}(A \times \widehat{A}, \mathbb{Z})$$

and let $S \in \mathrm{CH}_2(A \times \widehat{A})$ be such that $cl(S) = \sigma_A$. The squares in the following diagram commute:

$$\begin{tikzcd} \mathrm{CH}^1(A) \arrow[r, "\pi_1^*"] \arrow[d, "cl"] & \mathrm{CH}^1(A \times \widehat{A}) \arrow[r, "\cdot S"] \arrow[d, "cl"] & \mathrm{CH}^{2g-1}(A \times \widehat{A}) \arrow[r, "\pi_{2,*}"] \arrow[d, "cl"] & \mathrm{CH}_1(\widehat{A}) \arrow[d, "cl"] \\ \mathrm{H}^2(A,\mathbb{Z}) \arrow[r, "\pi_1^*"] & \mathrm{H}^2(A \times \widehat{A},\mathbb{Z}) \arrow[r, "\cdot\sigma_A"] & \mathrm{H}^{4g-2}(A \times \widehat{A},\mathbb{Z}) \arrow[r, "\pi_{2,*}"] & \mathrm{H}^{2g-2}(\widehat{A},\mathbb{Z}). \end{tikzcd} \tag{7.2}$$

Since $\mathscr{F}_A = \pi_{2,*}\left(\mathrm{ch}(\mathcal{P}_A) \cdot \pi_1^*(-)\right)$ restricts to an isomorphism

$$\mathscr{F}_A \colon \mathrm{H}^2(A,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{2g-2}(\widehat{A},\mathbb{Z})$$

by [Bea82, Proposition 1], the composition $\pi_{2,*} \circ (- \cdot \sigma_A) \circ \pi_1^*$ on the bottom row of (7.2) is an isomorphism. By Lefschetz $(1,1)$, the map $cl \colon \mathrm{CH}_1(\widehat{A}) \to \mathrm{Hdg}^{2g-2}(\widehat{A},\mathbb{Z})$ is therefore surjective.

It remains to prove the algebraicity of the classes $\theta^i/i! \in \mathrm{H}^{2i}(A,\mathbb{Z})$. This follows from Theorem 6.8 and the following equality, see [Bea82, Corollaire 2]):

$$\frac{\theta^i}{i!} = \frac{\gamma_\theta^{\star j}}{j!}, \quad \gamma_\theta = \frac{\theta^{g-1}}{(g-1)!} \in \mathrm{H}^{2g-2}(A,\mathbb{Z}), \quad i+j=g.$$

Therefore, the proof is finished. □

Corollary 7.8. *Let A and B be complex abelian varieties of respective dimensions g_A, g_B.*

- *The Hodge classes $\rho_A \in \mathrm{H}^{4g_A-2}(A \times \widehat{A},\mathbb{Z})$ and $\rho_B \in \mathrm{H}^{4g_B-2}(B \times \widehat{B},\mathbb{Z})$ are algebraic if and only if $A \times \widehat{A}$, $B \times \widehat{B}$, $A \times B$ and $\widehat{A} \times \widehat{B}$ satisfy the integral Hodge conjecture for one-cycles.*

- *If A and B are principally polarized, then the integral Hodge conjecture for one-cycles holds for $A \times B$ if and only if it holds for A and B.*

Proof. The first statement follows from Theorem 7.1 and Equation (6.10). The second statement follows from the fact that the minimal cohomology class of the product $A \times B$ is algebraic if and only if the minimal cohomology classes of the factors A and B are both algebraic. □

Proof of Theorem 7.2. By Corollary 7.8 we may assume $n = 1$, so let C be a smooth projective curve. Let $p \in C$ and consider the morphism $\iota \colon C \to J(C)$ defined by sending a point q to the isomorphism class of the degree zero line bundle $\mathcal{O}(p-q)$. Then $cl(\iota(C)) = \gamma_\theta \in \mathrm{H}^{2g-2}(J(C), \mathbb{Z})$ by Poincaré's formula [ACGH85], so γ_θ is algebraic and the result follows from Theorem 7.1. $\square$

Remarks 7.9. 1. Let us give another proof of Theorem 7.2 in the case $n = 1$, i.e. let C be a smooth projective curve of genus g and let us prove the integral Hodge conjecture for one-cycles on $J(C)$ in a way that does not use Fourier transforms. It is classical that any Abel-Jacobi map $C^{(g)} \to J(C)$ is birational. On the other hand, the integral Hodge conjecture for one-cycles is a birational invariant, see [Voi07, Lemma 15]. Therefore, to prove it for $J(C)$ it suffices to prove it for $C^{(g)}$. One then uses [Bn02, Corollary 5] which says that for each $n \in \mathbb{Z}_{\geq 1}$, there is a natural polarization η on the n-fold symmetric product $C^{(n)}$ such that for any $i \in \mathbb{Z}_{\geq 0}$, the map $\eta^{n-i} \cup (-) \colon \mathrm{H}^i(C^{(n)}, \mathbb{Z}) \to \mathrm{H}^{2n-i}(C^{(n)}, \mathbb{Z})$ is an isomorphism. In particular, the variety $C^{(n)}$ satisfies the integral Hodge conjecture for one-cycles for any positive integer n.

2. Along these lines, observe that the integral Hodge conjecture for one-cycles holds not only for symmetric products of smooth projective complex curves but also for any product $C_1 \times \cdots \times C_n$ of smooth projective curves C_i over $\mathbb{C}$. Indeed, this follows readily from the Künneth formula.

3. Let C be a smooth projective complex curve of genus g. Our proof of Theorem 7.1 provides an explicit description of $\mathrm{Hdg}^{2g-2}(J(C), \mathbb{Z})$ depending on $\mathrm{Hdg}^2(J(C), \mathbb{Z})$. More generally, let (A, θ) be a principally polarized abelian variety of dimension g, identify A and $\widehat{A}$ via the polarization, and let $\ell = c_1(\mathcal{P}_A) \in \mathrm{H}^2(A \times \widehat{A}, \mathbb{Z})$. Then $\ell = m^*(\theta) - \pi_1^*(\theta) - \pi_2^*(\theta)$, which implies that

$$\sigma_A = \frac{\ell^{2g-2}}{(2g-2)!} = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=2g-2}}^{2g-2} (-1)^{j+k} \cdot m^*\left(\frac{\theta^i}{i!}\right) \cdot \pi_1^*\left(\frac{\theta^j}{j!}\right) \cdot \pi_2^*\left(\frac{\theta^k}{k!}\right).$$

Any $\beta \in \mathrm{Hdg}^{2g-2}(A, \mathbb{Z})$ is of the form $\pi_{2,*}(\sigma_A \cdot \pi_1^*[D])$, where $[D] = cl(D)$ for a divisor D on A, as follows from (7.2). Therefore, any $\beta \in \mathrm{Hdg}^{2g-2}(A, \mathbb{Z})$ may be written as

$$\beta = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=2g-2}}^{2g-2} (-1)^{j+k} \cdot \pi_{2,*}\left(m^*\left(\frac{\theta^i}{i!}\right) \cdot \pi_1^*\left(\frac{\theta^j}{j!}\right) \cdot \pi_1^*[D]\right) \cdot \frac{\theta^k}{k!}. \tag{7.3}$$

Returning to the case of a Jacobian $J(C)$ of a smooth projective curve C of genus g, the classes $\theta^i/i!$ appearing in (7.3) are effective algebraic cycle classes. Indeed, for $p \in C$ and $d \in \mathbb{Z}_{\geq 1}$, the image of the morphism $C^d \to J(C)$, $(x_i) \mapsto \mathcal{O}(\sum_i x_i - d \cdot p)$ defines a subvariety $W_d(C) \subset J(C)$ and by Poincaré's formula [ACGH85, §I.5] one has $cl(W_d(C)) = \theta^{g-d}/(g-d)! \in \mathrm{H}^{2g-2d}(J(C), \mathbb{Z})$.

Apart from Theorem 7.2, we obtain the following corollary of Theorem 7.1:

Corollary 7.10. *Let $A \to S$ be a principally polarized abelian scheme over a proper, smooth and connected variety S over $\mathbb{C}$. Let $X \subset S(\mathbb{C})$ be the set of $x \in S(\mathbb{C})$ such that the abelian variety A_x satisfies the integral Hodge conjecture for one-cycles. Then $X = \cup_i Z_i(\mathbb{C})$ for some countable union of closed algebraic subvarieties $Z_i \subset S$. In particular, if the integral Hodge conjecture for one-cycles holds on $U(\mathbb{C})$ for a non-empty open subscheme U of S, then it holds on all of $S(\mathbb{C})$.*

Proof. Write $\mathcal{A} = A(\mathbb{C})$ and $B = S(\mathbb{C})$ and let $\pi\colon \mathcal{A} \to B$ be the induced family of complex abelian varieties. Let $g \in \mathbb{Z}_{\geq 0}$ be the relative dimension of π and define, for $t \in S(\mathbb{C})$,

$$\theta_t \in \mathrm{NS}(\mathcal{A}_t) \subset \mathrm{H}^2(\mathcal{A}_t, \mathbb{Z})$$

to be the polarization of $\mathcal{A}_t$. There is a global section $\gamma_\theta \in \mathrm{R}^{2g-2}\pi_*\mathbb{Z}$ such that for each $t \in B$,

$$\gamma_{\theta_t} = \theta_t^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(\mathcal{A}_t, \mathbb{Z}).$$

Note that γ_θ is Hodge everywhere on B. For those $t \in B$ for which γ_{θ_t} is algebraic, write γ_{θ_t} as the difference of effective algebraic cycle classes on $\mathcal{A}_t$. This gives a countable disjoint union

$$\phi\colon \sqcup_{ij} H_i \times_S H_j \to S$$

of products of relative Hilbert schemes H_i/S. By Lemma 7.11 below, γ_{θ_t} is algebraic precisely for closed points t in the image $Y \subset S$ of ϕ. By Theorem 7.1, $X = Y$. $\square$

Lemma 7.11. *Let S be an integral variety over $\mathbb{C}$, let $\mathcal{A} \to S$ be a principally polarized abelian scheme of relative dimension g over S and let $\mathcal{C}_i \subset \mathcal{A}$ for $i = 1, \ldots, k$ be relative curves in $\mathcal{A}$ over S. Let $n_1, \ldots, n_k$ be integers and let $y \in S(\mathbb{C})$ be a point that satisfies $\sum_{i=1}^k n_i \cdot cl(C_{i,y}) = \gamma_{\theta_y} \in \mathrm{H}^{2g-2}(A_y, \mathbb{Z})$. Then for every $x \in S(\mathbb{C})$, one has the equality $\sum_{i=1}^k n_i \cdot cl(C_{i,x}) = \gamma_{\theta_x} \in \mathrm{H}^{2g-2}(A_x, \mathbb{Z})$.*

Proof. Since it suffices to prove the lemma for any open affine $U \subset S$ that contains y, we may assume that S is quasi-projective. Fix $x \in S(\mathbb{C})$. After replacing S by a suitable base change containing x and y, we may assume that S is a smooth connected curve. For $t \in S$, denote by $\theta_{\bar{t}} \in \mathrm{H}^2_{\text{ét}}(A_{\bar{t}}, \mathbb{Z}_\ell)$ the class of the polarization and $\gamma_{\theta_{\bar{t}}} = \theta_{\bar{t}}^{g-1}/(g-1)!$. Let $\eta = \mathrm{Spec}(K)$ be the generic point of S. The elements $\sum_i n_i \cdot cl(C_{i,\bar{\eta}})$ and $\gamma_{\theta_{\bar{\eta}}}$ in $\mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Z}_\ell)$ both map to

$$\sum_i n_i \cdot cl(C_{i,y}) = \gamma_{\theta_y} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_y, \mathbb{Z}_\ell)$$

under the specialization homomorphism

$$s \colon \mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Z}_\ell) \to \mathrm{H}^{2g-2}_{\text{ét}}(A_y, \mathbb{Z}_\ell)$$

by [Ful98, Example 20.3.5]. Since s is an isomorphism, we have $\sum_i n_i \cdot cl(C_{i,\bar{\eta}}) = \gamma_{\theta_{\bar{\eta}}}$, which implies that $\sum_i n_i \cdot cl(C_{x,i}) = \gamma_{\theta_x} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_x, \mathbb{Z}_\ell)$. □

7.2.2 *Density of abelian varieties satisfying the integral Hodge conjecture for one-cycles*

The goal of this section is to prove that Conditions 1 − 3 in Theorem 7.1 are satisfied on a dense subset of the moduli space of complex abelian varieties. To do so, will we state yet another criterion that a complex abelian variety may satisfy. In some sense this criterion provides a bridge between abelian varieties outside the Torelli locus and those lying within, thereby implying the integral Hodge conjecture for one-cycles for the abelian variety under consideration.

Definition 7.12. Let A and B be a complex abelian varieties and let p a prime number. We say that A is *prime-to-p isogenous to a B* if there is an isogeny $\alpha \colon A \to B$ whose degree $\deg(\alpha)$ is not divisible by p. We say that A is *p-power isogenous to B* if A is isogenous to B for some isogeny α whose degree is a power of p.

The following proposition shows in particular that to prove the density part of the statement in Theorem 7.3, it suffices to prove that for any prime number ℓ, those abelian varieties that are ℓ-power isogenous to a product of elliptic curves are dense in their moduli space.

Proposition 7.13. *Let A be a complex abelian variety of dimension g. Let $\widehat{A}$ be the dual abelian variety and let $\mathcal{P}_A$ be the Poincaré bundle. Let κ be a non-zero integer such that the cohomology class $\kappa \cdot c_1(\mathcal{P}_A)/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic. Consider the following statements:*

1. *The abelian variety A satisfies the integral Hodge conjecture for one-cycles.*
2. *For every prime p, there is an abelian variety B such that the abelian variety $A \times B$ is prime-to-p isogenous to the Jacobian of a smooth projective curve.*
3. *For every prime p that divides κ, there is an abelian variety B such that the abelian variety $A \times B$ is prime-to-p isogenous to a Jacobian of a smooth projective curve.*
4. *For every prime p, there is an abelian variety B such that the abelian variety $A \times B$ is prime-to-p isogenous to a product of Jacobians of smooth projective curves.*
5. *For every prime number p dividing κ, there exists an abelian variety B such that the abelian variety $A \times B$ is prime-to-p isogenous to a product of Jacobians of smooth projective curves.*

The [$2 \implies 3 \implies 5 \implies 1$] and [$2 \implies 4 \implies 5$]. Moreover, if A is principally polarized by $\theta_A \in \mathrm{NS}(A)$, then 1 is implied by

6. *For any prime number $p | (g-1)!$ there exists a smooth projective curve C and a morphism of abelian varieties $\phi \colon A \to J(C)$ such that $\phi^* \theta_{J(C)} = m \cdot \theta_A$ for $m \in \mathbb{Z}_{\geq 1}$ with $\gcd(m, p) = 1$.*

Finally, if A is principally polarized of Picard rank one, then the statements $1 - 6$ are equivalent.

Proof. **Step one**: [$2 \implies 3 \implies 5$] *and* [$2 \implies 4 \implies 5$]. This is trivial.

Step two: [$5 \implies 1$]. Let g be the dimension of A. We want to prove that the class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic. Let p be any prime number that divides κ. Then by Condition 5, there exists an abelian variety B and an isogeny $\alpha\colon A \times B \to Y$ to the product $Y = \prod_i J(C_i)$ of Jacobians $J(C_i)$ of smooth projective curves C_i such that $\gcd(\deg(\alpha), p) = 1$. Define $X = A \times B$. Let g_B be the dimension of B, let $h = g + g_B = \dim(X) = \dim(Y)$, and let $m_p = \deg(\alpha)$. There exists an isogeny $\beta\colon Y \to X$ such that $\beta \circ \alpha = [m_p]_X$. If we define $n_p = \deg(\beta)$ then

$$m_p \cdot n_p = \deg(\alpha) \cdot \deg(\beta) = \deg(\alpha \circ \beta) = m_p^{2h},$$

hence $(\beta \circ \alpha) \times (\widehat{\alpha} \circ \widehat{\beta}) = [m_p]_{X \times \widehat{X}}$. For $N_p = 2h \cdot (4h-2)$, the homomorphism

$$[m_p^{2h}]^* = (m_p^{N_p} \cdot (-))\colon \mathrm{H}^{4h-2}(X \times \widehat{X}, \mathbb{Z}) \to \mathrm{H}^{4h-2}(X \times \widehat{X}, \mathbb{Z})$$

will therefore factor through $\mathrm{H}^{4h-2}(Y \times \widehat{Y}, \mathbb{Z})$. Since $Y \times \widehat{Y}$ satisfies the integral Hodge conjecture by Theorem 7.2, the Hodge class

$$m_p^{N_p} \cdot c_1(P_X)^{2h-1}/(2h-1)! \in \mathrm{H}^{4h-2}(X \times \widehat{X}, \mathbb{Z})$$

is algebraic. Let $f\colon A \times B \times \widehat{A} \times \widehat{B} \to A \times \widehat{A}$ and $g\colon A \times B \times \widehat{A} \times \widehat{B} \to B \times \widehat{B}$ be the canonical projections. Then $\mathcal{P}_X \cong f^*\mathcal{P}_A \otimes g^*\mathcal{P}_B$. Using this and denoting $\mu = c_1(\mathcal{P}_A)$ and $\nu = c_1(\mathcal{P}_B)$ we have

$$\frac{c_1(\mathcal{P}_X)^{2h-1}}{(2h-1)!} = f^*\left(\frac{\mu^{2g-1}}{(2g-1)!}\right) \cdot g^*\left(\frac{\nu^{2g_B}}{(2g_B)!}\right) + f^*\left(\frac{\mu^{2g}}{(2g)!}\right) \cdot g^*\left(\frac{\nu^{2g_B-1}}{(2g_B-1)!}\right).$$

This implies that

$$f_*\left(c_1(\mathcal{P}_X)^{2h-1}/(2h-1)!\right) = (-1)^{g_b}\mu^{2g-1}/(2g-1)!.$$

In particular, the class $m_p^{N_p} \cdot c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic. Let $p_1, \ldots, p_n$ be all prime divisors of κ and observe that

$$\gcd(\kappa, m_{p_1}^{N_{p_1}}, m_{p_2}^{N_{p_2}}, \ldots, m_{p_n}^{N_{p_n}}) = 1.$$

Thus, there are integers $a, b_1, \ldots, b_n$ such that $a \cdot \kappa + \sum_{i=1}^n b_i \cdot m_{p_i}^{N_{p_i}} = 1$, and

$$\frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} = a \cdot \kappa \cdot \frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} + \sum_{i=1}^n b_i \cdot m_{p_i}^{N_{p_i}} \cdot \frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z}).$$

This proves that $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is a $\mathbb{Z}$-linear combination of algebraic classes, hence algebraic. Condition 1 follows then from Theorem 7.1.

Step three: [6 $\implies$ 1] *for A principally polarized by $\theta_A \in \mathrm{NS}(A)$.* Let $p_1, \dots, p_k$ be the prime factors of $(g-1)!$ and let $C_1, \dots, C_k$ be smooth proper curves for which there exist homomorphisms $\phi_i \colon A \to J(C_i)$ such that $\phi^*\theta_{J(C_i)} = m_i \cdot \theta_A$ for some $m_i \in \mathbb{Z}_{\geq 1}$ with $p_i \nmid m$. Since $\theta_{J(C_i)}^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(J(C_i), \mathbb{Z})$ is algebraic for each i, the classes

$$\phi_i^*(\theta_{J(C_i)}^{g-1}/(g-1)!) = m_i^{g-1} \cdot \theta_A^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$$

are algebraic. Since $\gcd((g-1)!, m_1, \dots, m_k) = 1$, this implies that $\theta_A^{g-1}/(g-1)!$ is algebraic. Condition 1 follows then from Theorem 7.1.

Step four: [6 $\impliedby$ 1 $\implies$ 2] *for (A, θ_A) principally polarized with $\rho(A) = 1$.* Write $\theta = \theta_A$. Let $Z_1, \dots, Z_n$ be integral curves $Z_i \subset A$ and let $e_1, \dots, e_n \in \mathbb{Z}$ with $e_i \neq 0$ for all i be such that

$$\theta^{g-1}/(g-1)! = \sum_{i=1}^{n} e_i \cdot [Z_i] \in \mathrm{H}^{2g-2}(A, \mathbb{Z}).$$

Since $\rho(A) = 1$, the group $\mathrm{Hdg}^{2g-2}(A, \mathbb{Z})$ is generated by $\theta^{g-1}/(g-1)!$. Consequently, we have $[Z_i] = f_i \cdot \left(\theta^{g-1}/(g-1)!\right)$ for some non-zero $f_i \in \mathbb{Z}$. Hence we can write

$$\theta^{g-1}/(g-1)! = \sum_{i=1}^{n} e_i \cdot [Z_i] = \sum_{i=1}^{n} e_i \cdot f_i \cdot \theta^{g-1}/(g-1)!$$

which implies $\sum_{i=1}^{n} e_i \cdot f_i = 1$. Now let p be any prime number. Then there exists an integer i with $1 \leq i \leq n$ such that p does not divide f_i. Let $C_i \to Z_i$ be the normalization of Z_i and let $\lambda_A = \varphi_\theta \colon A \to \widehat{A}$ be the polarization corresponding to θ. This gives a diagram

$$\begin{tikzcd}
C_i \arrow[rr, "\varphi"] \arrow[dr, "\iota"'] & & A \arrow[r, "\lambda_A", "\sim"'] \arrow[rrr, bend right, "\phi"'] & \widehat{A} \arrow[r, "\varphi^*"] & \mathrm{Pic}^0(C_i) \arrow[r, "a", "\sim"'] & J(C_i), \\
& J(C_i) \arrow[ur, "\psi"'] & & & &
\end{tikzcd} \tag{7.4}$$

where

$$\iota\colon C_i \to J(C_i) = H^0(C, \Omega_C)^*/H_1(C, \mathbb{Z})$$

is the Abel–Jacobi map (for some $p \in C$), and $\varphi^*\colon \widehat{A} = \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(C_i)$ is the pullback of line bundles along $\varphi\colon C_i \to A$. The natural homomorphism $a\colon \mathrm{Pic}^0(C_i) \to J(C_i)$ is an isomorphism by the Abel–Jacobi theorem. Since the triangle on the left in Diagram (7.4) commutes and $[Z_i] \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ is non-zero, the morphism $\psi\colon J(C_i) \to A$ is non-zero. As $\rho(A) = 1$, the map $\psi\colon J(C_i) \to A$ must be surjective, the Picard rank of a non-simple abelian variety being greater than one. Dually, ψ gives rise to a non-zero homomorphism $\widehat{\psi}\colon \widehat{A} \to \widehat{J(C_i)}$, and the simpleness of $\widehat{A}$ implies that $\widehat{\psi}$ is finite onto its image.

We claim that the same is true for ϕ. To prove this, it suffices to show that the kernel of $\varphi^*\colon \widehat{A} \to \mathrm{Pic}^0(C_i)$ is finite. Since the homomorphism $\iota^*\colon \widehat{J(C_i)} \to \mathrm{Pic}^0(C_i)$ induced by the embedding $\iota\colon C_i \to J(C_i)$ is an isomorphism, dualizing the triangle on the left in Diagram (7.4) proves our claim.

By construction, we have

$$\varphi_*[C_i] = [Z_i] = f_i \cdot \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(A, \mathbb{Z}).$$

By a version of Welters' Criterion (see [BL04, Lemma 12.2.3]), this implies that $\phi^*\left(\theta_{J(C_i)}\right) = f_i \cdot \theta \in \mathrm{H}^2(A, \mathbb{Z})$, where $\theta_{J(C_i)} \in \mathrm{H}^2(J(C_i), \mathbb{Z})$ is the canonical principal polarization. In particular, 6 holds.

We claim that also 2 holds. Let $j\colon A_0 \hookrightarrow J(C_i)$ be the embedding of $A_0 = \phi(A)$ into $J(C_i)$ and let $\lambda_0\colon A_0 \to \widehat{A}_0$ be the polarization on A_0 induced by j. We have $\phi^*(\lambda) = \varphi_{f_i\cdot\theta} = f_i \cdot \varphi_\theta = f_i \cdot \lambda_A$. We obtain a commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "\pi"] \arrow[dl, "{[f_i]_A}"'] \arrow[d, "f_i\cdot\lambda_A"] & A_0 \arrow[r, "j"] \arrow[d, "\lambda_0"] & J(C_i) \arrow[d, "\lambda"] \\
A & \widehat{A} \arrow[l, "\lambda_{\widehat{A}}"] & \widehat{A}_0 \arrow[l, "\widehat{\pi}"] & \widehat{J(C_i)}. \arrow[l, "\widehat{j}"]
\end{tikzcd}$$

Let G be the kernel of π. Define

$$K = \mathrm{Ker}([f_i]_A) = \mathrm{Ker}(f_i \cdot \lambda_A) \cong (\mathbb{Z}/f_i)^{2g} \subset A,$$

and $U = \mathrm{Ker}(\widehat{\pi} \circ \lambda_0) \subset A_0$. Also define $H = \mathrm{Ker}(\lambda_0)$, and observe that $H \subset U$. The exact sequence

$$0 \to G \to K \to U \to 0$$

shows that if a, k, u and h are the respective orders of G, K, U and H, then one has

$$h|u|k|f_i \quad \text{and} \quad a|k|f_i. \tag{7.5}$$

Then define $B = \mathrm{Ker}(\widehat{j} \circ \lambda) \subset J(C_i)$ with inclusion $i \colon B \hookrightarrow J(C_i)$. It is easy to see that B is connected. Moreover, we have $A_0 \cap B = H$ and, therefore, an exact sequence of commutative group schemes

$$0 \to H \to A_0 \times B \xrightarrow{\psi} J(C_i) \to 0.$$

The morphism $\alpha \colon A \times B \to J(C_i)$, defined as the composition

$$A \times B \xrightarrow{\pi \times \mathrm{id}} A_0 \times B \xrightarrow{\psi} J(C_i),$$

is an isogeny. Since the degree of an isogeny is multiplicative in compositions, we have

$$\deg(\alpha) = \deg\left(\psi \circ (\pi \times \mathrm{id})\right) = \deg(\psi) \cdot \deg(\pi \times \mathrm{id}) = h \cdot \deg(\pi) = h \cdot a.$$

In particular, p does not divide $\deg(\alpha)$ because h and a divide f_i by Equation (7.5). □

Proof of Theorem 7.3. According to Theorem 7.1, it suffices to show that the cohomology class

$$\frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$$

is algebraic for $[(A, \lambda)]$ in a dense subset X of $\mathsf{A}_{g,\delta}(\mathbb{C})$ as in the statement. Define $D = \mathrm{diag}(\delta_1, \dots, \delta_g)$ and define, for each subring R of $\mathbb{C}$, a group

$$\mathrm{Sp}_{2g}^{\delta}(R) = \left\{ M \in \mathrm{GL}_{2g}(R) \mid M \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} M^t = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \right\}. \tag{7.6}$$

The isomorphism

$$\mathrm{Sp}_{2g}^{\delta}(\mathbb{R}) \to \mathrm{Sp}_{2g}(\mathbb{R}), \quad M \mapsto \begin{pmatrix} 1_g & 0 \\ 0 & D \end{pmatrix}^{-1} M \begin{pmatrix} 1_g & 0 \\ 0 & D \end{pmatrix}$$

induces an action of $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z})$ on the genus g Siegel space $\mathbb{H}_g$, and the period map defines an isomorphism of complex analytic spaces $\mathsf{A}_{g,\delta}(\mathbb{C}) \cong \mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}) \setminus \mathbb{H}_g$ [BL04, Theorem 8.2.6]. Pick any prime number $\ell > (2g-1)!$ and consider, for a period matrix $x \in \mathbb{H}_g$, the orbit $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell]) \cdot x \subset \mathbb{H}_g$. Let (A, λ) be a polarized abelian variety admitting a period matrix equal to x. The image of $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell]) \cdot x$ in $\mathsf{A}_{g,\delta}(\mathbb{C})$ is the *Hecke-ℓ-orbit* of $[(A,\lambda)] \in \mathsf{A}_{g,\delta}(\mathbb{C})$, i.e. the set of isomorphism classes of polarized abelian varieties $[(B,\mu)] \in \mathsf{A}_{g,\delta}(\mathbb{C})$ for which there exists integers $n, m \in \mathbb{Z}_{\geq 0}$ and an isomorphism of polarized rational Hodge structures $\phi \colon \mathrm{H}_1(B,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_1(A,\mathbb{Q})$ such that $\ell^n \cdot \phi$ and $\ell^m \cdot \phi^{-1}$ are morphisms of integral Hodge structures (Hecke orbits were studied in positive characteristic in e.g. [Cha95, CO19]). The degree of the isogeny $\alpha = \ell^n \phi$ must be ℓ^k for some nonnegative integer k. In particular, if one abelian variety in a Hecke-ℓ-orbit happens to be isomorphic to a Jacobian, then every abelian variety in that orbit is ℓ-power isogenous to a Jacobian, see Definition 7.12. The decomposition of a polarized abelian variety into non-decomposable polarized abelian subvarieties is unique [Deb96, Corollaire 2], which implies that the morphism

$$\begin{aligned} \pi \colon \prod_{i=1}^{g} \mathsf{A}_{1,1} &\to \mathsf{A}_{g,\delta}, \\ \big([(E_1,\lambda_1)], \ldots, [(E_g,\lambda_g)]\big) &\mapsto \big([E_1 \times \cdots \times E_g, \delta_1 \cdot \lambda_1 \times \cdots \times \delta_g \cdot \lambda_g)\big] \end{aligned}$$

is finite onto its image. Thus $\mathsf{A}_{g,\delta}$ contains a g-dimensional subvariety on which the integral Hodge conjecture for one-cycles holds. Let $V = \pi\left(\prod_{i=1}^{g} \mathsf{A}_{1,1}\right) \subset \mathsf{A}_{g,\delta}$. Then

$$X' := \mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell]) \cdot V = \cup_i Z_i \subset \mathsf{A}_{g,\delta}(\mathbb{C})$$

is a countable union of closed analytic subsets $Z_i \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ of dimension $\dim Z_i \geq g$ such that, by Lemma 7.14 below, $X' \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ is dense in the analytic topology, and that $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic for every polarized abelian variety (A, λ) of polarization type δ whose isomorphism class lies in X'. To prove the theorem, we are reduced to proving that there exists a similar countable union $X \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ whose components are algebraic. For this, it suffices to prove:

Claim: The locus of $[(A,\lambda)] \in \mathsf{A}_{g,\delta}(\mathbb{C})$ such that

$$c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})_{\mathrm{alg}}$$

is a countable union

$$W = \cup_j Y_j \subset \mathsf{A}_{g,\delta}(\mathbb{C})$$

of closed algebraic subsets $Y_j \subset \mathsf{A}_{g,\delta}(\mathbb{C})$.

Indeed, let $U \to \mathcal{A}_{g,\delta}$ be a finite étale cover of the moduli stack $\mathcal{A}_{g,\delta}$ and let $\mathcal{X} \to U$ be the pullback of the universal family of abelian varieties along $U \to \mathcal{A}_{g,\delta}$. This gives an abelian scheme $\mathcal{X} \times \widehat{\mathcal{X}} \to U$ carrying a relative Poincaré line bundle $\mathcal{P}_{\mathcal{X}/U}$ and arguments similar to those used to prove Lemma 7.11 show that indeed, for each irreducible component $U' \subset U$, the locus in $U'(\mathbb{C})$ where $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is algebraic is a countable union of closed algebraic subvarieties of $U'(\mathbb{C})$. We have $X' \subset W$ and since each $Z_i \subset X$ is irreducible, each Z_i is contained in an irreducible component $Y_j \subset W$. We may then define X as the union of those $Y_j \subset W$ that contain some Z_i.

Finally, Theorem 7.1 implies that for each $[(A,\lambda)] \in X$, the integral Hodge conjecture for one-cycles holds for the abelian variety A, so we are done. □

Lemma 7.14. *Let ℓ be a prime number and let $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell]) \subset \mathrm{Sp}_{2g}(\mathbb{R})$ be the group defined in (7.6). Then $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell])$ is analytically dense in $\mathrm{Sp}_{2g}(\mathbb{R})$.*

Proof. Since $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Q})$ arises as the group of rational points of an algebraic subgroup $\mathrm{Sp}_{2g}^{\delta}$ of GL_{2g} over $\mathbb{Q}$ [PR94, Chapter 2, §2.3.2], which is isomorphic to Sp_{2g} over $\mathbb{Q}$, the lemma follows from the well-known fact that for $S = \{\ell\} \subset \mathrm{Spec}(\mathbb{Z})$, the algebraic group Sp_{2g} satisfies the strong approximation property with respect to S [PR94, Chapter 7, §7.1]. The latter is classical and follows from the non-compactness of $\mathrm{Sp}_{2g}(\mathbb{Q}_\ell)$, see [PR94, Theorem 7.12]. □

Remark 7.15. Using level structures one can show that whenever $\gcd(\prod_i \delta_i, (2g-1)!) = 1$ (or, more generally, $\gcd(\prod_i \delta_i, (2g-2)!) = 1$, see Section 7.2.3 below), there is a countable union $X = \cup_i Z_i \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ as in Theorem 7.3 such that $\dim Z_i \geq 3g-3$. Indeed, let $\mathsf{A}^*_{g,\delta_g}$ be the moduli space of principally polarized abelian varieties of dimension g with δ_g-level structure. Then there is a natural morphism $\phi\colon \mathsf{A}^*_{g,\delta_g} \to \mathsf{A}_{g,\delta}$ such that for any $x = [(A,\lambda)] \in \mathsf{A}^*_{g,\delta_g}(\mathbb{C})$ with $[(B,\mu)] = \phi(x) \in \mathsf{A}_{g,\delta}(\mathbb{C})$, there exists an isogeny $\alpha\colon A \to B$ of degree $\prod_{i=1}^{g} \delta_i$, see [Mum71].

Remark 7.16. In the principally polarized case, the density in the moduli space of those abelian varieties that satisfy the integral Hodge conjecture for one-cycles admits another proof which might be interesting for comparison. Let A_g be the coarse moduli space of principally polarized complex abelian varieties of dimension g and let $[(A,\theta)]$ be a closed point of A_g. Then by [BL04, Exercise 5.6.(10)], the following are equivalent: (i) A is isogenous to the g-fold self-product E^g for an elliptic curve E with complex multiplication, (ii) A has maximal Picard rank $\rho(A) = g^2$, (iii) A is *isomorphic* to the product $E_1 \times \cdots \times E_g$ of pairwise isogenous elliptic curves E_i with complex multiplication. If any of these conditions is satisfied, then A satisfies the integral Hodge conjecture for one-cycles by Theorem 7.2, and the set of isomorphism classes of principally polarized abelian varieties (A,θ) for which this holds is dense in A_g by [Lan75]. For an explicit example in dimension $g = 4$ of a principally polarized abelian variety (A,θ) that satisfies one of the equivalent conditions above, but is not isomorphic to a Jacobian, see [Deb87, §5].

7.2.3 *The integral Hodge conjecture for one-cycles up to factor n*

In this section, we study a property of a smooth projective complex variety that lies somewhere in between the integral Hodge conjecture and the usual (i.e. rational) Hodge conjecture. The key will be the following:

Definition 7.17. Let $d,k,n \in \mathbb{Z}_{\geq 1}$ and let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Recall the definition of the degree $2d-2k$ *Voisin group* of X [Voi16, Per22]:

$$\begin{aligned} \mathrm{Z}^{2d-2k}(\mathrm{X}) &:= \mathrm{Hdg}^{2d-2k}(\mathrm{X},\mathbb{Z})/\mathrm{H}^{2d-2k}(\mathrm{X},\mathbb{Z})_{\mathrm{alg}} \\ &= \mathrm{Coker}\left(\mathrm{CH}_k(\mathrm{X}) \to \mathrm{Hdg}^{2d-2k}(\mathrm{X},\mathbb{Z})\right). \end{aligned}$$

We say that X *satisfies the integral Hodge conjecture for k-cycles up to factor n* if $\mathrm{Z}^{2d-2k}(X)$ is annihilated by n (in other words, if $n \cdot x \in \mathrm{H}^{2d-2k}(X,\mathbb{Z})_{\mathrm{alg}}$ for every $x \in \mathrm{Hdg}^{2d-2k}(X,\mathbb{Z})$).

Lemma 7.18. *Let A be a complex abelian variety of dimension g. Define $\sigma_A \in \mathrm{H}^{4g-4}(A \times \widehat{A},\mathbb{Z})$ to be the class $c_1(\mathcal{P}_A)^{2g-2}/(2g-2)!$.*

1. *Let n be a positive integer and let $\mathcal{F}_n \colon \mathrm{CH}^1(\widehat{A}) \to \mathrm{CH}_1(A)$ be a group homomorphism such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{CH}^1(\widehat{A}) \arrow[r, "\mathcal{F}_n"] \arrow[d] & \mathrm{CH}_1(A) \arrow[d] \\ \mathrm{H}^2(\widehat{A}, \mathbb{Z}) \arrow[r, "n \cdot \mathscr{F}_{\widehat{A}}"] & \mathrm{H}^{2g-2}(A, \mathbb{Z}). \end{tikzcd}$$

Then A satisfies the integral Hodge conjecture for one-cycles up to factor n.

2. *Let $n \in \mathbb{Z}_{\geq 1}$ be such that $n \cdot \sigma_A$ is algebraic. A homomorphism $\mathcal{F}_n$ as in 1 exists.*

Proof. Statement 1 follows immediately from the fact that $\mathrm{CH}^1(\widehat{A}) \to \mathrm{Hdg}^2(\widehat{A}, \mathbb{Z})$ is surjective by Lefschetz $(1,1)$. To prove 2, first observe that if

$$\sigma_{\widehat{A}} := c_1(\mathcal{P}_{\widehat{A}})^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}(\widehat{A} \times A, \mathbb{Z}),$$

then $n \cdot \sigma_{\widehat{A}}$ is algebraic since $n \cdot \sigma_A$ is. Let $\Sigma_n \in \mathrm{CH}_2(\widehat{A} \times A)$ be such that $cl(\Sigma_n) = n \cdot \sigma_{\widehat{A}}$. This gives a commutative diagram:

$$\begin{tikzcd} \mathrm{CH}^1(\widehat{A}) \arrow[r, "\pi_1^*"] \arrow[d, "cl"] & \mathrm{CH}^1(\widehat{A} \times A) \arrow[r, "\cdot \Sigma_n"] \arrow[d, "cl"] & \mathrm{CH}^{2g-1}(\widehat{A} \times A) \arrow[r, "\pi_{2,*}"] \arrow[d, "cl"] & \mathrm{CH}_1(A) \arrow[d, "cl"] \\ \mathrm{H}^2(\widehat{A}, \mathbb{Z}) \arrow[r, "\pi_1^*"] & \mathrm{H}^2(\widehat{A} \times A, \mathbb{Z}) \arrow[r, "\cdot n \cdot \sigma_{\widehat{A}}"] & \mathrm{H}^{4g-2}(\widehat{A} \times A, \mathbb{Z}) \arrow[r, "\pi_{2,*}"] & \mathrm{H}^{2g-2}(A, \mathbb{Z}). \end{tikzcd}$$

Since

$$\pi_{2,*} \circ \left((-) \cdot n \cdot \sigma_{\widehat{A}}\right) \circ \pi_1^* = n \cdot \mathscr{F}_{\widehat{A}},$$

the homomorphism $\mathcal{F}_n := \pi_{2,*} \circ ((-) \cdot \Sigma_n) \cdot \pi_1^*$ has the required property. $\square$

Theorem 7.19. *Consider a complex abelian variety A of dimension g. Define the cycle σ_A in $\mathrm{H}^{4g-4}(A \times \widehat{A}, \mathbb{Z})$ as before and define $\rho_A = c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$.*

1. *Let $n \in \mathbb{Z}_{\geq 1}$ be such that $n \cdot \rho_A$ is algebraic. Then $n^2 \cdot \sigma_A$ is algebraic. In particular, A satisfies the integral Hodge conjecture up to factor $\gcd(n^2, (2g-2)!)$ in this case.*

2. *If A is principally polarized, and $n \in \mathbb{Z}_{\geq 1}$ is such that $n \cdot \gamma_\theta \in \mathrm{Hdg}^{2g-2}(A, \mathbb{Z})$ is algebraic, then $n \cdot \rho_A \in \mathrm{Hdg}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic.*

3. *A satisfies the integral Hodge conjecture for one-cycles up to factor $(2g-2)!$, and Prym varieties satisfy the integral Hodge conjecture for one-cycles up to factor 4.*

Proof. 1. By Lemma 6.5, one has

$$\sigma_A = c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! = (-1)^g \cdot (\rho_A)^{\star 2}/2! \in \mathrm{H}^{4g-4}(A \times \widehat{A}, \mathbb{Z}).$$

By Theorem 6.8, this implies that if $n \cdot \rho_A$ is algebraic, then $n^2 \cdot \sigma_A$ is algebraic. Since $(2g-2)! \cdot \sigma_A$ is algebraic, it follows that $\gcd(n^2, (2g-2)!) \cdot \sigma_A$ is algebraic. Thus we are done by Lemma 7.18.

2. This follows from Lemma 6.6.

3. This follows from Lemma 7.18, parts 1 and 2 and the fact that if A is a g-dimensional Prym variety with principal polarization $\theta \in \mathrm{Hdg}^2(A, \mathbb{Z})$, then $2 \cdot \gamma_\theta \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ is algebraic. □

7.3 THE INTEGRAL TATE CONJECTURE

Let X be a smooth projective variety over the separable closure k of a finitely generated field. Let k_0 be a finitely generated field of definition of X. A class $u \in \mathrm{H}^{2i}_{\text{ét}}(X, \mathbb{Z}_\ell(i))$ is an *integral Tate class* if it is fixed by some open subgroup of $\mathrm{Gal}(k/k_0)$. Totaro has shown that for codimension-one cycles on X, the Tate conjecture over k implies the integral Tate conjecture over k [Tot21, Lemma 6.2]. This means that every integral Tate class is the class of an algebraic cycle over k with $\mathbb{Z}_\ell$-coefficients.

Suppose that $A_{/k}$ is an abelian variety, defined over a finitely generated field $k_0 \subset k$ such that k is the separable closure of k_0. Then the Tate conjecture for codimension-one cycles holds for A over k by results of Tate [Tat66], Faltings [Fal83, FWG+86], and Zarhin [Zar74a, Zar74b]. By the above, A satisfies the integral Tate conjecture for codimension-one cycles over k. On the other hand, the Fourier transform defines an isomorphism

$$\mathscr{F}_A \colon \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^{2g-2}_{\text{ét}}(\widehat{A}, \mathbb{Z}_\ell(g-1)), \tag{7.7}$$

see [Tot21, Section 7]. Since (7.7) is Galois-equivariant (the Poincaré bundle being defined over k_0) it sends integral Tate classes to integral Tate classes. Therefore, to prove the integral Tate conjecture for one-cycles on A, it suffices to lift (7.7) to a homomorphism $\mathrm{CH}^1(A)_{\mathbb{Z}_\ell} \to \mathrm{CH}_1(\widehat{A})_{\mathbb{Z}_\ell}$.

Proof of Theorem 7.6. This follows from the above and Proposition 6.13.2. □

Corollary 7.20. *Let A and B be abelian varieties defined over the separable closure k of a finitely generated field, of respective dimensions g_A and g_B.*

1. *The classes*

$$\rho_A \in \mathrm{H}^{4g_A-2}_{\text{ét}}(A \times \widehat{A}, \mathbb{Z}_\ell(2g_A - 1)), \quad \text{and} \quad \rho_B \in \mathrm{H}^{4g_B-2}_{\text{ét}}(B \times \widehat{B}, \mathbb{Z}_\ell(2g_A - 1))$$

are algebraic if and only if $A \times \widehat{A}$, $B \times \widehat{B}$, $A \times B$ and $\widehat{A} \times \widehat{B}$ satisfy the integral Tate conjecture for one-cycles over k.

2. *If A and B are principally polarized, then the integral Tate conjecture for one-cycles over k holds for $A \times B$ if and only if it holds for both A and B.*

3. *If $\theta \in \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1))$ is the first Chern class of an ample line bundle that induces a principal polarization on A, and if $\gamma_\theta = \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}_{\text{ét}}(A, \mathbb{Z}_\ell(g-1))$ is algebraic, then $\theta^i/i!$ in $\mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell(i))$ is algebraic for each $i \in \mathbb{Z}_{\geq 1}$.*

Proof. 1. See Equation (6.10).
2. This is true because the minimal cohomology class of the product is algebraic if and only if the minimal cohomology classes of the factors are algebraic.
3. One has $\theta^i/i! = \gamma_\theta^{\star(g-i)}/(g-i)!$ by [Bea82, Corollaire 2]. □

Combining Theorems 7.1 and 7.6, we obtain:

Corollary 7.21. *Let A_K be a principally polarized abelian variety over a number field $K \subset \mathbb{C}$. Its base change $A_\mathbb{C}$ over $\mathbb{C}$ satisfies the integral Hodge conjecture for one-cycles if and only if $A_{\bar{K}}$ satisfies the integral Tate conjecture for one-cycles over $\bar{K} = \bar{\mathbb{Q}}$.*

Proof. We view $\bar{\mathbb{Q}}$ as a subfield of $\mathbb{C}$ in a way compatible with the inclusion $K \hookrightarrow \mathbb{C}$. For a prime number ℓ, let $\theta_\ell = c_1(\mathcal{L}) \in \mathrm{H}^2_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(1))$ be the ℓ-adic étale cohomology class of $\mathcal{L}$. On the other hand, define $\theta_\mathbb{C} \in \mathrm{NS}(A) \subset \mathrm{H}^2(A_\mathbb{C}, \mathbb{Z})$ to be the polarization of the complex abelian variety $A_\mathbb{C}$. By Theorems 7.1 and 7.6, it suffices to show that $\gamma_{\theta_\mathbb{C}} \in \mathrm{H}^{2g-2}(A_\mathbb{C}, \mathbb{Z})$ is algebraic if and only if $\gamma_{\theta_\ell} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(g-1))$ is in the image of (7.1) for each prime number ℓ. Define $\mathrm{R}^{2g-2}(A) = \mathrm{Coker}\left(\mathrm{CH}_1(A_\mathbb{C}) \to \mathrm{H}^{2g-2}(A_\mathbb{C}, \mathbb{Z})\right)$. This implies that

$$\mathrm{R}^{2g-2}(A) \otimes \mathbb{Z}_\ell = \mathrm{Coker}\left(\mathrm{CH}_1(A_\mathbb{C})_{\mathbb{Z}_\ell} \to \mathrm{H}^{2g-2}(A_\mathbb{C}, \mathbb{Z}_\ell)\right).$$

Suppose that γ_{θ_ℓ} is in the image of (7.1) for every prime number ℓ. Then the image of $\gamma_{\theta_\mathbb{C}}$ in $\mathrm{R}^{2g-2}(A) \otimes \mathbb{Z}_\ell$ is zero for each prime number ℓ, which implies that the image of $\gamma_{\theta_{\mathbb{C}}}$ in $\mathrm{R}^{2g-2}(A)$ is zero, i.e. $\gamma_{\theta_{\mathbb{C}}}$ is algebraic. Conversely, suppose that $\gamma_{\theta_{\mathbb{C}}} = \sum_{i=1}^{k} n_i \cdot cl(C_i)$ for some smooth projective curves C_i over $\mathbb{C}$. The Hilbert scheme $\mathcal{H} = \mathrm{Hilb}_{A_K/K}$ is defined over K; for each $i = 1, \dots, k$ we pick a $\bar{\mathbb{Q}}$-point in the connected component of $\mathcal{H}$ containing $[C_i \subset A]$. This gives smooth projective curves $C_i' \subset A_{\bar{\mathbb{Q}}}$ over $\bar{\mathbb{Q}}$ and we define $\Gamma = \sum_i n_i \cdot [C_i'] \in \mathrm{CH}_1(A_{\bar{\mathbb{Q}}})$. On the one hand, $cl(\Gamma_{\mathbb{C}}) = \gamma_{\theta_{\mathbb{C}}}$ by Lemma 7.11. On the other hand, the Artin comparison theorem gives an isomorphism of $\mathbb{Z}_\ell$-algebras

$$\phi\colon \mathrm{H}^\bullet_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell) = \mathrm{H}^\bullet_{\text{ét}}(A_{\mathbb{C}}, \mathbb{Z}_\ell) \cong \mathrm{H}^\bullet(A_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell.$$

Since ϕ is compatible with the cycle class maps $\mathrm{CH}(A_{\bar{\mathbb{Q}}}) \to \mathrm{H}^\bullet_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell)$ and $\mathrm{CH}(A_{\mathbb{C}}) \to \mathrm{H}^\bullet(A_{\mathbb{C}}, \mathbb{Z})$, we have

$$\phi(\gamma_{\theta_\ell}) = \gamma_{\theta_{\mathbb{C}}} \quad \text{and} \quad \phi(cl(\Gamma)) = cl(\Gamma_{\mathbb{C}}) = \gamma_{\theta_{\mathbb{C}}}.$$

Therefore, $cl(\Gamma) = \gamma_{\theta_\ell}$. □

Another corollary of Theorem 7.6 is that the integral Tate conjecture for one-cycles on abelian varieties is stable under specialization. For example, one has the following (c.f. Corollary 7.10):

Corollary 7.22. *Let A_K be a principally polarized abelian variety over a number field K. Suppose that $A_{\bar{K}}$ satisfies the integral Tate conjecture for one-cycles over $\bar{K}$. Let $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K at which A_K has good reduction and write $\kappa = \mathcal{O}_K/\mathfrak{p}$. The abelian variety $A_{\bar{\kappa}}$ over $\bar{\kappa}$ satisfies the integral Tate conjecture for one-cycles over $\bar{\kappa}$.*

Proof. Write $S = \mathrm{Spec}(\mathcal{O}_K)$ and let $A \to S$ be the Néron model of A_K. Let R (resp. $K_{\mathfrak{p}}$) be the completion of $\mathcal{O}_K$ (resp. K) at the prime $\mathfrak{p}$. The natural composition $K \to K_{\mathfrak{p}} \to \bar{K}_{\mathfrak{p}}$ induces an embedding $\bar{K} \to \bar{K}_{\mathfrak{p}}$, where $\bar{K}_{\mathfrak{p}}$ is an algebraic closure of $K_{\mathfrak{p}}$. This gives a commutative diagram, where the square on the right is provided in [Ful98, Example 20.3.5]:

$$\begin{tikzcd}
\mathrm{CH}(A_{\bar{K}})_{\mathbb{Z}_\ell} \arrow[r] \arrow[d] & \mathrm{CH}(A_{\bar{K}_{\mathfrak{p}}})_{\mathbb{Z}_\ell} \arrow[r] \arrow[d] & \mathrm{CH}(A_{\bar{\kappa}})_{\mathbb{Z}_\ell} \arrow[d] \\
\oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(A_{\bar{K}}, \mathbb{Z}_\ell(r)) \arrow[r, "\sim"] & \oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(A_{\bar{K}_{\mathfrak{p}}}, \mathbb{Z}_\ell(r)) \arrow[r, "\sim"] & \oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(A_{\bar{\kappa}}, \mathbb{Z}_\ell(r)).
\end{tikzcd} \tag{7.8}$$

Now the principal polarization $\lambda_K\colon A_K \xrightarrow{\sim} \widehat{A}_K$ extends uniquely to a homomorphism $\lambda\colon A \to \widehat{A}$ by the Néron mapping property [BLR90, Section 1.2, Definition 1] and since the same is true for the inverse $\lambda_K^{-1}\colon \widehat{A}_K \xrightarrow{\sim} A_K$ we find that λ is an isomorphism. In particular, we see that $A_{\bar{\kappa}}$ is principally polarized and that the class in $\mathrm{CH}^1(A_{\bar{K}})_{\mathbb{Z}_\ell}$ of a theta divisor on $A_{\bar{K}}$ is sent to the class in $\mathrm{CH}^1(A_{\bar{\kappa}})_{\mathbb{Z}_\ell}$ of a theta divisor on $A_{\bar{\kappa}}$. Thus, the minimal class $\gamma_{\theta_{\bar{K}}} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{K}}, \mathbb{Z}_\ell(g-1))$ is sent to the minimal class $\gamma_{\theta_{\bar{\kappa}}} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{\kappa}}, \mathbb{Z}_\ell(g-1))$ by the isomorphism on the bottom of (7.8). It follows that $\gamma_{\theta_{\bar{\kappa}}}$ is algebraic; by Theorem 7.6, we are done. □

Finally, let us prove Theorem 7.7. The theorem follows from Theorem 7.6 together with a result of Chai on the density of an ordinary isogeny class in positive characteristic [Cha95].

Proof of Theorem 7.7. For any $t \in A_g(k)$, let (A_t, λ_t) be a principally polarized abelian variety such that $[(A_t, \lambda_t)] = t$. Let

$$A = E_1 \times \cdots \times E_g$$

be the product of g ordinary elliptic curves E_i over k and provide A with its natural principal polarization. Let $x \in \mathsf{A}_g(k)$ be the point corresponding to the isomorphism class of A. Let $q > (g-1)!$ be a prime, different from p, and let

$$\mathscr{G}_q(x) \subset \mathsf{A}_g(k)$$

be the set of isomorphism classes $y = [(A_y, \lambda_y)]$ that admit an isogeny $\phi\colon A_y \to A_x$ with $\phi^*\lambda_x = q^N \cdot \lambda_y$ for some nonnegative integer N.

We claim that A_y satisfies the integral Tate conjecture for one-cycles over k for any $y \in \mathscr{G}_q(x)$. Indeed, for such y there exists a nonnegative integer N such that the isogeny $[q^N]\colon A_y \to A_y$ factors through A_x. Consequently, $q^{(2g-2)\cdot N} \cdot \gamma_\theta$ is algebraic for the first Chern class θ of the principal polarization on A_y, which implies that γ_θ is algebraic (as $q > (g-1)!$). Thus, the claim follows from Theorem 7.6.

Now $\mathscr{G}_q(z)$ is dense in A_g for any ordinary principally polarized abelian variety (A_z, λ_z) by a result of Chai [Cha95, Theorem 2]. Therefore, $\mathscr{G}_q(x)$ is dense in A_g and the proof is finished. □

8

CURVES ON REAL ABELIAN THREEFOLDS

8.1 INTRODUCTION

In the final Chapter 8 of our thesis, we will, as in the previous Chapter 7, provide applications of the results of Chapter 6. In Chapter 7, we used integral Fourier transforms on complex abelian varieties A of any dimension g to establish an algebraic link between Hodge classes of degree two and Hodge classes of degree $2g-2$. We saw that this is possible if *one* specific Hodge class is algebraic, namely the minimal Poincaré class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ on $A \times \widehat{A}$ (see Theorem 7.1). In Chapter 8 we likewise study one-cycles, but this time we study them over $\mathbb{R}$.

A natural question is whether there is again a preferred cohomology class, whose algebraicity would imply the real integral Hodge conjecture for one-cycles on the real abelian variety under consideration. It turns out that if one considers the real integral Hodge conjecture *modulo torsion*, then this is indeed the case (see Theorem 8.20). As in the complex case, the proof of this statement relies on integral Fourier transforms, i.e. on the theory developed in of Chapter 6.

Apart from this similarly with the complex situation, two differences stand out:

1. Proving the algebraicity of minimal classes is more difficult over $\mathbb{R}$ than over $\mathbb{C}$. For example, for every complex Jacobian variety $(J(C), \theta)$, one has $\theta^{g-1}/(g-1)! = [C]$ for any Abel-Jacobi embedding of the curve C into $J(C)$. If C is defined over $\mathbb{R}$, but $C(\mathbb{R}) = \emptyset$, then it is not clear why the minimal class $\theta^{g-1}/(g-1)!$ in $\mathrm{Hdg}^{2g-2}(J(C), \mathbb{Z}(g-1))^G$ should be algebraic.

2. The G-equivariant cohomology group $\mathrm{H}_G^{2g-2}(A(\mathbb{C}), \mathbb{Z}(g-1))$ of a real abelian variety A often contains non-zero *torsion* classes. For some of these torsion classes, there are no topological obstructions to be algebraic.

It is because of these two subtleties 1 and 2 that when working with algebraic cycles on abelian varieties over the real numbers, new methods have to be employed, methods specific to real algebraic geometry. The goal of Chapter 8 is to introduce such techniques and provide some first positive results, mostly in dimension three.

To explain the main results of Chapter 8, let us recall and continue the discussion of Section 1.2.3. Let X be a smooth projective variety over $\mathbb{R}$, and remember that $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ (Definition 1.1). Building on work of Krasnov [Kra91, Kra94] and Van Hamel [Ham97], Benoist and Wittenberg define a subgroup $\mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0$ of the G-equivariant cohomology group $\mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$ in the sense of Borel, and study the cycle class map [Kra91], [BW20a, §1.6.1]

$$\mathrm{CH}_i(X) \to \mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0 \qquad (i + k = \dim(X)). \tag{8.1}$$

The *real integral Hodge conjecture for i-cycles* refers to the property that (8.1) is surjective. Just as in the complex situation, this property holds for every smooth projective variety X over $\mathbb{R}$ if $i \in \{\dim(X), \dim(X) - 1, 0\}$ [Kra91, MvH98, Ham97, BW20a], but may fail for other values of i.

Complex uniruled threefolds, as well as threefolds X over $\mathbb{C}$ with $K_X = 0$, satisfy the integral Hodge conjecture by work of Grabowski, Voisin and Totaro [Gra04, Voi06, Tot21]. In [BW20a, Question 2.16], Benoist and Wittenberg ask whether the same is true over $\mathbb{R}$. In fact, in [BW20b] they provide positive answers for various classes of uniruled threefolds. For real Calabi-Yau varieties, however, nothing seems to be known. In Chapter 8, we address the following:

Question 8.1. Does a real abelian threefold satisfy the real integral Hodge conjecture?

Our goal is to provide evidence towards a positive answer to Question 8.1.

Let us start with explaining our results. If A is a real abelian variety and $k = \dim(A) - 1$, there is an exact sequence (see Lemma 8.12):

$$0 \to \mathrm{Hdg}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))_0[2] \to \mathrm{Hdg}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))_0 \to \mathrm{Hdg}^{2k}(A(\mathbb{C}), \mathbb{Z}(k))^G \to 0.$$

Definition 8.2. A real abelian variety A of dimension g satisfies the *real integral Hodge conjecture for one-cycles modulo torsion* if the following map is surjective:

$$\mathrm{CH}_1(A) \to \mathrm{Hdg}^{2g-2}(A(\mathbb{C}), \mathbb{Z}(g-1))^G. \tag{8.2}$$

If $g = 3$, we say that A satisfies the *real integral Hodge conjecture modulo torsion* if the homomorphism (8.2) is surjective.

The first main result of Chapter 8 is as follows.

Theorem 8.3. *Every abelian threefold over $\mathbb{R}$ satisfies the real integral Hodge conjecture modulo torsion.*

By using the Hochschild-Serre spectral sequence to calculate the torsion rank of the equivariant cohomology of a real abelian threefold, we obtain:

Corollary 8.4. *Let A be an abelian threefold over $\mathbb{R}$ such that $A(\mathbb{R})$ is connected. Then A satisfies the real integral Hodge conjecture.*

Our proof of Theorem 8.3 is inspired by Grabowski's proof of the integral Hodge conjecture for complex abelian threefolds [Gra04]. It consists of two steps:

1. Reduce the real integral Hodge conjecture for one-cycles modulo torsion for abelian varieties of dimension g to the algebraicity of the class

$$\gamma_\theta = \theta^{g-1}/(g-1)! \in \mathrm{Hdg}^{2g-2}(A(\mathbb{C}), \mathbb{Z}(g-1))^G$$

   for every principally polarized real abelian variety (A, θ) of dimension g.

2. Reduce the algebraicity of γ_θ on a principally polarized real abelian threefold A to the case where $A = J(C)$ is the Jacobian of a real algebraic curve C with non-empty real locus, where this is clear.

An essential ingredient for reduction step (1) is the fact that any polarized abelian variety over $\mathbb{R}$ is isogenous to a principally polarized one. Although the analogue of this statement over any algebraically closed field is classical [Mum74], it fails over general fields [How01]. We will prove this fact in Theorem 8.18.

As for step (2), our strategy is to use Hecke orbits. For integers α, β, define the (α, β)-*Hecke orbit* of a moduli point $[(A, \lambda)] \in \mathcal{A}_g(\mathbb{R})$ as the set of $[(B, \mu)] \in \mathcal{A}_g(\mathbb{R})$ admitting an isogeny $A \to B$ preserving the polarizations up to a product of powers of α and β (see Definition 8.17). Hecke orbits are well-known to be dense in $\mathcal{A}_g(\mathbb{C})$ (see Lemma 7.14); we obtain the following real analogue.

Theorem 8.5. *Let p and q be distinct odd prime numbers. The (p, q)-Hecke orbit of any $x \in \mathcal{A}_g(\mathbb{R})$ is analytically dense in the connected component of $\mathcal{A}_g(\mathbb{R})$ containing x.*

Corollary 8.6. *Let p and q be as above. Every principally polarized abelian threefold over $\mathbb{R}$ is isogenous, via an isogeny that preserves the polarizations up to a product of powers of p and q, to the Jacobian of a non-hyperelliptic curve with non-empty real locus.*

Reduction step (2) follows because if an odd multiple of $\theta^2/2$ is algebraic for a principally polarized real abelian threefold (A, θ), then $\theta^2/2$ is algebraic. Corollary 8.6 turns out to be useful for the general principally polarized case as well:

Theorem 8.7. *Let $\mathcal{A}_3(\mathbb{R})^+$ be a component of the moduli space of principally polarized real abelian threefolds. Suppose that the real integral Hodge conjecture holds for every Jacobian $J(C)$ such that $[J(C)] \in \mathcal{A}_3(\mathbb{R})^+$ and the real locus $C(\mathbb{R})$ of C is non-empty. Then the real integral Hodge conjecture holds for every real abelian variety in $\mathcal{A}_3(\mathbb{R})^+$.*

In view of Theorems 8.3 and 8.7, Question 8.1 can (in the principally polarized case) be rephrased as follows. Let C be a real algebraic curve of genus three with non-empty real locus. Is the torsion subgroup of $\mathrm{Hdg}^4_G(J(C)(\mathbb{C}), \mathbb{Z}(2))_0$ algebraic?

Our final result reduces this question further. The theorem concerns torsion cohomology classes of degree four on real abelian varieties of any dimension g. For a smooth projective variety X over $\mathbb{R}$, the Hochschild-Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q(X(\mathbb{C}), \mathbb{Z}(k)) \implies \mathrm{H}^{p+q}_G(X(\mathbb{C}), \mathbb{Z}(k))$$

induces a filtration $F^\bullet$ on $\mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0 \subset \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$ (see Section 8.2).

Theorem 8.8. *Let A be an abelian variety over $\mathbb{R}$. The group $F^3\mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0$ is zero and the group $F^2\mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0$ is algebraic.*

The proof of Theorem 8.8 relies on (a slightly more general version of) Proposition 6.12 in Chapter 6 and an analysis of the Abel-Jacobi map for zero-cycles.

For an abelian variety A over $\mathbb{R}$, the Hochschild-Serre spectral sequence degenerates [Kra83]. For $p \in \mathbb{Z}_{\geq 0}$, define $\mathrm{H}^p(G, \mathrm{H}^{4-p}(A(\mathbb{C}), \mathbb{Z}(2)))_0$ as the image of the canonical homomorphism

$$F^p\mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0 \to \mathrm{H}^p(G, \mathrm{H}^{4-p}(A(\mathbb{C}), \mathbb{Z}(2))).$$

Combining Theorems 8.3 and 8.8, we obtain:

Corollary 8.9. *Let A be an abelian threefold over $\mathbb{R}$. Then A satisfies the real integral Hodge conjecture if and only if the canonical homomorphism*

$$\mathrm{CH}_1(A)_{\mathrm{hom}} \to \mathrm{H}^1(G, \mathrm{H}^3(A(\mathbb{C}), \mathbb{Z}(2)))_0$$

is surjective, where $\mathrm{CH}_1(A)_{\mathrm{hom}}$ denotes the kernel of $\mathrm{CH}_1(A) \to \mathrm{H}^4(A(\mathbb{C}), \mathbb{Z}(2))$.

Using the Künneth formula and the Abel-Jacobi map for zero-cycles, we show that for a real abelian threefold B and a real elliptic curve E whose real locus $E(\mathbb{R})$ is connected, the homomorphism $\mathrm{CH}_1(B \times E)_{\mathrm{hom}} \to \mathrm{H}^1(G, \mathrm{H}^3(B(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Z}(2)))$ is surjective (see Proposition 8.34). Together with Corollary 8.9, this implies:

Proposition 8.10. *Let B be a real abelian surface, and E a real elliptic curve whose real locus $E(\mathbb{R})$ is connected. Then $A = B \times E$ satisfies the real integral Hodge conjecture.*

Real abelian threefolds A come in four different types, corresponding to the G-equivariant diffeomorphism type of the complex locus $A(\mathbb{C})$ (see [GH81, §1]). Proposition 8.10 shows that for abelian threefolds of three out of these four types, there are no topological obstructions to the real integral Hodge conjecture.

8.2 THE REAL INTEGRAL HODGE CONJECTURE

8.2.1 *Generalities*

Let X be a smooth projective variety over $\mathbb{R}$. The group

$$G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \sigma\}$$

acts on $X(\mathbb{C})$ via the canonical anti-holomorphic involution $\sigma\colon X(\mathbb{C}) \to X(\mathbb{C})$. For $k \in \mathbb{Z}$, we denote by $\mathbb{Z}(k)$ the G-module that has $\mathbb{Z}$ as underlying $\mathbb{Z}$-module, on which G acts by $\sigma(1) = (-1)^k$. Thus, $\mathbb{Z}(k) = \mathbb{Z}(q)$ for every $q \in \mathbb{Z}$ with $k \equiv q \bmod 2$. By abuse of notation, we also denote by $\mathbb{Z}(k)$ the constant G-sheaf on $X(\mathbb{C})$ attached to the G-module $\mathbb{Z}(k)$. For $k, q \in \mathbb{Z}_{\geq 0}$, the G-action on the group $\mathrm{H}^k(X(\mathbb{C}), \mathbb{Z}(q))$ is understood to be the one induced by the involution $\mathrm{H}^k(\sigma) \circ F_\infty$, where $F_\infty = \sigma^*$ is the pull-back of the anti-holomorphic involution σ on $X(\mathbb{C})$, and $\mathrm{H}^k(\sigma)$ is the involution on cohomology induced by $\sigma\colon \mathbb{Z}(q) \to \mathbb{Z}(q)$.

Let $k \in \mathbb{Z}_{\geq 0}$. Attached to X is also the so-called degree $2k$ *equivariant cohomology group* with coefficients in $\mathbb{Z}(k)$, see [Gro57]. It is denoted by $\mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$, and relates to singular cohomology via canonical homomorphisms

$$\varphi \colon \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k)) \to \mathrm{H}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))^G. \tag{8.3}$$

A real subvariety $Z \subset X$ of codimension k induces a class $[Z] \in \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$, whose image $\varphi([Z])$ in $\mathrm{H}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))^G$ is the Hodge class $[Z_{\mathbb{C}}]$. It turns out that such algebraic cycle classes satisfy an additional condition, discovered by Kahn and Krasnov [Kah87, Kra94]. It depends only the structure of $X(\mathbb{C})$ as a topological G-space. For $i \in \{0, \dots, 2k\}$, define

$$\phi_i \colon \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k)) \to \mathrm{H}^i(X(\mathbb{R}), \mathbb{Z}/2)$$

as the composition

$$\begin{aligned} \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k)) &\xrightarrow{\text{mod } 2} \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}/2) \\ &\xrightarrow{\text{restriction}} \mathrm{H}^{2k}_G(X(\mathbb{R}), \mathbb{Z}/2) = \mathrm{H}^{2k}(X(\mathbb{R}) \times BG, \mathbb{Z}/2) \\ &\xrightarrow[\sim]{\text{Künneth}} \mathrm{H}^0(X(\mathbb{R}), \mathbb{Z}/2) \oplus \cdots \oplus \mathrm{H}^{2k}(X(\mathbb{R}), \mathbb{Z}/2) \\ &\xrightarrow{\text{projection}} \mathrm{H}^i(X(\mathbb{R}), \mathbb{Z}/2). \end{aligned}$$

For $\alpha \in \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$, define $\alpha_i = \phi_i(\alpha) \in \mathrm{H}^i(X(\mathbb{R}), \mathbb{Z}/2)$.

Definition 8.11 (Benoist–Wittenberg)**.** The subgroup

$$\mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0 \subset \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$$

is the group of classes $\alpha \in \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$ that satisfy the following conditions:

1. The class α lies in the subgroup $\mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0$, which means that

$$(\alpha_0, \alpha_1, \dots, \alpha_k, \dots, \alpha_{2k}) = \left(0, \dots, 0, \alpha_k, Sq^1(\alpha_k), Sq^2(\alpha_k), \dots, Sq^k(\alpha_k)\right). \tag{8.4}$$

Here, the Sq^i are the Steenrod operations

$$Sq^i : \mathrm{H}^p(X(\mathbb{R}), \mathbb{Z}/2) \to \mathrm{H}^{p+i}(X(\mathbb{R}), \mathbb{Z}/2).$$

2. The image of α in $\mathrm{H}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))$ under (8.3) is a Hodge class.

By [BW20a, §1.6.4], Definition 8.11 is functorial in X.

8.2.2 *Hochschild-Serre*

For a smooth variety X over $\mathbb{R}$, the *Hochschild-Serre* spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q(X(\mathbb{C}), \mathbb{Z}(k)) \implies \mathrm{H}_G^{p+q}(X(\mathbb{C}), \mathbb{Z}(k)) \tag{8.5}$$

is obtained by viewing $\mathrm{H}^i_G(X(\mathbb{C}), -)$ as the right-derived functor of the composition of taking global sections and G-invariants on the category of G-sheaves on $X(\mathbb{C})$.

Let A be an abelian variety over $\mathbb{R}$. Then (8.5) degenerates by [Kra83, §5.7]. Consequently, for every non-negative integer k, there are canonical identifications

$$\begin{aligned} \mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))_{\mathrm{tors}} &= \mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))[2] \\ &= \mathrm{Ker}\left(\mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k)) \to \mathrm{H}^{2k}(A(\mathbb{C}), \mathbb{Z}(k))\right) \\ &= F^1\mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k)). \end{aligned} \tag{8.6}$$

Moreover, these $\mathbb{Z}/2$-modules are (non-canonically) isomorphic to the $\mathbb{Z}/2$-module

$$\bigoplus_{\substack{p+q=k \\ p>0}} \mathrm{H}^p(G, \mathrm{H}^q(A(\mathbb{C}), \mathbb{Z}(k)).$$

8.2.3 *The topological condition*

Let X be a smooth projective variety of dimension n over $\mathbb{R}$. The following sequence is exact:

$$\begin{aligned} 0 \to \mathrm{H}^{2n-2}_G(X(\mathbb{C}), \mathbb{Z}(n-1))_0 \to \mathrm{H}^{2n-2}_G(X(\mathbb{C}), \mathbb{Z}(n-1)) \to \\ \to \bigoplus_{\substack{0 \le p < n-2 \\ p \equiv n-1 \bmod 2}} \mathrm{H}^p(X(\mathbb{R}), \mathbb{Z}/2) \to 0. \end{aligned} \tag{8.7}$$

This follows from [BW20a, Proposition 1.8, Equation (1.33) & Remark 1.20.(i)].

The following result will be useful for us. Let n be a positive integer and let $k = n-1$. For a smooth projective variety X of dimension n over $\mathbb{R}$, define

$$\mathrm{H}^{2\bullet}_{\star}(X(\mathbb{R}),\mathbb{Z}/2) = \bigoplus_{\substack{0\leq p<k-1 \\ p\equiv k \bmod 2}} \mathrm{H}^p(X(\mathbb{R}),\mathbb{Z}/2).$$

Lemma 8.12. *Let X be a smooth projective variety of dimension n over $\mathbb{R}$, and let $k = n-1$. Suppose that $X(\mathbb{C})$ has torsion-free degree $2k$ integral singular cohomology and that the Hochschild-Serre spectral sequence (8.5) degenerates. Then each row and each column in the following commutative diagram is exact:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^{2k}_G(X(\mathbb{C}),\mathbb{Z}(k))_0[2] \arrow[r] \arrow[d, hook] & \mathrm{H}^{2k}_G(X(\mathbb{C}),\mathbb{Z}(k))_0 \arrow[r, "\varphi"] \arrow[d, hook] & \mathrm{H}^{2k}(X(\mathbb{C}),\mathbb{Z}(k))^G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{H}^{2k}_G(X(\mathbb{C}),\mathbb{Z}(k))[2] \arrow[r] \arrow[d, two heads] & \mathrm{H}^{2k}_G(X(\mathbb{C}),\mathbb{Z}(k)) \arrow[r, "\varphi"] \arrow[d, two heads] & \mathrm{H}^{2k}(X(\mathbb{C}),\mathbb{Z}(k))^G \arrow[r] & 0 \\
& \mathrm{H}^{2\bullet}_{\star}(X(\mathbb{R}),\mathbb{Z}/2) \arrow[r, equal] & \mathrm{H}^{2\bullet}_{\star}(X(\mathbb{R}),\mathbb{Z}/2). & &
\end{tikzcd} \tag{8.8}$$

Proof. By the degeneration of the Hochschild-Serre spectral sequence (8.5), the map

$$\varphi\colon \mathrm{H}^{2k}_G(X(\mathbb{C}),\mathbb{Z}(k)) \to \mathrm{H}^{2k}(X(\mathbb{C}),\mathbb{Z}(k))^G$$

is a surjective homomorphism between abelian groups of the same rank. The target of φ is torsion-free, so its kernel is $\mathrm{H}^{2k}(X(\mathbb{C}),\mathbb{Z}(k))[2]$, which explains the horizontal exact sequence in the middle of diagram (8.8). By the proof of [BW20a, Proposition 1.8], the exact sequence (8.7) is *split*. This implies that, in diagram (8.8), the vertical arrow on the bottom left and the horizontal map φ on the top right are both surjective. □

Note that the first and second horizontal sequence in diagram (8.8) remain exact after restricting to Hodge classes.

8.2.4 *Threefolds*

Now let X be a smooth projective threefold over $\mathbb{R}$. The topological condition (8.4) on degree four classes takes a particularly simple form: a class $\alpha \in \mathrm{H}^4_G(X(\mathbb{C}), \mathbb{Z}(2))$ lies in $\mathrm{H}^4_G(X(\mathbb{C}), \mathbb{Z}(2))_0$ if and only if

$$\alpha|_x = 0 \in \mathrm{H}^4_G(\{x\}, \mathbb{Z}(2)) = \mathbb{Z}/2 \quad \text{for any} \quad x \in X(\mathbb{R}).$$

The conditions $\alpha|_x = 0$ for x in different connected components of $X(\mathbb{R})$ are *linearly independent* over $\mathbb{Z}/2$: for $n = 3$, the sequence (8.7) is the split exact sequence

$$0 \to \mathrm{H}^4_G(X(\mathbb{C}), \mathbb{Z}(2))_0 \to \mathrm{H}^4_G(X(\mathbb{C}), \mathbb{Z}(2)) \xrightarrow{\phi_0} \mathrm{H}^0(X(\mathbb{R}), \mathbb{Z}/2) \to 0. \tag{8.9}$$

Finally, since X satisfies the real integral Hodge conjecture for d-cycles whenever $d \in \{0, 2, 3\}$ (see [BW20a, §2.3.1 and §2.3.2]), the real integral Hodge conjecture for X is equivalent to the surjectivity of the homomorphism

$$\mathrm{CH}_1(X) \to \mathrm{Hdg}^4_G(X(\mathbb{C}), \mathbb{Z}(2))_0.$$

8.3 DENSITY OF HECKE ORBITS

8.3.1 *Polarized real abelian varieties*

Let A be a real abelian variety. As before, the dual abelian variety of A is denoted by $\widehat{A}$. Define $\Lambda = \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$. Denote by $\sigma\colon A(\mathbb{C}) \to A(\mathbb{C})$ the canonical anti-holomorphic involution, and by $F_\infty\colon \Lambda \to \Lambda$ its push-forward.

There is a canonical bijection between:

- Symmetric isogenies $\lambda\colon A \to \widehat{A}$ such that $\lambda_{\mathbb{C}} = \varphi_{\mathcal{L}}$ is the homomorphism $\varphi_{\mathcal{L}}\colon A_{\mathbb{C}} \to \widehat{A}_{\mathbb{C}}$ induced by an ample line bundle $\mathcal{L}$ on $A_{\mathbb{C}}$ as in [Mum74].
- Alternating forms $E\colon \Lambda \times \Lambda \to \mathbb{Z}$ such that $F^*_\infty(E) = -E$ and such that the following hermitian form is positive definite:

$$H\colon \Lambda_{\mathbb{R}} \times \Lambda_{\mathbb{R}} \to \mathbb{C}, \quad H(x, y) = E(ix, y) + iE(x, y).$$

- Classes of ample line bundles $\theta \in \mathrm{NS}(A_{\mathbb{C}})^G = \mathrm{Hdg}^2(A(\mathbb{C}), \mathbb{Z}(1))^G$.

In the sequel, a *polarization* on A will be an element in either one of the three sets above; the context will make clear which structure is meant.

8.3.2 *Moduli of real abelian varieties*

Let (A, λ) be a principally polarized complex abelian variety of dimension g. By Galois descent (see Section 1.1), to give a model of (A, λ) over $\mathbb{R}$ is to give an anti-holomorphic involution

$$\sigma\colon A(\mathbb{C}) \to A(\mathbb{C}) \quad \text{such that} \quad \sigma(0) = 0 \quad \text{and} \quad F_\infty^*(E) = -E.$$

By [Sil89, Chapter IV, Theorem (4.1)] (or [GH81, Section 9]), such an anti-holomorphic involution σ exists if and only if the complex principally polarized abelian variety (A, λ) admits a period matrix of the form

$$(I_g, \frac{1}{2}M + \mathrm{i}N). \tag{8.10}$$

Here N is a positive definite real matrix and M is a symmetric $g \times g$-matrix with integral coefficients such that if $r = \mathrm{rank}(M) \leq g$, then M is of the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

or of the form

$$\begin{pmatrix} 0 & \dots & 1 & 0 \\ \vdots & \iddots & \iddots & 0 \\ 1 & 0 & \iddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2}$$

Definition 8.13 (Silhol)**.** The *type* $(r, \alpha) \in \mathbb{Z}^2$ of a principally polarized real abelian variety (A, λ) is defined as follows. If $(I_g, \frac{1}{2}M + \mathrm{i}N)$ is a period matrix for $(A_\mathbb{C}, \lambda_\mathbb{C})$ as above, then $r = \mathrm{rank}(M)$. Define $\alpha \in \{0, 1, 2\}$ in the following way:

- If r is odd, then $\alpha = 1$. Thus the type of (A, λ) is $(r, 1)$.
- If r is zero, then $\alpha = 0$. Thus the type of (A, λ) is $(0, 1)$.

- If r is even, but non-zero, then $\alpha = 1$ if M is of the form (1) and $\alpha = 2$ if M is of the form (2).

This definition makes sense, because of the following:

Proposition 8.14 (Silhol). *The type (r, α) of a principally polarized real abelian variety (A, λ) does not depend on the chosen period matrix (8.10) for $(A_\mathbb{C}, \lambda_\mathbb{C})$ nor on the isomorphism class of (A, λ). If (A, λ) is of type (r, α), there exists a period matrix $(I_g, \frac{1}{2}M + \mathrm{i}N)$ for (A, λ) such that M is of the form (1) or (2), according to whether α equals 1 or 2.*

Proof. See [Sil89, Chapter IV, Corollaries (4.3) and (4.5)]. □

Definition 8.15. Let $\mathscr{T}(g)$ be the set of types (r, α) of principally polarized abelian varieties of dimension g over $\mathbb{R}$. For any type $\tau \in \mathscr{T}(g)$, define $M(\tau)$ to be the integral $g \times g$-matrix (1) or (2) above, according to whether α equals 1 or 2. Then define $\mathrm{GL}_g^\tau(\mathbb{Z})$ to be the subgroup of $\mathrm{GL}_g(\mathbb{Z})$ of matrices $T \in \mathrm{GL}_g(\mathbb{Z})$ that satisfy

$$T^t \cdot M(\tau) \cdot T \equiv M(\tau) \mod 2 \qquad (T^t = \text{ transpose of } T).$$

Finally, let H_g be the set of symmetric positive definite real matrices of rank g.

Theorem 8.16 (Silhol). *Let g be a positive integer. For $\tau \in \mathscr{T}(g)$, define $|\mathcal{A}_g(\mathbb{R})|^\tau$ to be the set of isomorphism classes of real principally polarized abelian varieties of type τ. For each type $\tau \in \mathscr{T}(g)$, the period map induces a bijection*

$$|\mathcal{A}_g(\mathbb{R})|^\tau = \mathrm{GL}_g^\tau(\mathbb{Z}) \setminus \mathrm{H}_g,$$

where $\mathrm{GL}_g^\tau(\mathbb{Z})$ acts on H_g by $N \mapsto T^t \cdot N \cdot T$. Therefore,

$$|\mathcal{A}_g(\mathbb{R})| = \bigsqcup_{\tau \in \mathscr{T}(g)} \mathrm{GL}_g^\tau(\mathbb{Z}) \setminus \mathrm{H}_g. \tag{8.11}$$

Proof. See [Sil89, Chapter IV, Theorem (4.6)]. □

See Theorem 2.15 for the fact that bijection (8.11) is actually a *homeomorphism*, with respect to the real-analytic topology on $|\mathcal{A}_g(\mathbb{R})|$ (see Definition 2.7).

8.3.3 *Density of Hecke orbits over the real numbers*

Before we prove Theorem 8.5, let us properly introduce the notion of Hecke orbits over the real numbers.

Definition 8.17. Let (A, λ) be a principally polarized abelian variety of dimension g over $\mathbb{R}$, and let $x = [(A, \lambda)] \in |\mathcal{A}_g(\mathbb{R})|$ the corresponding moduli point. For a tuple of integers (α, β), the (α, β)*-Hecke orbit of* x is the subset $\mathcal{G}_{\alpha,\beta}(x) \subset |\mathcal{A}_g(\mathbb{R})|$ of isomorphism classes $[(B, \nu)] \in |\mathcal{A}_g(\mathbb{R})|$ of principally polarized abelian varieties (B, ν) of dimension g over $\mathbb{R}$, for which there exist $n, m \in \mathbb{Z}_{\geq 0}$ and an isogeny

$$\phi\colon A \to B \quad \text{such that} \quad \phi^*(\nu) = \alpha^n \beta^m \cdot \lambda.$$

Proof of Theorem 8.5. Let p and q be distinct odd prime numbers.

Step 1: *If $x = [(A, \lambda)] \in |\mathcal{A}_g(\mathbb{R})|$ and $\tau \in \mathscr{T}(g)$ is the type of (A, λ), then $\mathcal{G}(x) \subset |\mathcal{A}_g(\mathbb{R})|^\tau$.* Indeed, for any $y = [(B, \mu)] \in \mathcal{G}(x)$, there exists an isogeny $\phi\colon A \to B$ such that $\phi^*(\mu) = n \cdot \lambda$ for some odd positive integer n. Such a map ϕ induces an isomorphism $\mathrm{H}_1(A(\mathbb{C}), \mathbb{Z}/2) \cong \mathrm{H}_1(B(\mathbb{C}), \mathbb{Z}/2)$ as symplectic spaces with involution. Since $x \in |\mathcal{A}_g(\mathbb{R})|^\tau$, this implies that $y \in |\mathcal{A}_g(\mathbb{R})|^\tau$ as well, see [GH81, Section 9].

Step 2: Define $R = \mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right]$. The ring homomorphism $R \to R/2R$ induces a group homomorphism $\mathrm{GL}_g(R) \to \mathrm{GL}_g(R/2R)$. For $\tau \in \mathscr{T}(g)$, we define

$$\mathrm{GL}_g^\tau(R) = \{T \in \mathrm{GL}_g(R)\colon T^t \cdot M(\tau) \cdot T \equiv M(\tau) \bmod 2\}.$$

Fix one such $\tau \in \mathscr{T}(g)$. Observe that the action of $\mathrm{GL}_g^\tau(\mathbb{Z})$ on H_g extends to a transitive action of $\mathrm{GL}_g(\mathbb{R})$ on H_g. We claim:

Let $x = [(A_x, \lambda_x)] \in |\mathcal{A}_g(\mathbb{R})|^\tau$, and lift x to a point $y \in \mathrm{H}_g$. Consider the orbit $\mathrm{GL}_g^\tau(R) \cdot y \subset \mathrm{H}_g$ as well as its image $\mathrm{GL}_g^\tau(\mathbb{Z}) \setminus \left(\mathrm{GL}_g^\tau(R) \cdot y\right)$ in $|\mathcal{A}_g(\mathbb{R})|^\tau = \mathrm{GL}_g^\tau(\mathbb{Z}) \setminus \mathrm{H}_g$. Then

$$\mathrm{GL}_g^\tau(\mathbb{Z}) \setminus \left(\mathrm{GL}_g^\tau(R) \cdot y\right) = \mathcal{G}_{p,q}(x).$$

Indeed, if $\mathbb{H}_g$ is the genus g Siegel space of symmetric, complex $g \times g$ matrices $Z = X + \mathrm{i}Y$ whose imaginary part Y is positive definite, then the inclusion

$$\rho_\tau\colon \mathrm{H}_g \hookrightarrow \mathbb{H}_g, \quad N \mapsto \frac{1}{2} \cdot M(\tau) + \mathrm{i}N$$

is equivariant for the embedding

$$f_\tau\colon \mathrm{GL}_g(\mathbb{R}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R}), \quad T \mapsto \begin{pmatrix} T^t & \frac{1}{2}\left(M(\tau)\cdot T^{-1} - T^t \cdot M(\tau)\right) \\ 0 & T^{-1} \end{pmatrix}.$$

Moreover, the action of the group $\mathrm{Sp}_{2g}(R)$ on $\mathbb{H}_g$ has the following geometric meaning: if we consider $\mathbb{H}_g$ as a moduli space of g-dimensional, principally polarized complex abelian varieties with symplectic basis, then two points $y = [A_y]$ and $z = [A_z] \in \mathbb{H}_g$ are in the same $\mathrm{Sp}_{2g}(R)$-orbit if and only if there exists an isogeny $\phi\colon A_y \to A_z$ that preserves the polarizations up to a product of powers of p and q. Since the intersection

$$f_\tau\left(\mathrm{GL}_g(\mathbb{R})\right) \cap \mathrm{Sp}_{2g}(R) = f_\tau\left(\mathrm{GL}_g^\tau(R)\right)$$

equals the subgroup of $\mathrm{Sp}_{2g}(R)$ that preserves the locus

$$\rho_\tau(\mathrm{H}_g) = \left\{\frac{1}{2}M(\tau) + \mathrm{i}N\right\} \subset \mathbb{H}_g$$

of real abelian varieties of type τ, this concludes Step 2.

Step 3: *For any $\tau \in \mathscr{T}(g)$, the subgroup $\mathrm{GL}_g^\tau(R) \subset \mathrm{GL}_g(\mathbb{R})$ is dense in the analytic topology.*

Define

$$\mathrm{SL}_g^\tau(R) = \mathrm{SL}_g(R) \cap \mathrm{GL}_g^\tau(R) = \{T \in \mathrm{SL}_g(R)\colon T^t \cdot M(\tau) \cdot T \equiv M(\tau) \bmod 2\}.$$

We claim that

$$\mathrm{SL}_g(\mathbb{R}) = \overline{\mathrm{SL}_g(R)} = \overline{\mathrm{SL}_g^\tau(R)} \subset \overline{\mathrm{GL}_g^\tau(R)} \subset \mathrm{GL}_g(\mathbb{R}). \tag{8.12}$$

Indeed, this follows from the following two statements:

a) The closure of $\mathrm{SL}_g(R)$ in $\mathrm{GL}_g(\mathbb{R})$ is $\mathrm{SL}_g(\mathbb{R})$.

b) The subgroup $\mathrm{SL}_g^\tau(R) \subset \mathrm{SL}_g(R)$ has finite index.

To prove (a), observe that the subgroup $\mathrm{SL}_g(\mathbb{R}) \subset \mathrm{GL}_g(\mathbb{R})$ is closed, which implies that the closure of $\mathrm{SL}_g(R)$ in $\mathrm{GL}_g(\mathbb{R})$ equals the closure of $\mathrm{SL}_g(R)$ in $\mathrm{SL}_g(\mathbb{R})$. Thus, (a) follows from the density of $\mathrm{SL}_g(R)$ in $\mathrm{SL}_g(\mathbb{R})$, which is true by strong approximation; see the proof of Lemma 7.14 for the precise argument. As for (b), the group $\mathrm{SL}_g(R)$ acts on $\mathrm{M}_g(R/2R)$ via $M \mapsto A^t \cdot M \cdot A$ mod 2, so there is an injection

$$\mathrm{SL}_g^\tau(R) \setminus \mathrm{SL}_g(R) \hookrightarrow \mathrm{M}_g(R/2R) = \mathrm{M}_g(\mathbb{Z}/2), \quad T \mapsto T^t \cdot M(\tau) \cdot T \mod 2.$$

Now (b) implies that the index of $\overline{\mathrm{SL}_g^\tau(R)} \subset \overline{\mathrm{SL}_g(R)}$ is finite. By (a), we have $\overline{\mathrm{SL}_g(R)} = \mathrm{SL}_g(\mathbb{R})$; thus $\overline{\mathrm{SL}_g^\tau(R)} \subset \mathrm{SL}_g(\mathbb{R})$ is a closed subgroup of finite index, hence open. Therefore $\overline{\mathrm{SL}_g^\tau(R)} = \mathrm{SL}_g(\mathbb{R})$ by connectivity of $\mathrm{SL}_g(\mathbb{R})$, proving Claim (8.12).

Write $G = \mathrm{GL}_g^\tau(R)$. If $H \subset \mathrm{GL}_g(\mathbb{R})$ is any Lie subgroup such that $\mathrm{SL}_g(\mathbb{R}) \subset H$, then $H = \det^{-1}(\det(H))$. Consequently, using (8.12), we obtain:

$$\overline{G} = \det^{-1}(\det(\overline{G})) \subset \mathrm{GL}_g(\mathbb{R}). \tag{8.13}$$

The equality (8.13) implies that in order to prove Step 3, it suffices to show that $\det^{-1}(\det(\overline{G})) = \mathrm{GL}_g(\mathbb{R})$; for this, it suffices to show that $\det(\overline{G}) = \mathbb{R}^*$. Now the morphism $\det\colon \mathrm{GL}_g(\mathbb{R}) \to \mathbb{R}^*$ is open, since its differential at the identity matrix $I_g \in \mathrm{GL}_g(\mathbb{R})$ is the trace homomorphism $\mathrm{M}_g(\mathbb{R}) \to \mathbb{R}$. Writing

$$\mathrm{GL}_g(\mathbb{R}) = \det^{-1}(\mathbb{R}^*) = \det^{-1}(\det(\overline{G}) \sqcup \det(\overline{G})^c) = \overline{G} \sqcup \det^{-1}\left(\det(\overline{G})^c\right),$$

it follows that $\det(\overline{G})$ is closed in $\mathbb{R}^*$. From this, we conclude that

$$\overline{\det(G)} \subset \overline{\det(\overline{G})} = \det(\overline{G}).$$

Thus, to show that $\det(\overline{G}) = \mathbb{R}^*$, it suffices to show that $\det(G)$ is dense in $\mathbb{R}^*$. The homomorphism $\det\colon \mathrm{GL}_g^\tau(R) \to R^*$ is surjective because it admits the section

$$R^* \to \mathrm{GL}_g^\tau(R), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & I_{g-1} \end{pmatrix}.$$

Therefore,

$$\det(G) = \det(\mathrm{GL}_g^\tau(R)) = R^* = \{\pm p^n q^m \colon n, m \in \mathbb{Z}\},$$

and it remains to prove that the latter is dense in $\mathbb{R}^*$. To see this, note that $R^*_{>0} = \{p^n q^m \colon n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}_{>0}$ because $\log(R^*_{>0}) = \mathbb{Z}\log(p) + \mathbb{Z}\log(q)$ is dense in $\mathbb{R}$; the latter holds since $\log(R^*_{>0})$ is not a cyclic subgroup of $\mathbb{R}$.

Step 4: *Finish the proof.* Let $x = [(A, \lambda)] \in |\mathcal{A}_g(\mathbb{R})|$, and let $\tau \in \mathscr{T}(g)$ be the type of the principally polarized real abelian variety (A, λ). Lift x to a point $y \in \mathrm{H}_g$. By Step 3, we know that the orbit $\mathrm{GL}_g^\tau(R) \cdot y$ is dense in $\mathrm{GL}_g(\mathbb{R}) \cdot y = \mathrm{H}_g$. Consequently, the image of $\mathrm{GL}_g^\tau(R) \cdot y$ under the projection $\mathrm{H}_g \to |\mathcal{A}_g(\mathbb{R})|^\tau$ is dense in $|\mathcal{A}_g(\mathbb{R})|^\tau$. By Step 2, this image is precisely $\mathcal{G}_{p,q}(x)$. Thus $\mathcal{G}_{p,q}(x)$ is dense in $|\mathcal{A}_g(\mathbb{R})|^\tau$ as desired.

$\square$

8.4 PRINCIPAL POLARIZATIONS IN REAL ISOGENY CLASSES

The goal of this section is to prove the following:

Theorem 8.18. *Let (A, λ_A) be a polarized abelian variety over $\mathbb{R}$. Then there exists a principally polarized abelian variety (B, λ_B) over $\mathbb{R}$ and an isogeny*

$$\phi \colon A \to B \quad \text{such that} \quad \phi^*(\lambda_B) = \lambda_A.$$

Proof. Let $K \subset A(\mathbb{C})$ be the kernel of the analytified polarization $\lambda_A \colon A(\mathbb{C}) \to \widehat{A}(\mathbb{C})$. Then K is a finite group of order d^2, where d^2 is the degree of λ_A, such that the real structure $\sigma \colon A(\mathbb{C}) \to A(\mathbb{C})$ restricts to an involution

$$\sigma \colon K \to K.$$

We may assume that $K \neq (0)$. Let p be any prime number that divides the order of K. We claim that there exists a subgroup $K_1 \subset K$ of order p such that $\sigma(K_1) = K_1$. To see this, let $H[p] \subset K$ be the p-torsion subgroup of K. Then $H[p]$ is preserved by σ, so that $H[p]$ is an $\mathbb{F}_p$-vector space of finite rank equipped with a linear involution σ. Therefore, $H[p]$ contains a one-dimensional $\mathbb{F}_p$-subspace K_1 preserved by σ, which proves our claim.

The group $K_1 \subset A(\mathbb{C})$ descends to a finite subgroup scheme $K_1 \subset A$ over $\mathbb{R}$; define A_1 to be the abelian variety A/K_1 over $\mathbb{R}$. Let $\Lambda = \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$ and $M = \mathrm{H}_1(A_1(\mathbb{C}), \mathbb{Z})$; the projection $A \to A_1$ induces an exact sequence

$$0 \to \Lambda \to M \to K_1 \to 0.$$

Let $E\colon \Lambda \times \Lambda \to \mathbb{Z}$ be the alternating form attached to the polarization λ_A of A. Since $M/\Lambda = K_1 \subset K = \Lambda^\vee/\Lambda$, we have inclusions

$$\Lambda \subset M \subset \Lambda^\vee, \quad \text{where} \quad \Lambda^\vee = \{x \in \Lambda \otimes \mathbb{Q} \mid E(x,\Lambda) \subset \Lambda\}.$$

Now the $\mathbb{Z}$-valued alternating form E on the lattice Λ gives rise to a bilinear form

$$\overline{E}\colon \Lambda^\vee/\Lambda \times \Lambda^\vee/\Lambda \to \mathbb{Q}/\mathbb{Z},$$

which vanishes on M/Λ because $M/\Lambda \cong \mathbb{Z}/p$ and $\overline{E}$ is alternating. This means that $E\colon \Lambda^\vee \times \Lambda^\vee \to \mathbb{Q}$ restricts to an *integer*-valued form E_1 on M. The latter induces a polarization $\lambda_{A_1} : A_1 \to \widehat{A}_1$ that makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\lambda_A"] & A_1 \arrow[d, "\lambda_{A_1}"] \\ \widehat{A} & \widehat{A}_1. \arrow[l, "\widehat{\pi}"'] \end{tikzcd}$$

Here π is the quotient map $A \to A_1$ and $\widehat{\pi}$ its dual. Since the degree of an isogeny is multiplicative in compositions, and $\deg(\pi) = p$, we have

$$p^2 \cdot \deg(\lambda_{A_1}) = \deg(\pi)^2 \cdot \deg(\lambda_{A_1}) = \deg(\lambda_A) = d^2.$$

If $d = p$, we are finished – otherwise, we repeat the above procedure until the real abelian variety $A_n = A_{n-1}/K_{n-1}$ becomes principally polarized. □

8.5 ALGEBRAIC CYCLES ON REAL ABELIAN VARIETIES

8.5.1 *The Fourier transform*

Let A be a real abelian variety of dimension g, and consider the Poincaré bundle $\mathcal{P}_A$ on $A \times \widehat{A}$. Let $a_1, \ldots, a_{2g}$ be integers. The Chern character

$$\mathrm{ch}(\mathcal{P}_{A_\mathbb{C}}) = \exp(c_1(\mathcal{P}_{A_\mathbb{C}})) \in \mathrm{H}^{2\bullet}(A(\mathbb{C}) \times \widehat{A}(\mathbb{C}), \mathbb{Z}(\bullet))$$

defines the *Fourier transform*

$$\mathscr{F}_A\colon \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathrm{H}^i(A(\mathbb{C}),\mathbb{Z}(a_i)) \to \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathrm{H}^i(\widehat{A}(\mathbb{C}),\mathbb{Z}(a_{2g-i}-g+i)) \tag{8.14}$$

It is defined as $\mathscr{F}_A(x) = \pi_{2,*}(\mathrm{ch}(\mathcal{P}_A)\cdot\pi_1^*(x))$, where π_i is the projection of $A\times\widehat{A}$ onto the i-th factor. By [Bea82], the map (8.14) is an isomorphism, inducing isomorphisms

$$\mathscr{F}_A\colon \mathrm{H}^i(A(\mathbb{C}),\mathbb{Z}(a_i)) \xrightarrow{\sim} \mathrm{H}^{2g-i}(\widehat{A}(\mathbb{C}),\mathbb{Z}(a_i+g-i)). \tag{8.15}$$

Since $\mathrm{ch}(\mathcal{P}_{A_\mathbb{C}})$ is fixed by G, these maps are isomorphisms of G-modules.

8.5.2 *Divisors*

Let A be an abelian variety over $\mathbb{R}$. Both homomorphisms in the following composition are surjective:

$$\mathrm{CH}^1(A) \to \mathrm{Hdg}^2_G(A(\mathbb{C}),\mathbb{Z}(1)) \to \mathrm{Hdg}^2(A(\mathbb{C}),\mathbb{Z}(1))^G. \tag{8.16}$$

Indeed, the first map is surjective by the real integral Hodge conjecture for divisors (see [BW20a, Proposition 2.8]), and the second by the degeneration of the Hochschild-Serre spectral sequence (see Section 8.2.2).

8.5.3 *The real integral Hodge conjecture for one-cycles modulo torsion*

The goal of this section is to provide an application of Theorems 8.5 and 8.18 combined: the real integral Hodge conjecture for one-cycles modulo torsion follows, in some cases, from the real integral Hodge conjecture for divisors modulo torsion (see Section 8.5.2). The proof uses Fourier transforms for real abelian varieties (see Section 8.5.1) in a way similar to the way in which we used Fourier transforms for complex abelian varieties in Section 7.2.1. Thus, we will need the results on integral Fourier transforms obtained in Chapter 6. The theory in Chapter 6 was developed for abelian varieties over a general field k – to apply it, we take $k=\mathbb{R}$.

We will need the following:

Definition 8.19. Let A be a real abelian variety, and k a non-negative integer. An element $\alpha \in \mathrm{Hdg}^{2k}(A(\mathbb{C}), \mathbb{Z}(k))^G$ is called *algebraic* if it is in the image of

$$\mathrm{CH}^k(A) \to \mathrm{Hdg}^{2k}(A(\mathbb{C}), \mathbb{Z}(k))^G.$$

Recall that by the main theorem of Chapter 7, the integral Hodge conjecture for one-cycles on a fixed principally polarized abelian variety (A, θ) over $\mathbb{C}$ is equivalent to the algebraicity of the minimal class $\gamma_\theta = \theta^{g-1}/(g-1)!$ on A (see Theorem 7.1). On the other hand, Grabowski reduced the integral Hodge conjecture for one-cycles for every complex abelian variety of dimension g to the algebraicity of γ_θ for every principally polarized abelian variety of dimension g [Gra04].

We have the following real analogue of these results:

Theorem 8.20. *Fix a positive integer g. Let (A, θ) be a principally polarized abelian variety of dimension g over $\mathbb{R}$. The following are equivalent:*

1. *The real abelian variety A satisfies the real integral Hodge conjecture for one-cycles modulo torsion.*
2. *The minimal class*
$$\gamma_\theta = \frac{\theta^{g-1}}{(g-1)!} \in \mathrm{Hdg}^{2g-2}(A(\mathbb{C}), \mathbb{Z}(g-1))^G \quad \textit{is algebraic.}$$
3. *The Chern character*
$$\mathrm{ch}(\mathcal{P}_{A_\mathbb{C}}) = \exp(c_1(\mathcal{P}_{A_\mathbb{C}})) \in \mathrm{Hdg}^{2\bullet}(A(\mathbb{C}) \times \widehat{A}(\mathbb{C}), \mathbb{Z}(\bullet))^G \quad \textit{is algebraic.}$$

Moreover, if the real integral Hodge conjecture for one-cycles modulo torsion holds for every principally polarized abelian variety of dimension g over $\mathbb{R}$, then it holds for every abelian variety of dimension g over $\mathbb{R}$.

Proof. The direction (1) $\implies$ (2) is trivial; let us assume that (2) holds. Let (A, θ) be a principally polarized abelian variety of dimension g over $\mathbb{R}$, and suppose that $\gamma_\theta \in \mathrm{Hdg}^{2g-2}(A(\mathbb{C}), \mathbb{Z}(g-1))^G$ is algebraic. By the proof of Proposition 6.12, the abelian variety A admits a motivic integral Fourier transform up to homology, see

Definition 6.2. This means the following. Let ℓ be a prime number. There exists a cycle

$$\Gamma \in \mathrm{CH}(A \times \widehat{A}) \quad \text{such that} \quad [\Gamma_{\mathbb{C}}] = \mathrm{ch}(\mathcal{P}_{A_{\mathbb{C}}}) \quad \in \quad \mathrm{H}^{2\bullet}_{\text{ét}}(A_{\mathbb{C}} \times \widehat{A}_{\mathbb{C}}, \mathbb{Z}_\ell(\bullet)).$$

As a consequence, $[\Gamma_{\mathbb{C}}] = \mathrm{ch}(\mathcal{P}_{A_{\mathbb{C}}}) \in \mathrm{H}^{2\bullet}(A(\mathbb{C}) \times \widehat{A}(\mathbb{C}), \mathbb{Z}(\bullet))^G$, which proves (3).

Let us now assume that (3) holds, and let $\Gamma \in \mathrm{CH}(A \times \widehat{A})$ be a cycle that induces $\mathrm{ch}(\mathcal{P}_{A_{\mathbb{C}}})$ in Betti cohomology. The correspondence Γ defines a group homomorphism $\Gamma_* : \mathrm{CH}^\bullet(A) \to \mathrm{CH}^\bullet(\widehat{A})$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{CH}^1(A) \arrow[r] \arrow[d] & \mathrm{CH}^\bullet(A) \arrow[r, "\Gamma_*"] \arrow[d] & \mathrm{CH}^\bullet(\widehat{A}) \arrow[r] \arrow[d] & \mathrm{CH}_1(\widehat{A}) \arrow[d] \\
\mathrm{Hdg}^2(A(\mathbb{C}), \mathbb{Z}(1))^G \arrow[r] & \mathrm{Hdg}^{2\bullet}(A(\mathbb{C}), \mathbb{Z}(\bullet))^G \arrow[r, "\mathscr{F}_A", "\sim"'] & \mathrm{Hdg}^{2\bullet}(\widehat{A}(\mathbb{C}), \mathbb{Z}(\bullet))^G \arrow[r] & \mathrm{Hdg}^{2k}(\widehat{A}(\mathbb{C}), \mathbb{Z}(k))^G.
\end{tikzcd}$$

Here $k = g - 1$, the composition on the bottom row is an isomorphism (see (8.15) in Section 8.5.1), and the left vertical arrow is surjective (see (8.16) in Section 8.5.2). Therefore, the right vertical arrow is surjective, which implies (1).

Next, suppose that (1) holds for every principally polarized real abelian variety of dimension g, and let A be any real abelian variety of dimension g. We would like to show that A satisfies the real integral Hodge conjecture for one-cycles modulo torsion. The isomorphism (8.15) induces an isomorphism

$$\mathscr{F}_A : \mathrm{Hdg}^2(A(\mathbb{C}), \mathbb{Z}(1))^G \xrightarrow{\sim} \mathrm{Hdg}^{2g-2}(\widehat{A}(\mathbb{C}), \mathbb{Z}(g-1))^G.$$

Since $\mathrm{Hdg}^2(A(\mathbb{C}), \mathbb{Z}(1))^G$ is algebraic by Section 8.5.2, it suffices to show that $\mathscr{F}_A([L_{\mathbb{C}}])$ is algebraic for every line bundle L on A. By [Har77, II, Exercice 7.5], there is an ample line bundle M on A such that $L \otimes M^{\otimes n}$ is ample for $n \gg 0$; we may thus assume that L is ample. By Theorem 8.18, there is a principally polarized abelian variety (B, λ), and an isogeny

$$\phi \colon A \to B$$

such that, if $\theta \in \mathrm{NS}(B_{\mathbb{C}})^G = \mathrm{Hdg}^2(B(\mathbb{C}), \mathbb{Z}(1))^G$ is the class corresponding to the principal polarization $\lambda \colon B \to \widehat{B}$, then

$$\phi^*(\theta) = [L_{\mathbb{C}}] \in \mathrm{Hdg}^2(A(\mathbb{C}), \mathbb{Z}(1))^G.$$

On the other hand, the following diagram commutes by [Bea82, Proposition 3]:

$$\begin{tikzcd}
\mathrm{Hdg}^2(A(\mathbb{C}),\mathbb{Z}(1))^G \arrow[r, "\mathscr{F}_A"] & \mathrm{Hdg}^{2g-2}(\widehat{A}(\mathbb{C}),\mathbb{Z}(g-1))^G \\
\mathrm{Hdg}^2(B(\mathbb{C}),\mathbb{Z}(1))^G \arrow[r, "\mathscr{F}_B"] \arrow[u, "\phi^*"] & \mathrm{Hdg}^{2g-2}(\widehat{B}(\mathbb{C}),\mathbb{Z}(g-1))^G. \arrow[u, "\hat{\phi}_*"']
\end{tikzcd}$$

Moreover, by [Bea82, Proposition 5], we have

$$\mathscr{F}_B(\theta) = (-1)^{g-1} \cdot \frac{\hat{\theta}^{g-1}}{(g-1)!} \in \mathrm{H}^{2g-2}(\widehat{B}(\mathbb{C}),\mathbb{Z}(g-1))^G,$$

where $\hat{\theta} \in \mathrm{Hdg}^2(\widehat{B}(\mathbb{C}),\mathbb{Z}(1))^G$ denotes the dual polarization class. We conclude that $F_B(\theta)$ is algebraic, so that $\mathscr{F}_A([L_{\mathbb{C}}]) = \widehat{\phi}_*(\mathscr{F}_B(\theta))$ is algebraic as well. □

Corollary 8.21. *Let $C_1, \ldots, C_n$ be smooth projective geometrically integral curves over $\mathbb{R}$ such that $C_i(\mathbb{R}) \neq \emptyset$ for each i. The real abelian variety $A = J(C_1) \times \cdots \times J(C_n)$ satisfies the real integral Hodge conjecture for one-cycles modulo torsion.*

Proof. The minimal class on a product of principally polarized abelian varieties over $\mathbb{R}$ is the sum of the pull-backs of the minimal classes on the factors, so by Theorem 8.20, it suffices to treat the case $n = 1$. For a real algebraic curve C whose real locus is non-empty, any Abel-Jacobi map gives an embedding of real varieties $\iota\colon C \hookrightarrow J(C)$. By Poincaré's formula, one has

$$[\iota(C)_{\mathbb{C}}] = \frac{\theta^{g-1}}{(g-1)!} \in \mathrm{Hdg}^{2g-2}(J(C)(\mathbb{C}),\mathbb{Z}(g-1))^G,$$

where the class on the right hand side of the equality is the minimal cohomology class γ_θ of $J(C)$. Thus γ_θ is algebraic, so we are done by Theorem 8.20. □

8.5.4 *Integral Hodge classes modulo torsion on real abelian threefolds*

Recall (see Definition 2.13) that a real algebraic curve is a smooth projective geometrically connected curve over $\mathbb{R}$. Let $\mathcal{M}_3(\mathbb{R})$ be the moduli space of real algebraic curves of genus three, and consider the Torelli map

$$t\colon \mathcal{M}_3(\mathbb{R}) \to \mathcal{A}_3(\mathbb{R}).$$

Let $\mathcal{N}_3(\mathbb{R}) \subset \mathcal{M}_3(\mathbb{R})$ be the non-hyperelliptic locus. Let us first consider:

Claim: The subset $t(\mathcal{N}_3(\mathbb{R}))$ is open in $\mathcal{A}_3(\mathbb{R})$.

Indeed, on the level of stacks, $\mathcal{T}\colon \mathcal{M}_3 \to \mathcal{A}_3$ is an open immersion when restricted to the non-hyperelliptic locus $\mathcal{N}_3 \subset \mathcal{M}_3$. Thus $\mathcal{T}(\mathcal{N}_3) \subset \mathcal{A}_3$ is an open substack. Recall that, for any algebraic stack $\mathscr{X}$ of finite type over $\mathbb{R}$, the set $|\mathscr{X}(\mathbb{R})|$ of isomorphism classes of $\mathbb{R}$-points of $\mathscr{X}$ admits a topology, called the real-analytic topology (see Definition 2.7). In this topology, the subset

$|\mathcal{T}(\mathcal{N}_3)(\mathbb{R})| \subset |\mathcal{A}_3(\mathbb{R})|$ is indeed open: this follows from Corollary 2.8.2.

Lemma 8.22. *Every connected component of the moduli space $\mathcal{A}_3(\mathbb{R})$ contains a non-empty open subset of non-hyperelliptic curves of genus three with non-empty real locus.*

Proof. Let $\mathcal{A}_3(\mathbb{R})^\tau$ be a connected component of $\mathcal{A}_3(\mathbb{R})$. By [GH81, page 182], there is a (unique) connected component $\mathcal{M}_3(\mathbb{R})^\tau$ of $\mathcal{M}_3(\mathbb{R})$ that satisfies the following two conditions: $C(\mathbb{R})$ is not empty for each $[C] \in \mathcal{M}_3(\mathbb{R})^\tau$, and

$$t\left(\mathcal{M}_3(\mathbb{R})^\tau\right) \subset \mathcal{A}_3(\mathbb{R})^\tau.$$

Now $\mathcal{M}_3(\mathbb{R})^\tau$ contains a component $\mathcal{N}_3(\mathbb{R})^\tau$ of $\mathcal{N}_3(\mathbb{R})$ by [GH81, Proposition 3.1] and the table on page 174 of *loc.cit.*, and $\mathcal{N}_3(\mathbb{R})^\tau$ is open in $\mathcal{A}_3(\mathbb{R})$ by the above. □

Proof of Corollary 8.6. This follows directly from Theorem 8.5 and Lemma 8.22. □

We are now in the position to prove Theorem 8.3.

Proof of Theorem 8.3. By Theorem 8.20, it suffices to show that for any principally polarized abelian threefold (A, θ) over $\mathbb{R}$, the class $\gamma_\theta = \theta^2/2 \in \mathrm{Hdg}^4(A(\mathbb{C}), \mathbb{Z}(2))^G$ is algebraic. Let us prove this. Let p and q be two distinct odd prime numbers. By Corollary 8.6, there exists a real algebraic curve C of genus three such that $C(\mathbb{R}) \neq \emptyset$ together with an isogeny

$$\phi\colon A \to J(C)$$

such that $\phi^*(\lambda_C) = p^n q^m \cdot \lambda_A$ for some non-negative integers n and m. Here λ_C denotes the canonical polarization on $J(C)$. Let $\theta_C \in \mathrm{Hdg}^2(J(C)(\mathbb{C}), \mathbb{Z}(1))^G$ be the corresponding cohomology class; then $\phi^*(\theta_C) = p^n q^m \cdot \theta$. Since $C(\mathbb{R}) \neq \emptyset$, the minimal class $\theta_C^2/2$ on $J(C)$ is algebraic; see Corollary 8.21. Therefore, the class

$$p^{2n}q^{2m} \cdot \theta^2/2 = \phi^*(\theta_C^2/2) \in \mathrm{Hdg}^4(A(\mathbb{C}), \mathbb{Z}(2))^G$$

is algebraic. We are done because θ^2 is algebraic and the integer $p^n q^m$ is odd. □

Proof of Corollary 8.4. For each $i \in \{1, \dots, 6\}$, cup-product defines a canonical isomorphism of G-modules:

$$\bigwedge^i \mathrm{H}^1(A(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^i(A(\mathbb{C}), \mathbb{Z}).$$

This allows us to calculate the G-module structure on $\mathrm{H}^i(A(\mathbb{C}), \mathbb{Z})$, for we know that $\mathrm{H}^1(A(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}[G]^3$ as G-modules [Ham97, Example 3.1]. It turns out that the group $\mathrm{H}^p(G, \mathrm{H}^q(A(\mathbb{C}), \mathbb{Z}))$ vanishes whenever $p + q = 4$ and $p > 0$. Therefore,

$$\mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0 = \mathrm{Hdg}^4(A(\mathbb{C}), \mathbb{Z}(2))^G$$

by Section 8.2.2 and Lemma 8.12. By Theorem 8.3, we are done. □

8.5.5 *Isogenies and torsion cohomology classes*

Let X be a smooth projective variety over $\mathbb{R}$, and let k be any non-negative integer.

Definition 8.23. 1. Let us call a class $\alpha \in \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$ *topologically distinguished* if $\alpha \in \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0$. This means that α is topologically distinguished if and only if α satisfies the topological condition (8.4).

2. For any subgroup $K \subset \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$, define

$$K_0 = K \cap \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0.$$

3. In case the Hochschild-Serre spectral sequence (8.5) degenerates, we define, for $p > 0$,

$$\begin{aligned}&\mathrm{H}^p(G, \mathrm{H}^{2k-p}(X(\mathbb{C}), \mathbb{Z}(k)))_0 \\ &= \mathrm{Im}(F^p\mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0 \to \mathrm{H}^p(G, \mathrm{H}^{2k-p}(X(\mathbb{C}), \mathbb{Z}(k)))).\end{aligned}$$

4. Let us call a subgroup $K \subset \mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$ *algebraic* if every element of K is in the image of the cycle class map (8.1). If the Hochschild-Serre spectral sequence (8.5) degenerates, we call a subgroup $L \subset \mathrm{H}^p(G, \mathrm{H}^{2k-p}(X(\mathbb{C}), \mathbb{Z}(k)))$ *algebraic* if L is the image of an algebraic subgroup $K \subset F^p\mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$ under the canonical map $F^p\mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k)) \to \mathrm{H}^p(G, \mathrm{H}^{2k-p}(X(\mathbb{C}), \mathbb{Z}(k)))$.

The following lemma provides the key to our proof of Theorem 8.7.

Lemma 8.24. *Let A and B be real abelian varieties, and let ϕ be an isogeny $A \to B$ of odd degree. For each pair of non-negative integers (k, i), the homomorphism*

$$\phi^*\colon \mathrm{H}^k_G(B(\mathbb{C}), \mathbb{Z}(i))[2] \to \mathrm{H}^k_G(A(\mathbb{C}), \mathbb{Z}(i))[2]$$

is an isomorphism. If k is even and $i = k/2$, then ϕ^* restricts to an isomorphism between the subgroups of topologically distinguished classes.

Proof. Let us first prove that for an odd integer n, possibly negative, the homomorphisms

$$[n]^*\colon \mathrm{H}^k_G(A(\mathbb{C}), \mathbb{Z}(i))[2] \to \mathrm{H}^k_G(A(\mathbb{C}), \mathbb{Z}(i))[2] \qquad \text{and}$$
$$[n]^*\colon \mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))_0[2] \to \mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))_0[2]$$

are isomorphisms. Write $\mathrm{H} = \mathrm{H}^k_G(A(\mathbb{C}), \mathbb{Z}(i))$. The Hochschild-Serre spectral sequence (8.5) degenerates, so $F^1\mathrm{H} = \mathrm{H}[2]$ and for each $p \in \{0, \ldots, k\}$ there is an exact sequence

$$0 \to F^{p+1}\mathrm{H} \to F^p\mathrm{H} \to \mathrm{H}^p(G, \mathrm{H}^{k-p}(A(\mathbb{C}), \mathbb{Z}(i))) \to 0.$$

We have $F^{k+1}\mathrm{H} = (0)$, and for each $p \in \{0, \ldots, k\}$, the morphism

$$[n]^*\colon \mathrm{H}^{k-p}(A(\mathbb{C}), \mathbb{Z}(i)) \to \mathrm{H}^{k-p}(A(\mathbb{C}), \mathbb{Z}(i))$$

induces the identity on $\mathrm{H}^p(G, \mathrm{H}^{k-p}(A(\mathbb{C}), \mathbb{Z}(i)))$. By the snake lemma and descending induction on p, we find that $[n]^*$ restricts to an isomorphism on $F^p\mathrm{H}$ for each $p > 0$, and in particular on $F^1\mathrm{H} = \mathrm{H}[2]$.

Write $\mathrm{M} = \mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))$. By [BW20a, §1.6.4], the class $[n]^*(\alpha)$ is topologically distinguished whenever $\alpha \in \mathrm{M}$ is topologically distinguished, thus $[n]^*$ induces an embedding $\mathrm{M}_0[2] \to \mathrm{M}_0[2]$, which is an isomorphism since $\mathrm{M}_0[2]$ is finite.

Next, let $\psi\colon B \to A$ be an isogeny such that $\psi \circ \phi = [d]_A$ and $\phi \circ \psi = [d]_B$, where d is the degree of the isogeny ϕ. The resulting equalities

$$\psi^* \circ \phi^* = [d]_A^* \quad \text{and} \quad \psi^* \circ \phi^* = [d]_B^*$$

together the fact $[d]_A^*$ and $[d]_B^*$ are isomorphisms that identify the subgroups of topologically distinguished classes if k is even and $i = k/2$, imply that

$$\begin{aligned} &\phi^*\colon \mathrm{H}^k_G(B(\mathbb{C}), \mathbb{Z}(i))[2] \to \mathrm{H}^k_G(A(\mathbb{C}), \mathbb{Z}(i))[2] \qquad \text{and} \\ &\phi^*\colon \mathrm{H}^{2k}_G(B(\mathbb{C}), \mathbb{Z}(k))_0[2] \to \mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))_0[2] \end{aligned}$$

are isomorphisms as well. □

Corollary 8.25. *Let A and B be real abelian varieties, and ϕ an isogeny $A \to B$ of odd degree. For any pair of integers (k, p) with $p > 0$, the pull-back ϕ^* defines an isomorphism*

$$F^p\mathrm{H}^{2k}_G(B(\mathbb{C}), \mathbb{Z}(k))_0 \xrightarrow{\sim} F^p\mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k))_0. \tag{8.17}$$

In particular, the left-hand side of (8.17) is spanned by algebraic classes if and only if the right-hand side of (8.17) is spanned by algebraic classes. □

8.5.6 *Reduction to the Jacobian case*

Proof of Theorem 8.7. Let $\mathcal{A}_3(\mathbb{R})^+$ be a connected component of $\mathcal{A}_3(\mathbb{R})$, and let $x = [(A, \lambda)] \in \mathcal{A}_3(\mathbb{R})^+$. By Corollary 8.6, there is a real algebraic curve C with non-empty real locus together with an isogeny

$$\phi\colon A \to J(C)$$

that preserves the polarizations up to an odd positive integer. By [GH81, page 180], we have $[J(C)] \in \mathcal{A}_3(\mathbb{R})^+$ for the isomorphism class $[J(C)]$ of the Jacobian $J(C)$ of C. The following sequence is exact (see Lemma 8.12):

$$0 \to \mathrm{H}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0[2] \to \mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0 \to \mathrm{Hdg}^4(A(\mathbb{C}), \mathbb{Z}(2))^G \to 0.$$

By Theorem 8.3, we know that $\mathrm{Hdg}^4(A(\mathbb{C}), \mathbb{Z}(2))^G$ is generated by classes of one-cycles on A. Therefore, A satisfies the real integral Hodge conjecture if and only if the group $\mathrm{H}^4_G(A(\mathbb{C}),\mathbb{Z}(2))_0[2]$ is algebraic. By Corollary 8.25, this is equivalent to the algebraicity of $\mathrm{H}^4_G(J(C)(\mathbb{C}),\mathbb{Z}(2))_0[2]$. $\square$

8.5.7 *Analysis of the Hochschild-Serre filtration*

Let A be an abelian variety over $\mathbb{R}$, define $\mathrm{H} = \mathrm{H}^4_G(A(\mathbb{C}),\mathbb{Z}(2))$, and consider the Hochschild-Serre filtration (c.f. Section 8.2.2):

$$0 \subset \mathrm{H}^4(G,\mathrm{H}^0(A(\mathbb{C}),\mathbb{Z}(2))) = F^4\mathrm{H} \subset F^3\mathrm{H} \subset F^2\mathrm{H} \subset F^1\mathrm{H} = \mathrm{H}[2] \subset \mathrm{H}. \tag{8.18}$$

The pull-back π^* along the structural morphism $\pi\colon A \to \mathrm{Spec}(\mathbb{R})$ defines a section

$$\mathbb{Z}/2 = \mathrm{H}^4_G(\{x\},\mathbb{Z}(2)) = \mathrm{H}^4(G,\mathrm{H}^0(\{x\},\mathbb{Z}(2))) \to \mathrm{H}^4(G,\mathrm{H}^0(A(\mathbb{C}),\mathbb{Z}(2))) \subset \mathrm{H}$$

of the restriction $\mathrm{H} \to \mathrm{H}^4_G(\{x\},\mathbb{Z}(2)) = \mathbb{Z}/2$ to the equivariant cohomology of any $\mathbb{R}$-point $x \in X(\mathbb{R})$. By Section 8.2.4, this implies that $F^4\mathrm{H}_0 = (0)$, so that the intersection of (8.18) with the group of topologically distinguished classes becomes

$$0 \subset F^3\mathrm{H}_0 \subset F^2\mathrm{H}_0 \subset F^1\mathrm{H}_0 \subset \mathrm{H}_0. \tag{8.19}$$

Continue to consider our abelian variety A over $\mathbb{R}$. *Assume that there exists a cycle*

$$\Gamma \in \mathrm{CH}(A \times \widehat{A}) \quad \textit{such that} \quad [\Gamma_{\mathbb{C}}] = \mathrm{ch}(\mathcal{P}_{A_{\mathbb{C}}}) \in \mathrm{H}^{2\bullet}(A(\mathbb{C}) \times \widehat{A}(\mathbb{C}),\mathbb{Z}(\bullet))^G.$$

Under this assumtion, we may define a homomorphism Γ_* as in the following diagram:

$$\begin{tikzcd} \mathrm{H}^{2\bullet}_G(A(\mathbb{C}),\mathbb{Z}(\bullet)) \arrow[r, "\pi_1^*"] \arrow[d, "\Gamma_*"] & \mathrm{H}^{2\bullet}_G(A(\mathbb{C}) \times \widehat{A}(\mathbb{C}),\mathbb{Z}(\bullet)) \arrow[d, "([\Gamma]\cdot -)"] \\ \mathrm{H}^{2\bullet}_G(\widehat{A}(\mathbb{C}),\mathbb{Z}(\bullet)) & \mathrm{H}^{2\bullet}_G(A(\mathbb{C}) \times \widehat{A}(\mathbb{C}),\mathbb{Z}(\bullet)). \arrow[l, "\pi_{2,*}"] \end{tikzcd} \tag{8.20}$$

Then Γ_* preserves the topologically distinguished classes by [BW20a, Theorem 1.21]. Define

$$\Gamma^{i,j}_*\colon \mathrm{H}^{2i}_G(A(\mathbb{C}),\mathbb{Z}(i)) \to \mathrm{H}^{2j}_G(\widehat{A}(\mathbb{C}),\mathbb{Z}(j)) \quad (i,j \in \mathbb{Z}_{\geq 0}) \tag{8.21}$$

as the composition of Γ_* with the natural inclusion and projection morphisms.

We are now in the position to prove:

Proposition 8.26. *Let A be an abelian variety over $\mathbb{R}$. Then*

$$F^3\mathrm{H}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0 = (0).$$

Proof. Let $g = \dim(A)$. By [Ham97, Chapter IV, Example 3.1], we have that $A(\mathbb{C})$ is G-equivariantly homeomorphic to the g-fold product of copies of the torus $S^1 \times S^1$, where $S^1 \subset \mathbb{C}$ is the unit circle and where G acts on each copy $S^1 \times S^1$ in one of the following ways: either $\sigma(x, y) = (x, \bar{y})$ for $(x, y) \in S^1 \times S^1$, or $\sigma(x, y) = (y, x)$ for $(x, y) \in S^1 \times S^1$. In particular, there exist g elliptic curves E_i over $\mathbb{R}$ and a G-equivariant homeomorphism

$$A(\mathbb{C}) \cong E_1(\mathbb{C}) \times \cdots \times E_g(\mathbb{C}). \tag{8.22}$$

Since the conclusion of Proposition 8.26 is a statement that only depends on the structure of $A(\mathbb{C})$ as a topological G-space, we may therefore assume that A is principally polarized by $\theta \in \mathrm{Hdg}^2(A(\mathbb{C}), \mathbb{Z}(1))^G$ and the following class is algebraic:

$$\frac{\theta^{g-1}}{(g-1)!} \in \mathrm{Hdg}^{2g-2}(A(\mathbb{C}), \mathbb{Z}(g-1))^G.$$

By Theorem 8.20, this means that the Chern character $\mathrm{ch}(\mathcal{P}_{A_\mathbb{C}})$ is algebraic, which allows us to define a homomorphism $\Gamma_* \colon \mathrm{H}^{2\bullet}_G(A(\mathbb{C}), \mathbb{Z}(\bullet)) \to \mathrm{H}^{2\bullet}_G(\widehat{A}(\mathbb{C}), \mathbb{Z}(\bullet))$ as in (8.20), and homomorphisms Γ_*^{ij} as in (8.21). We have a commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^4_G(A(\mathbb{C}), \mathbb{Z}(2)) \arrow[r, "\Gamma_*^{4,2g+2}"] & \mathrm{H}^{2g+2}_G(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+1)) \arrow[r, "\Gamma_*^{2g+2,4}"] & \mathrm{H}^4_G(A(\mathbb{C}), \mathbb{Z}(2)) \\
F^3\mathrm{H}^4_G(A(\mathbb{C}), \mathbb{Z}(2)) \arrow[u, hook] \arrow[r, "\Gamma_*^{4,2g+2}"] \arrow[d] & F^3\mathrm{H}^{2g+2}_G(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+1)) \arrow[u, hook] \arrow[r, "\Gamma_*^{2g+2,4}"] \arrow[d] & F^3\mathrm{H}^4_G(A(\mathbb{C}), \mathbb{Z}(2)) \arrow[u, hook] \arrow[d] \\
\mathrm{H}^3(G, \mathrm{H}^1(A(\mathbb{C}), \mathbb{Z}(2))) \arrow[r, "\mathrm{H}^3(G, \mathscr{F}_A)"] & \mathrm{H}^3(G, \mathrm{H}^{2g-1}(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+1))) \arrow[r, "\mathrm{H}^3(G, \mathscr{F}_{\widehat{A}})"] & \mathrm{H}^3(G, \mathrm{H}^1(A(\mathbb{C}), \mathbb{Z}(2))).
\end{tikzcd}$$

Here $\mathscr{F}_A \colon \mathrm{H}^1(A(\mathbb{C}), \mathbb{Z}(2)) \to \mathrm{H}^{2g-1}(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+1))$ is the Fourier transform considered in Section 8.5.1. This map is an isomorphism, with inverse

$$(-1)^{1+g} \cdot \mathscr{F}_{\widehat{A}} \colon \mathrm{H}^{2g-1}(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+1)) \to \mathrm{H}^1(A(\mathbb{C}), \mathbb{Z}(2)),$$

see [Huy06, Corollary 9.24]. Consequently, $\mathrm{H}^3(G, \mathscr{F}_A)$ is an isomorphism, with inverse $\mathrm{H}^3(G, \mathscr{F}_{\widehat{A}})$. By the compatibility of Γ_* with the topological condition (8.4), we obtain an isomorphism

$$\mathrm{H}^3(G, \mathscr{F}_A) \colon \mathrm{H}^3(G, \mathrm{H}^1(A(\mathbb{C}), \mathbb{Z}(2)))_0 \xrightarrow{\sim} \mathrm{H}^3(G, \mathrm{H}^{2g-1}(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+1)))_0. \quad (8.23)$$

Because $\mathrm{H}^{2g+2}_G(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+1))_0 = (0)$ by [BW20a, §2.3.2], the group on the right of equation (8.23) vanishes. Therefore, the group on the left must be zero as well. □

In fact, the argument used in the above proposition can be generalized to prove the following lemma, which we record for later use:

Lemma 8.27. *Let A be an abelian variety of dimension g over $\mathbb{R}$. Let p and q be non-negative integers such that $p + q = 2k \in 2\mathbb{Z}_{\geq 0}$. The isomorphism on group cohomology*

$$\mathrm{H}^p(G, \mathscr{F}_A) \colon \mathrm{H}^p(G, \mathrm{H}^q(A(\mathbb{C}), \mathbb{Z}(k))) \to \mathrm{H}^p(G, \mathrm{H}^{2g-q}(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+k-q)))$$

identifies $\mathrm{H}^p(G, \mathrm{H}^q(A(\mathbb{C}), \mathbb{Z}(k)))_0$ with $\mathrm{H}^p(G, \mathrm{H}^{2g-q}(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+k-q)))_0$.

Proof. This is a topological statement, so we may and do assume that A is a product of elliptic curves (see (8.22)). In this case, we may lift $\mathrm{H}^p(G, \mathscr{F}_A)$ to an algebraic homomorphism

$$\Gamma_*^{2k, 2g+2k-2q} \colon F^p\mathrm{H}^{2k}_G(A(\mathbb{C}), \mathbb{Z}(k)) \to F^p\mathrm{H}^{2g+2k-2q}_G(\widehat{A}(\mathbb{C}), \mathbb{Z}(g+k-q))$$

as above (see Section 8.5.7). We can do the same thing for $\mathrm{H}^p(G, \mathscr{F}_{\widehat{A}})$; the lemma follows from arguments similar to those used to prove Proposition 8.26. □

8.5.8 *Codimension-two cycles on real abelian varieties*

Let us now begin the proof of Theorem 8.8. The proof consists of two steps: first we prove it for the Jacobian of a real algebraic curve with non-empty real locus, and then we reduce the general case to this particular case.

Proof of Theorem 8.8 (Jacobian case). Let us prove Theorem 8.8 in the case where A is the Jacobian $J = J(C)$ of a real algebraic curve C of genus $g \in \mathbb{Z}_{\geq 1}$ such that $C(\mathbb{R}) \neq \emptyset$. By Proposition 8.26, we have

$$F^2\mathrm{H}^4_G(J(\mathbb{C}), \mathbb{Z}(2))_0 = \mathrm{H}^2(G, \mathrm{H}^2(J(\mathbb{C}), \mathbb{Z}(2)))_0$$

and it remains to prove that $F^2\mathrm{H}^4_G(J(\mathbb{C}),\mathbb{Z}(2))_0$ is algebraic. By Theorem 8.20, the Chern character of the Poincaré bundle of J in integral Betti cohomology $\mathrm{H}^{2\bullet}(J(\mathbb{C})\times\widehat{J}(\mathbb{C}),\mathbb{Z}(\bullet))^G$ is algebraic. By Section 8.5.7 and [Huy06, Corollary 9.24], we obtain a commutative diagram

$$\begin{tikzcd} F^2\mathrm{H}^4_G(J(\mathbb{C}),\mathbb{Z}(2))_0 \arrow[r, hook, "\Gamma_*^{4,6}"] \arrow[d, equal] & F^2\mathrm{H}^{2g}_G(\widehat{J}(\mathbb{C}),\mathbb{Z}(g))_0 \arrow[r, two heads, "\Gamma_*^{6,4}"] \arrow[d] & F^2\mathrm{H}^4_G(J(\mathbb{C}),\mathbb{Z}(2))_0 \arrow[d, equal] \\ \mathrm{H}^2(G,\mathrm{H}^2(J(\mathbb{C}),\mathbb{Z}(2)))_0 \arrow[r, "\sim"] & \mathrm{H}^2(G,\mathrm{H}^{2g-2}(\widehat{J}(\mathbb{C}),\mathbb{Z}(g)))_0 \arrow[r, "\sim"] & \mathrm{H}^2(G,\mathrm{H}^2(J(\mathbb{C}),\mathbb{Z}(2)))_0 \end{tikzcd} \tag{8.24}$$

such that the composition on the bottom row of (8.24) is the identity. Therefore, the composition on the top row of (8.24) is the identity. Since J satisfies the real integral Hodge conjecture for zero-cycles by [BW20a, Proposition 2.10], the group $F^2\mathrm{H}^{2g}_G(J(\mathbb{C}),\mathbb{Z}(g))_0$ – and hence also the group $F^2\mathrm{H}^4_G(J(\mathbb{C}),\mathbb{Z}(2))_0$ – is algebraic. $\square$

Proof of Theorem 8.8 (general case). To prove that $F^2\mathrm{H}^4_G(A(\mathbb{C}),\mathbb{Z}(2))_0$ is algebraic for a real abelian variety A, we would like to reduce to the Jacobian case. The fact that this can be done rests on the following proposition in combination with Lemma 8.33 below.

Proposition 8.28. *Let $0\to B\to J\xrightarrow{f} A\to 0$ be an exact sequence of abelian varieties over $\mathbb{R}$ such that $f_{\mathbb{R}}\colon J(\mathbb{R})\to A(\mathbb{R})$ is surjective. Let $d=\dim(J)$ and $g=\dim(A)$. Then*

$$f_*\colon F^2\mathrm{H}^{2d}_G(J(C)(\mathbb{C}),\mathbb{Z}(d))_0\to F^2\mathrm{H}^{2g}_G(A(\mathbb{C}),\mathbb{Z}(g))_0 \qquad \textit{and}$$
$$\hat{f}^*\colon F^2\mathrm{H}^4_G(\widehat{J}(C)(\mathbb{C}),\mathbb{Z}(2))_0\to F^2\mathrm{H}^4_G(\widehat{A}(\mathbb{C}),\mathbb{Z}(2))_0$$

are surjective, where $\widehat{f}\colon\widehat{A}\to\widehat{J}$ is the dual homomorphism of $f\colon J\to A$.

To prove Proposition 8.28, we will need three lemmas (Lemmas 8.29, 8.30 and 8.31 below). We will first state and prove these lemmas, and then use them to prove Proposition 8.28. After that, we will state and prove Lemma 8.33, and then use Proposition 8.28 and Lemma 8.33 to prove Theorem 8.8.

Consider a projective variety X, smooth over $\mathbb{R}$. Define $\mathrm{CH}_0(X)_{\mathrm{tors}}$ to be the group of torsion zero-cycles modulo rational equivalence on X, and define $\Lambda_X=\mathrm{H}^{2d-1}(X(\mathbb{C}),\mathbb{Z}(d))/(\text{torsion})$. If X is endowed with a real point $x\in X(\mathbb{R})$, then there is an abelian variety $\mathrm{Alb}(X)$, the Albanese variety of X, equipped with a morphism $u\colon (X, x) \to \mathrm{Alb}(X)$ which is initial in the category of morphisms from (X, x) to abelian varieties over $\mathbb{R}$ [Wit08, Theorem A.1]. We have, as G-modules:

$$\mathrm{H}_1(\mathrm{Alb}(X)(\mathbb{C}), \mathbb{Z}) = \mathrm{H}_1(X(\mathbb{C}), \mathbb{Z})/(\text{torsion}) = \Lambda_X.$$

See [Sil89, Chapter IV, §1] for the former and [Man17, Corollaire 3.1.9] for the latter isomorphism. The map u induces a universal regular homomorphism $\mathrm{CH}_0(X_{\mathbb{C}})_{\mathrm{hom}} \to \mathrm{Alb}(X)(\mathbb{C})$, see [Mur85], which is G-equivariant thus induces a homomorphism $\mathrm{AJ}\colon \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X)(\mathbb{R})$. See also [Ham00].

Lemma 8.29. *Let X be a smooth projective variety of dimension d over $\mathbb{R}$ with $x \in X(\mathbb{R})$. Suppose that the Hochschild-Serre spectral sequence (8.5) degenerates. Define $\mathrm{CH}_0(X)_{\mathrm{tors}}$ and Λ_X as above. Then the following diagram commutes, and its arrows are surjective:*

$$\begin{tikzcd}
\mathrm{CH}_0(X)_{\mathrm{tors}} \arrow[rr, two heads] \arrow[d, two heads, "\mathrm{AJ}"] & & F^1\mathrm{H}^{2d}_G(X(\mathbb{C}), \mathbb{Z}(d))_0 \arrow[d, two heads] \\
\mathrm{Alb}(X)(\mathbb{R})_{\mathrm{tors}} \arrow[r, two heads] & \pi_0\left(\mathrm{Alb}(X)(\mathbb{R})\right) \arrow[r, "\sim"] & \mathrm{H}^1(G, \Lambda_X).
\end{tikzcd} \tag{8.25}$$

Here, the horizontal isomorphism on the right of the bottom row is the map induced by the boundary map attached to the canonical exact sequence of G-modules

$$0 \to \Lambda_X \to \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{R}(d)) \to \mathrm{Alb}(X)(\mathbb{C}) \to 0. \tag{8.26}$$

Proof. We claim that we can complete diagram (8.25) in the following way:

$$\begin{tikzcd}
\mathrm{CH}_0(X)_{\mathrm{tors}} \arrow[rr] \arrow[dr, "\sim"] \arrow[ddd, "\mathrm{AJ}"] & & F^1\mathrm{H}^{2d}_G(X(\mathbb{C}), \mathbb{Z}(d))_0 \arrow[dd] \\
& \mathrm{H}^{2d-1}_G(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))_0 \arrow[ur] \arrow[d] & \\
& \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))^G \arrow[r] & \mathrm{H}^1(G, \Lambda_X) \arrow[d, equal] \\
\mathrm{Alb}(X)(\mathbb{R})_{\mathrm{tors}} \arrow[ur, "\sim"] \arrow[rr] & & \pi_0\left(\mathrm{Alb}(X)(\mathbb{R})\right).
\end{tikzcd} \tag{8.27}$$

Let us define the groups and arrows occurring in this diagram, prove that each arrow is surjective, and that each quadrilateral and triangle commutes. Observe that this is indeed enough to prove the lemma.

Let us start with the diagram on the bottom of (8.27). The exact sequence of (sheaves of) G-modules

$$0 \to \mathbb{Z}(d) \to \mathbb{Q}(d) \to \mathbb{Q}/\mathbb{Z}(d) \to 0 \tag{8.28}$$

induces an exact sequence of G-modules

$$0 \to \Lambda_X \to \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}(d)) \to \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d)) \to 0.$$

This gives a canonical isomorphism $\mathrm{Alb}(X)(\mathbb{C})_{\mathrm{tors}} \cong \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))$, and the boundary map

$$\mathrm{Alb}(X)(\mathbb{R})_{\mathrm{tors}} \cong \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))^G \to \mathrm{H}^1(G, \Lambda_X)$$

is surjective because $\mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}(d))$ has no higher group cohomology. Thus the diagram on the bottom of (8.27) exists, commutes and its arrows are surjective.

Let us continue with the diagram on the left of (8.27). A canonical isomorphism between $\mathrm{CH}_0(X_{\mathbb{C}})_{\mathrm{tors}}$ and $\mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))$ is constructed in [CT93]. This map is G-equivariant, thus induces an isomorphism

$$\mathrm{CH}_0(X_{\mathbb{C}})^G_{\mathrm{tors}} \cong \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))^G.$$

As shown by Van Hamel [Ham00], the composition of the latter with the natural map $\mathrm{CH}_0(X)_{\mathrm{tors}} \to \mathrm{CH}_0(X_{\mathbb{C}})^G_{\mathrm{tors}}$ factors through a composition of morphisms

$$\mathrm{CH}_0(X)_{\mathrm{tors}} \to \mathrm{H}^{2d-1}_G(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d)) \to \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))^G. \tag{8.29}$$

On the other hand, the composition

$$\mathrm{CH}_0(X)_{\mathrm{tors}} \to \mathrm{H}^{2d-1}_G(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d)) \xrightarrow{\delta} \mathrm{H}^{2d}_G(X(\mathbb{C}), \mathbb{Z}(d))$$

coincides with the cycle class map by [CTSS83, Corollaire 1], where the map δ is the boundary map induced by the exact sequence (8.28). Therefore, the triangle on the top of (8.27) commutes, where we define $\mathrm{H}^{2d-1}_G(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))_0$ to be the inverse image of $\mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d))_0$ under this boundary map δ. Note that the image of δ lies indeed in $F^1\mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d))$, because the composition of δ with the map $\mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d)) \to \mathrm{H}^{2d}(X(\mathbb{C}),\mathbb{Z}(d))$ factors through the following composition, which is zero:

$$\mathrm{H}^{2d-1}_G(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z}(d)) \to \mathrm{H}^{2d-1}(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z}(d)) \to \mathrm{H}^{2d}(X(\mathbb{C}),\mathbb{Z}(d)) = \mathbb{Z}.$$

We claim that the thus-obtained map

$$\mathrm{H}^{2d-1}_G(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z}(d))_0 \to F^1\mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d))_0$$

is surjective. Indeed, this follows since $\mathrm{H}^{2d-1}_G(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z}(d)) \to F^1\mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d))$ is surjective, which is in turn a consequence of the commutativity of the diagram

$$\begin{tikzcd}
 & \mathrm{H}^{2d}(X(\mathbb{C}),\mathbb{Z}(d))^G \arrow[r] & \mathrm{H}^{2d}(X(\mathbb{C}),\mathbb{Q}(d))^G \\
\mathrm{H}^{2d-1}_G(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z}(d)) \arrow[r] \arrow[dr, "\delta"] & \mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d)) \arrow[r] \arrow[u] & \mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Q}(d)), \arrow[u, "\wr"'] \\
 & F^1\mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d)) \arrow[u, hook] &
\end{tikzcd}$$

and the fact that its middle row and column are exact.

The image of the first arrow in (8.29) equals $\mathrm{H}^{2d-1}_G(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z}(d))_0$ because of the following diagram with exact rows, see [Ham00, Theorem 3.2], [BW20a, Proposition 1.8]:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}_0(X)_{\mathrm{tors}} \arrow[r] \arrow[d, two heads] & \mathrm{H}^{2d-1}_G(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z}(d)) \arrow[r] \arrow[d, two heads] & \bigoplus_{i>0} \mathrm{H}^{d-2i}(X(\mathbb{R}),\mathbb{Z}/2) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F^1\mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d))_0 \arrow[r] & F^1\mathrm{H}^{2d}_G(X(\mathbb{C}),\mathbb{Z}(d)) \arrow[r] & \bigoplus_{\substack{0\le p<d \\ p\equiv d \bmod 2}} \mathrm{H}^p(X(\mathbb{R}),\mathbb{Z}/2) \arrow[r] & 0.
\end{tikzcd}$$

The Abel-Jacobi map AJ on the left of (8.27) is surjective by [Ham00, Corollary 4.5] and our assumption that the Hochschild-Serre spectral sequence (8.5) degenerates.

Moreover, the fact that the quadrilateral on the left of (8.27) commutes follows from the commutativity of

$$\begin{tikzcd} \mathrm{CH}_0(X_{\mathbb{C}})_{\mathrm{tors}} \arrow[r] \arrow[d] & \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d)) \arrow[dl, "\sim"'] \\ \mathrm{Alb}(X)(\mathbb{C})_{\mathrm{tors}} & \end{tikzcd}$$

which is [Blo79, Proposition 3.9].

To finish the proof, it remains to show that the quadrilateral on the top right of (8.27) commutes. This follows from the fact that the morphism

$$\mathrm{H} := \mathrm{H}^{2d-1}_G(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d)) \to \mathrm{H}^{2d}_G(X(\mathbb{C}), \mathbb{Z}(d)) =: \tilde{\mathrm{H}}$$

shifts the filtrations induced by the Hochschild-Serre spectral sequences by one degree, thereby inducing a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F^1\mathrm{H} \arrow[r] \arrow[d] & F^0\mathrm{H} \arrow[r] \arrow[d] & \mathrm{H}^0(G, \mathrm{H}^{2d}(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(d))) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F^2\tilde{\mathrm{H}} \arrow[r] & F^1\tilde{\mathrm{H}} \arrow[r] & \mathrm{H}^1(G, \mathrm{H}^{2d-1}(X(\mathbb{C}), \mathbb{Z}(d))) \arrow[r] & 0. \end{tikzcd}$$

□

Lemma 8.30. *Let $f\colon J \to A$ be a surjective homomorphism of real abelian varieties.*

1. *The induced homomorphism $f^0_{\mathbb{R}}\colon J(\mathbb{R})^0 \to A(\mathbb{R})^0$ is surjective.*
2. *Let $B = \mathrm{Ker}(f)$ and assume that B is an abelian variety (equivalently: that $B(\mathbb{C})$ is connected). The induced homomorphism $J(\mathbb{R})^0_{\mathrm{tors}} \to A(\mathbb{R})^0_{\mathrm{tors}}$ is also surjective.*

Proof. 1. The diagram

$$\begin{tikzcd} J(\mathbb{C}) \arrow[r, "\mathrm{Nm}"] \arrow[d, "f"] & J(\mathbb{R})^0 \arrow[d, "f^0_{\mathbb{R}}"] \\ A(\mathbb{C}) \arrow[r, "\mathrm{Nm}"] & A(\mathbb{R})^0 \end{tikzcd}$$

commutes, where $\mathrm{Nm}\colon X(\mathbb{C}) \to X(\mathbb{R})^0, x \mapsto x + \sigma(x)$ denotes the norm homomorphism of a real abelian variety X. By [GH81, Proposition 1.1], this map Nm is surjective. Thus $f^0_{\mathbb{R}}$ is surjective.

2. Let $x \in A(\mathbb{R})^0_{\mathrm{tors}}$ and suppose that $n \cdot x = 0$ for some $n \in \mathbb{Z}_{\geq 2}$. By part 1, there exists $y \in J(\mathbb{R})^0$ such that $f(y) = x$. We have that $n \cdot y \in B(\mathbb{R})$, hence $2n \cdot y \in B(\mathbb{R})^0$. Since the abelian group $B(\mathbb{R})^0$ is divisible, there exists $z \in B(\mathbb{R})^0$ such that $2n \cdot z = 2n \cdot y$. Define $\alpha = y - z \in J(\mathbb{R})^0$. Then $f(\alpha) = f(y) = x$. Moreover, $2n \cdot \alpha = 0$. We conclude that $J(\mathbb{R})^0_{\mathrm{tors}} \to A(\mathbb{R})^0_{\mathrm{tors}}$ is surjective.

$\square$

Lemma 8.31. *Consider an exact sequence of real abelian varieties* $0 \to B \to J \xrightarrow{f} A \to 0$. *If the restricted homomorphism* $f\colon J(\mathbb{R}) \to A(\mathbb{R})$ *is surjective, then the push-forward*

$$f_*\colon \mathrm{CH}_0(J)_{\mathrm{tors}} \to \mathrm{CH}_0(A)_{\mathrm{tors}} \qquad \textit{is also surjective.}$$

Proof. For an abelian group G, let G_{tors} be the torsion subgroup and $G_{\mathrm{tors,div}}$ the maximal divisible torsion subgroup of G. Following van Hamel [Hamoo], we may define, for an algebraic variety X over $\mathbb{R}$, a group $\mathrm{A}_0(X)^{\mathrm{top}}$ as the quotient

$$\mathrm{A}_0(X)^{\mathrm{top}} = \mathrm{A}_0(X)_{\mathrm{tors}}/\mathrm{A}_0(X)_{\mathrm{tors,div}} = \mathrm{CH}_0(X)_{\mathrm{tors}}/\mathrm{CH}_0(X)_{\mathrm{tors,div}},$$

where $\mathrm{A}_0(X)$ is the group of zero cycles of degree zero modulo rational equivalence on X.

In our case, we obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}_0(J)_{\mathrm{tors,div}} \arrow[r] \arrow[d] & \mathrm{CH}_0(J)_{\mathrm{tors}} \arrow[r] \arrow[d] & \mathrm{A}_0(J)^{\mathrm{top}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{CH}_0(A)_{\mathrm{tors,div}} \arrow[r] & \mathrm{CH}_0(A)_{\mathrm{tors}} \arrow[r] & \mathrm{A}_0(A)^{\mathrm{top}} \arrow[r] & 0.
\end{tikzcd}$$

To prove that the vertical arrow in the middle is surjective, it suffices to prove the surjectivity of the outer two vertical arrows. To prove the surjectivity of the left vertical arrow, one uses the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{CH}_0(J)_{\mathrm{tors,div}} \arrow[r, "\mathrm{AJ}", "\sim"'] \arrow[d] & J(\mathbb{R})^0_{\mathrm{tors}} \arrow[d] \\
\mathrm{CH}_0(A)_{\mathrm{tors,div}} \arrow[r, "\mathrm{AJ}", "\sim"'] & A(\mathbb{R})^0_{\mathrm{tors}}
\end{tikzcd}$$

and Lemma 8.30. The two Abel-Jacobi maps appearing in this diagram are isomorphisms by [Hamoo, Corollary 4.2].

It remains to prove that $A_0(J)^{\mathrm{top}} \to A_0(A)^{\mathrm{top}}$ is surjective. Now for any smooth projective variety X over $\mathbb{R}$, one has by [Hamoo, §3.2] that

$$\mathrm{A}_0(X)^{\mathrm{top}} = \mathrm{A}_0(X)/\mathrm{A}_0(X)_{\mathrm{div}}.$$

Thus the desired surjectivity follows from the surjectivity of $\mathrm{A}_0(J) \to \mathrm{A}_0(A)$. The latter is a consequence of the surjectivity of $J(\mathbb{C}) \to A(\mathbb{C})$ and $J(\mathbb{R}) \to A(\mathbb{R})$, and the fact that for an abelian variety X over a field k, the group $\mathrm{A}_0(X)$ is generated by zero-cycles of the form $[x] - \deg(k(x_i)/k) \cdot [0]$ for closed points x on X. □

Let us now prove Proposition 8.28. We will need some notation.

Definition 8.32. For an abelian variety A of dimension g over $\mathbb{R}$, define

$$\begin{aligned} F^2\mathrm{CH}_0(A)_{\mathrm{tors}} &= \mathrm{Ker}\left(\mathrm{CH}_0(A)_{\mathrm{tors}} \to \mathrm{H}^1(G, \mathrm{H}^{2g-1}(A(\mathbb{C}), \mathbb{Z}(g)))\right) \\ &= \left\{\alpha \in \mathrm{CH}_0(A)_{\mathrm{tors}} \mid \mathrm{AJ}(\alpha) \in A(\mathbb{R})^0\right\}. \end{aligned}$$

For the second equality in Definition 8.32, see Lemma 8.29.

Proof of Proposition 8.28. Let $d = \dim(J)$ and $g = \dim(A)$. Consider the following commutative diagram:

$$\begin{tikzcd} F^2\mathrm{CH}_0(J)_{\mathrm{tors}} \arrow[r] \arrow[d] & F^2\mathrm{CH}_0(A)_{\mathrm{tors}} \arrow[d] \\ F^2\mathrm{H}^{2d}_G(J(\mathbb{C}), \mathbb{Z}(d))_0 \arrow[r] & F^2\mathrm{H}^{2g}_G(A(\mathbb{C}), \mathbb{Z}(g))_0. \end{tikzcd}$$

Its vertical arrows are surjective by Lemma 8.29 and Definition 8.32. To prove the surjectivity of the lower horizontal arrow, it thus suffices to prove the surjectivity of the upper horizontal arrow. Let this be our first goal.

For a real abelian variety X of dimension n over $\mathbb{R}$, define $\Lambda_X = \mathrm{H}^{2n-1}(X(\mathbb{C}), \mathbb{Z}(n))$.

Recall the canonical identification $\pi_0(X(\mathbb{R})) = \mathrm{H}^1(G, \Lambda_X)$, see the sequence (8.26). By Lemma 8.29, the rows in the following commutative diagram are exact:

$$\begin{tikzcd}
0 \arrow[r] & F^2\mathrm{CH}_0(J)_{\mathrm{tors}} \arrow[r] \arrow[d] & \mathrm{CH}_0(J)_{\mathrm{tors}} \arrow[r] \arrow[d, "\phi"] & \mathrm{H}^1(G, \Lambda_J) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & F^2\mathrm{CH}_0(A)_{\mathrm{tors}} \arrow[r] & \mathrm{CH}_0(A)_{\mathrm{tors}} \arrow[r] & \mathrm{H}^1(G, \Lambda_A) \arrow[r] & 0.
\end{tikzcd} \tag{8.30}$$

The middle vertical arrow of diagram (8.30) is surjective by Lemma 8.31, and its right vertical arrow is surjective by the surjectivity of $\pi_0(J(\mathbb{R})) \to \pi_0(A(\mathbb{R}))$; the latter follows from the surjectivity of $J(\mathbb{R}) \to A(\mathbb{R})$.

Claim 1: The kernel of $\phi\colon \mathrm{CH}_0(J)_{\mathrm{tors}} \to \mathrm{CH}_0(A)_{\mathrm{tors}}$ surjects onto the kernel of $f\colon \pi_0(J(\mathbb{R})) \to \pi_0(A(\mathbb{R}))$.

Indeed, this follows from the following commutative diagram, in which the two-headed arrows are surjective:

$$\begin{tikzcd}
\mathrm{Ker}(\phi) \arrow[rrr] \arrow[dd] & & & \mathrm{Ker}(f) \arrow[dd] \\
& \mathrm{CH}_0(B)_{\mathrm{tors}} \arrow[r, two heads] \arrow[ul] \arrow[dl] & \pi_0(B(\mathbb{R})) \arrow[ur, two heads] \arrow[dr] & \\
\mathrm{CH}_0(J)_{\mathrm{tors}} \arrow[rrr] \arrow[d, "\phi"] & & & \pi_0(J(\mathbb{R})) \arrow[d, "f"] \\
\mathrm{CH}_0(A)_{\mathrm{tors}} \arrow[rrr] & & & \pi_0(A(\mathbb{R})).
\end{tikzcd} \tag{8.31}$$

The fact that $\mathrm{CH}_0(B)_{\mathrm{tors}} \to \pi_0(B(\mathbb{R}))$ is surjective follows from Lemma 8.29. The fact that $\pi_0(B(\mathbb{R}))$ surjects onto the kernel of $f\colon \pi_0(J(\mathbb{R})) \to \pi_0(A(\mathbb{R}))$ follows from the exact sequence in group cohomology

$$0 \to \Lambda_B^G \to \Lambda_J^G \to \Lambda_A^G \to \mathrm{H}^1(G, \Lambda_B) \to \mathrm{H}^1(G, \Lambda_J) \to \mathrm{H}^1(G, \Lambda_A) \to 0 \tag{8.32}$$

arising from the short exact sequence of G-modules $0 \to \Lambda_B \to \Lambda_J \to \Lambda_A \to 0$.

Claim 1 implies the surjectivity of the left vertical arrow in (8.30) by the snake lemma. It remains to prove the surjectivity of the map $\hat{f}^*$ in the proposition.

Claim 2: The maps f_* and $\hat{f}^*$ fit into the following commutative diagram, where the arrows $\twoheadrightarrow$ are surjective and the arrows $\xrightarrow{\sim}$ are isomorphisms:

$$\begin{tikzcd}
F^2\mathrm{H}^{2d}_G(J(\mathbb{C}),\mathbb{Z}(d))_0 \arrow[dr, two heads] \arrow[dd, two heads, "f_*"] & & F^2\mathrm{H}^4_G(\widehat{J}(\mathbb{C}),\mathbb{Z}(2))_0 \arrow[dr, "\sim"] \arrow[dd, "\hat{f}^*" near start] & \\
& \mathrm{H}^2(G,\mathrm{H}^{2d-2}(J(\mathbb{C}),\mathbb{Z}(d)))_0 \arrow[rr, "\sim", "\mathrm{H}^2(G,\mathscr{F}_J)"'] \arrow[dd, "\mathrm{H}^2(G,f_*)"] & & \mathrm{H}^2(G,\mathrm{H}^2(\widehat{J}(\mathbb{C}),\mathbb{Z}(2)))_0 \arrow[dd, "\mathrm{H}^2(G,\hat{f}^*)"] \\
F^2\mathrm{H}^{2g}_G(A(\mathbb{C}),\mathbb{Z}(g))_0 \arrow[dr, two heads] & & F^2\mathrm{H}^4(\widehat{A}(\mathbb{C}),\mathbb{Z}(2))_0 \arrow[dr, "\sim"] & \\
& \mathrm{H}^2(G,\mathrm{H}^{2g-2}(A(\mathbb{C}),\mathbb{Z}(g)))_0 \arrow[rr, "\sim", "\mathrm{H}^2(G,\mathscr{F}_A)"'] & & \mathrm{H}^2(G,\mathrm{H}^2(\widehat{A}(\mathbb{C}),\mathbb{Z}(2)))_0.
\end{tikzcd} \tag{8.33}$$

Since the surjectivity of f_* on the left of diagram (8.33) has already been proved, claim 2 follows from the functoriality of Fourier transforms (see [MP10, (3.7.1)]), Proposition 8.26 and Lemma 8.27.

We can now finish the proof. Consider the diagram (8.33). By the commutativity on the left, the morphism $\mathrm{H}^2(G, f_*)$ is surjective. By the commutativity of the square on the front, this implies that $\mathrm{H}^2(G, \hat{f}^*)$ is surjective which, by the commutativity on the right hand side of the diagram, implies that the morphism $\hat{f}^*\colon F^2\mathrm{H}^4_G(\widehat{J}(\mathbb{C}),\mathbb{Z}(2))_0 \to F^2\mathrm{H}^4(\widehat{A}(\mathbb{C}),\mathbb{Z}(2))_0$ is surjective. $\square$

Lemma 8.33. *Let A be an abelian variety over $\mathbb{R}$. Then A contains a smooth, proper, geometrically connected curve C over $\mathbb{R}$ that passes through $0 \in A(\mathbb{R})$ in such a way that the following two conditions hold: the induced homomorphism $f\colon J(C) \to A$ is surjective with connected kernel, and the homomorphism $f_{\mathbb{R}}\colon J(C)(\mathbb{R}) \to A(\mathbb{R})$ is surjective.*

Proof. Let $S \subset A(\mathbb{R})$ be a finite set of points containing $0 \in A(\mathbb{R})$ and at least one point of each connected component of $A(\mathbb{R})$. Since $\mathbb{R}$ is infinite, Bertini's theorem can be applied: there exists a smooth and geometrically connected hyperplane section $Z \subset A$ passing through S [Har77, II, Theorem 8.18], [Deb01, Footnote 12, page 32]. Let $g = \dim(A)$. By taking $g-1$ general such hyperplane sections, we get a smooth curve C in A that contains S. It remains to prove that C satisfies the requirements stated in the proposition.

Write $J = J(C)$ and consider the map $f\colon J \to A$ arising from the inclusion $(C, 0) \hookrightarrow (A, 0)$. By the proof of Theorem 10.1 in [Mil86], the homomorphism f is surjective. We claim that the kernel of $f\colon J_{\mathbb{C}} \to A_{\mathbb{C}}$ is connected. This follows from the fact that $\mathrm{H}_1(C(\mathbb{C}),\mathbb{Z}) \to \mathrm{H}_1(A(\mathbb{C}),\mathbb{Z})$ is surjective by the Lefschetz hyperplane theorem. Alternatively, see [Gab01, Proposition 2.4] for an algebraic argument.

We have that $f^0_{\mathbb{R}}\colon J(\mathbb{R})^0 \to A(\mathbb{R})^0$ is surjective by Lemma 8.30. Because S is contained in the image of $f_{\mathbb{R}}$, we conclude that $f_{\mathbb{R}}$ is surjective. □

Let us finish the proof of Theorem 8.8. Let A be an abelian variety over $\mathbb{R}$. The group $F^3\mathrm{H}^4_G(A(\mathbb{C}),\mathbb{Z}(2))_0$ is trivial by Proposition 8.26, so it remains to prove that $F^2\mathrm{H}^4_G(A(\mathbb{C}),\mathbb{Z}(2))_0$ is algebraic. Let $C \subset \widehat{A}$ be a real algebraic curve that satisfies the conditions of Lemma 8.33. By Proposition 8.28, the pull-back homomorphism

$$\hat{f}^*\colon F^2\mathrm{H}^4_G(\widehat{J}(C)(\mathbb{C}),\mathbb{Z}(2))_0 \to F^2\mathrm{H}^4_G(A(\mathbb{C}),\mathbb{Z}(2))_0$$

is surjective, where $f\colon J(C) \to \widehat{A}$ is the homomorphism induced by the inclusion of $(C,0)$ in $(\widehat{A},0)$. By the proof of Theorem 8.8 in the Jacobian case, we know that $F^2\mathrm{H}^4_G(\widehat{J}(C)(\mathbb{C}),\mathbb{Z}(2))_0$ is algebraic. Therefore $F^2\mathrm{H}^4_G(A(\mathbb{C}),\mathbb{Z}(2))_0$ is algebraic. □

8.5.9 *Reduction to the Abel-Jacobi map*

Let us prove Corollary 8.9 and Proposition 8.10.

Proof of Corollary 8.9. This follows Lemma 8.12, Theorem 8.3, equalities (8.6), filtration (8.19) and Theorem 8.8. □

Proposition 8.34. *Let B be a real abelian surface, and let E be a real elliptic curve whose real locus $E(\mathbb{R})$ is connected. The following homomorphism is surjective:*

$$\mathrm{CH}^2(B \times E)_{\mathrm{hom}} \to \mathrm{H}^1(G,\mathrm{H}^3(B(\mathbb{C}) \times E(\mathbb{C}),\mathbb{Z}(2))). \tag{8.34}$$

In particular, $B \times E$ satisfies the real integral Hodge conjecture, i.e. Proposition 8.10 holds.

Proof. First observe that

$$\mathrm{H}^1(G,\mathrm{H}^2(B(\mathbb{C}),\mathbb{Z}) \otimes \mathrm{H}^1(E(\mathbb{C}),\mathbb{Z})) = (0)$$

because $\mathrm{H}^1(E(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}[G]$ by our hypothesis that $E(\mathbb{R})$ is connected [Ham97, Chapter IV, Example 3.1]. By the Künneth formula, the canonical morphism

$$\begin{aligned} \mathrm{H}^1(G, \mathrm{H}^3(B(\mathbb{C}), \mathbb{Z}(2)) \bigoplus \ & \mathrm{H}^1(G, \mathrm{H}^1(B(\mathbb{C}), \mathbb{Z}(1)) \otimes \mathrm{H}^2(E(\mathbb{C}), \mathbb{Z}(1))) \\ \to \ & \mathrm{H}^1(G, \mathrm{H}^3(B(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Z}(2))) \end{aligned} \tag{8.35}$$

is therefore an isomorphism. Since $\mathrm{H}^2(E(\mathbb{C}), \mathbb{Z}(1)) \cong \mathbb{Z}$ as G-modules, the canonical map

$$\begin{aligned} \mathrm{H}^1(G, \mathrm{H}^1(B(\mathbb{C}), \mathbb{Z}(1))) \bigotimes \ & \mathrm{H}^0(G, \mathrm{H}^2(E(\mathbb{C}), \mathbb{Z}(1))) \\ \to \ & \mathrm{H}^1(G, \mathrm{H}^1(B(\mathbb{C}), \mathbb{Z}(1)) \otimes \mathrm{H}^2(E(\mathbb{C}), \mathbb{Z}(1))) \end{aligned} \tag{8.36}$$

is an isomorphism as well. To simplify notation, for any abelian variety X over $\mathbb{R}$, define

$$\mathrm{H}^i_X(j) = \mathrm{H}^i(X(\mathbb{C}), \mathbb{Z}(j)).$$

Consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{CH}^2(B)_{\mathrm{tors}} \oplus \left(\mathrm{CH}^1(B)_{\mathrm{hom}} \otimes \mathrm{CH}^1(E)\right) \arrow[r] \arrow[d] & \mathrm{CH}^2(B \times E)_{\mathrm{hom}} \arrow[d] \\ F^1\mathrm{H}^4_G(B(\mathbb{C}), \mathbb{Z}(2)) \oplus (F^1\mathrm{H}^2_G(B(\mathbb{C}), \mathbb{Z}(1)) \otimes \mathrm{H}^2_G(E(\mathbb{C}), \mathbb{Z}(1))) \arrow[r] \arrow[d] & F^1\mathrm{H}^4_G(B(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Z}(2)) \arrow[dd] \\ \mathrm{H}^1(G, \mathrm{H}^3_B(2)) \oplus (\mathrm{H}^1(G, \mathrm{H}^1_B(1)) \otimes \mathrm{H}^0(G, \mathrm{H}^2_E(1))) \arrow[d, "\wr"] & \\ \mathrm{H}^1(G, \mathrm{H}^3_B(2)) \bigoplus \mathrm{H}^1(G, \mathrm{H}^1_B(1) \otimes \mathrm{H}^2_E(1)) \arrow[r, "\sim"] & \mathrm{H}^1(G, \mathrm{H}^3_{B \times E}(2)). \end{tikzcd} \tag{8.37}$$

The indicated isomorphisms $\xrightarrow{\sim}$ in diagram (8.37) arise from the isomorphisms (8.35) and (8.36) above. The map $\mathrm{CH}^2(B)_{\mathrm{tors}} \to \mathrm{H}^1(G, \mathrm{H}^3_B(2))$ is surjective by Lemma 8.29, and the map $\mathrm{CH}^1(B)_{\mathrm{hom}} \to \mathrm{H}^1(G, \mathrm{H}^1_B(1))$ is surjective by the real integral Hodge conjecture for divisors (see Section 8.5.2) and the fact that the topological condition for degree two cohomology classes is trivial [BW20a, §2.3.1]. Therefore, the vertical composition on the left of (8.37) is surjective, which implies that the vertical composition on the right – which is (8.34) – is surjective. By Corollary 8.9, this implies that $B \times E$ satisfies the real integral Hodge conjecture. $\square$

The topological condition (8.4) does not appear on the right hand side of (8.34). This has the following corollary. To state it, recall the following fact. For a real abelian variety A of dimension g, one has $|\pi_0(A(\mathbb{R}))| \leq 2^g$ (see [GH81, §1]) and for every non-negative pair of integers (i, g) with $i \leq g$, there is a real abelian variety A of dimension g such that $|\pi_0(A(\mathbb{R}))| = 2^i$ (take a suitable product of elliptic curves). In particular, if $g = \dim(A) = 3$, then $|\pi_0(A(\mathbb{R}))| \in \{1, 2, 4, 8\}$.

Corollary 8.35. *Let A be a real abelian threefold, and suppose that $|\pi_0(A(\mathbb{R}))| \neq 8$. Then the canonical map*

$$F^1\mathrm{H}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0 \to \mathrm{H}^1(G, \mathrm{H}^3(A(\mathbb{C}), \mathbb{Z}(2))) \tag{8.38}$$

is surjective. Thus, A satisfies the real integral Hodge conjecture if and only if the Abel-Jacobi map

$$\mathrm{CH}_1(A)_{\mathrm{hom}} \to \mathrm{H}^1(G, \mathrm{H}^3(A(\mathbb{C}), \mathbb{Z}(2)))$$

is surjective.

Proof. To prove the surjectivity of (8.38), we may replace $A(\mathbb{C})$ by any differentiable G-manifold which is G-equivariantly diffeomorphic to $A(\mathbb{C})$. In particular, we may assume that $A = B \times E$, where B is an abelian surface and E an elliptic curve whose real locus is connected (see (8.22)). The surjectivity of (8.38) follows then from the fact that the group $\mathrm{H}^1(G, \mathrm{H}^3(B(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Z}(2)))$ is algebraic by Proposition 8.34. The second statement follows from the first, due to Corollary 8.9. □

Part III

Summary in French

9

RÉSUMÉ EN FRANÇAIS

9.1 INTRODUCTION

Cette thèse porte sur l'étude des cycles algébriques et des espaces de modules en géométrie algébrique réelle. Nous espérons que le présent manuscrit puisse apporter une contribution à la compréhension de ces concepts, tant pris individuellement que dans leurs interactions mutuelles. Afin de mener ce projet à bien, nous étudions au fil de ce texte les phénomènes suivants :

- La structure hyperbolique de l'espace de modules des quintiques binaires réelles.
- La distribution des variétés abéliennes non-simples dans une famille de variétés abéliennes réelles.
- Courbes algébriques sur des variétés abéliennes complexes et réelles.

Comme nous allons le voir, les résultats ainsi que les méthodes de ces différents sujets sont étroitement liés. Un outil omniprésente tout au long de ce projet est la *théorie de Hodge équivariante* : l'utilisation de la théorie de Hodge pour étudier la cohomologie, les modules et les cycles des variétés algébriques réelles.

9.2 RÉSUMÉ DU CHAPITRE 2: ESPACES DE MODULES RÉELLES

Notre premier résultat fournit une méthode générale de construction des espaces de modules en géométrie algébrique réelle. Dans le cadre classique de la géométrie complexe, l'espace de modules d'une catégorie donnée d'objets algébriques peut souvent être construit en appliquant le théorème de Keel-Mori au champ de modules correspondant. Par exemple, cela fonctionne lorsqu'un tel champ est séparé et de Deligne-Mumford sur $\mathbb{C}$. Pour un champ de modules défini sur les réels, il est *a priori* moins aisé d'obtenir une structure analytique sur l'ensemble des classes d'isomorphisme de son lieu réel, cet ensemble n'étant en général pas en bijection avec le lieu réel de l'espace de modules grossier. Pourtant, nous parvenons à établir le résultat suivant.

Théorème 9.1. *Soit $\mathscr{X}$ un champ algébrique localement de type fini sur $\mathbb{R}$. Il existe une topologie sur l'ensemble des classes d'isomorphisme $|\mathscr{X}(\mathbb{R})|$ du lieu réel de $\mathscr{X}$, fonctorielle en $\mathscr{X}$ et qui coincide avec la topologie analytique réelle lorsque $\mathscr{X}$ est un schéma. En outre, si $\mathscr{X}$ est lisse et de Deligne-Mumford sur $\mathbb{R}$, l'espace $|\mathscr{X}(\mathbb{R})|$ porte une structure d'orbifold fonctorielle en $\mathscr{X}$.*

Comment cette approche se compare-t-elle aux méthodes plus classiques pour construire des espaces de modules en géométrie algébrique réelle ? Considérons deux cas particuliers : soit $\mathcal{A}_g$ le champ des variétés abéliennes principalement polarisées de dimension g, et soit $\mathcal{M}_g$ le champ des courbes propres, lisses et géométriquement connexes de genre g. Il est bien connu qu'il existe des uniformisations analytiques complexes

$$\mathcal{A}_g(\mathbb{C}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}) \setminus \mathbb{H}_g, \quad \text{et} \quad \mathcal{M}_g(\mathbb{C}) \cong \Gamma_g \setminus \mathcal{T}_g.$$

Ici, $\mathbb{H}_g$ (resp. $\mathcal{T}_g$) est une variété complexe paramétrant les variétés abéliennes complexes (resp. des courbes complexes) dotées d'une certaine structure supplémentaire; $\mathrm{Sp}_{2g}(\mathbb{Z})$ (resp. Γ_g) est un groupe discret agissant holomorphiquement et proprement discontinûment sur $\mathbb{H}_g$ (resp. $\mathcal{T}_g$) en permutant lesdites structures supplémentaires; enfin, l'application $\mathbb{H}_g \to \mathcal{A}_g(\mathbb{C})$ (resp. $\mathcal{T}_g \to \mathcal{M}_g(\mathbb{C})$) est obtenue en oubliant ces structures.

Pour obtenir des espaces de modules réels pour ces problèmes de modules, on définit des ensembles d'involutions anti-holomorphes

$$\{\sigma\colon \mathbb{H}_g \to \mathbb{H}_g\} \quad \text{et} \quad \{\tau\colon \mathcal{T}_g \to \mathcal{T}_g\}$$

de sorte que toute variété abélienne réelle principalement polariée de dimension g (resp. toute courbe réelle de genre g) se trouve dans $\mathbb{H}_g^\sigma$ (resp. $\mathcal{T}_g^\tau$) pour un certain σ (resp. τ). Pour des sous-groupes $\mathrm{Sp}_{2g}(\mathbb{Z})(\sigma) \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ et $\Gamma_g(\tau) \subset \Gamma_g$ définis de manière appropriée, on obtient des bijections (voir [GH81, Sil89, SS89]):

$$\mathcal{A}_g(\mathbb{R}) \cong \bigsqcup_{\sigma} \mathrm{Sp}_{2g}(\mathbb{Z})(\sigma) \setminus \mathbb{H}_g^\sigma \quad \text{and} \quad \mathcal{M}_g(\mathbb{R}) \cong \bigsqcup_{\tau} \Gamma_g(\tau) \setminus \mathcal{T}_g^\tau. \tag{9.1}$$

Comme dans le cas complexe, il se trouve que ces applications des périodes sont des homéomorphismes.

Théorème 9.2. *Les bijections dans (9.1) sont des homéomorphismes, où les ensembles* $|\mathcal{A}_g(\mathbb{R})|$ *et* $|\mathcal{M}_g(\mathbb{R})|$ *sont munis de la topologie donnée par le Théorème 9.1.*

9.3 RÉSUMÉ DU CHAPITRE 3: LIEUX DE NOETHER-LEFSCHETZ RÉELS

Dans le Chapitre 3, nous développons un certain critère de densité, analogue au critère de Green–Voisin pour les lieux de Noether-Lefschetz d'une variation de structure de Hodge de poids deux (voir [Voi02, §17.3.4]). Pour formuler ce résultat et le comparer au critère bien connu de Green–Voisin, on considère une famille holomorphe de variétés abéliennes complexes polarisées

$$\left(\psi \colon A \to B, \quad s \colon B \to A, \quad E \in R^2\psi_*\mathbb{Z}\right). \tag{9.2}$$

C'est-à-dire que ψ est une submersion propre holomorphe, s une section holomorphe de ψ et on suppose que pour chaque $t \in B$, la fibre $A_t = \psi^{-1}(t)$ est une variété abélienne complexe polarisée d'origine $s(t)$ et de polarisation $E_t \in \mathrm{H}^2(A_t, \mathbb{Z})$. On suppose que B est connexe. Pour chaque entier positif $k < g$, notons $S_k \subset B$ l'ensemble des $t \in B$ tels que A_t contient une sous-variétié abélienne complexe de dimension k. Le résultat principal de [CP90] est le suivant :

Théorème (Colombo–Pirola)**.** *Si la Condition 3.1 du Chapitre 3 est vérifiée, alors* S_k *est dense pour la topologie euclidienne dans* B*.*

Supposons à notre tour qu'il existe des involutions anti-holomorphes $\sigma \colon B \to B$ et $\tau \colon A \to A$, qui commutent avec ψ et s et qui sont compatibles à la polarisation E. En posant

$$R_k = \{t \in B(\mathbb{R}) \mid A_t \text{ contient une sous-variété abélienne réelle de dimension } k\},$$

nous obtenons le résultat suivant :

Théorème 9.3. *Si la Condition 3.1 du Chapitre 3 est vérifée, alors* R_k *est dense pour la topologie euclidienne dans* $B(\mathbb{R})$*.*

Il est intéressant de comparer ces théorèmes avec des résultats analogues pour les lieux de Noether-Lefschetz des hypersurfaces dans $\mathbb{P}^3$. Fixons $d \geq 4$ un entier et considérons l'hypersurface lisse universelle $\mathbb{P}^3 \times \mathscr{B} \supset \mathscr{S} \to \mathscr{B}$ de degré d. Classiquement, le *lieu de Noether-Lefschetz* est défini comme l'ensemble des $t \in \mathscr{B}(\mathbb{C})$ tel que $\mathscr{S}_t$ contient une courbe qui n'est pas une intersection complète. Dans ce cas, le résultat principal de [CHM88] assure que ce lieu est dense pour la topologie euclidenne dans $\mathscr{B}(\mathbb{C})$, bien que son intérieur soit vide [Lef50]. La situation sur $\mathbb{R}$ s'avère être plus délicate que la situation sur $\mathbb{C}$. Par analogie avec ce qui précède, on peut définir le *lieu de Noether-Lefschetz réel* comme l'ensemble des $t \in \mathscr{B}(\mathbb{R})$ tel que $\operatorname{Pic}(\mathscr{S}_t) \ncong \mathbb{Z}$. En contraste avec la situation complexe, il existe pour $d = 4$ une composante connexe K de $\mathscr{B}(\mathbb{R})$ dont l'intersection avec le lieu de Noether-Lefschetz réel est vide. Il existe un critère de Green–Voisin sur les réels [Ben18, Proposition 1.1], mais son hypothèse est plus compliquée et il ne s'applique qu'à une composante connexe de $\mathscr{B}(\mathbb{R})$ à la fois.

Il est remarquable que pour le problème analogue pour les variétés abéliennes, aucune de ces difficultés n'apparaît. Les applications du Théorème 9.3 sont donc pléthores :

Théorème 9.4. *Soient k et g deux entiers positifs avec $k < g$.*

1. *Les points dans $|\mathcal{A}_g(\mathbb{R})|$ correspondant à des variétés abéliennes réelles contenant une sous-variété abélienne réelle de dimension k sont denses dans $|\mathcal{A}_g(\mathbb{R})|$.*
2. *Si $k \leq 3 \leq g$, les points dans $|\mathcal{M}_g(\mathbb{R})|$ correspondant à des courbes algébriques réelles C admettant un morphisme $f\colon C \to A$ vers une variété abélienne A de dimension k tel que $f(C(\mathbb{C}))$ engendre le groupe $A(\mathbb{C})$ sont denses dans $|\mathcal{M}_g(\mathbb{R})|$.*
3. *Soit $V \subset \mathbb{P}H^0(\mathbb{P}^2_{\mathbb{R}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(d))$ l'espace des courbes planes réelles lisses de degré d. Les points $t \in V$ tels que la courbe correspondante C_t admet un morphisme non constant $C_t \to E$ vers une courbe elliptique réelle E sont denses dans V.*

9.4 RÉSUMÉ DU CHAPITRE 4: RECOLLER DES QUOTIENTS DE LA BOULE

On verra plus tard (Section 9.5) que chaque composante connexe de l'espace de modules des quintiques binaires réelles lisses est isomorphe à un sous-ensemble ouvert d'un quotient du plan hyperbolique réel par un groupe discret d'isométries.

En fait, il se trouve que ces isomorphismes s'étendent en un isomorphisme entre l'espace de modules réel des quintiques binaires stables et le quotient du plan hyperbolique réel par un groupe de triangles non arithmétique (voir le Théorème 9.7 ci-dessous). Pour établir un tel résultat, nous développons une méthode générale de recollement des espaces quotients hyperboliques réels dans le Chapitre 4, s'appuyant sur les travaux de Allcock, Carlson et Toledo [ACT06, ACT10]. Les variétés unitaires de Shimura fournissent un cadre approprié pour ces techniques de collage ; nous expliquons ici sommairement et dans un langage plus élémentaire comment cela fonctionne.

Soit K un corps CM de degré $2g$ sur $\mathbb{Q}$ d'anneau des entiers $\mathcal{O}_K$. Soit Λ un $\mathcal{O}_K$-module libre de type fini, muni d'une forme hermitienne $h : \Lambda \times \Lambda \to \mathcal{O}_K$. Supposons que la signature de h soit $(n, 1)$ par rapport à un plongement $\tau \colon K \to \mathbb{C}$, et que h soit définie par rapport aux autres places infinies de K. Soit $\mathbb{C}H^n$ l'espace des droites négatives dans $\Lambda \otimes_{\mathcal{O}_K, \tau} \mathbb{C}$ et $P\Gamma = \mathrm{Aut}(\Lambda, h)/\mu_K$ où $\mu_K \subset \mathcal{O}_K^*$ le groupe des unités finies de K. Notons $P\mathscr{A}$ le quotient de l'ensemble des involutions anti-unitaires $\Lambda \to \Lambda$ par μ_K, et $\mathscr{H} = \cup_{h(r,r)=1} \langle r_{\mathbb{C}} \rangle^{\perp} \subset \mathbb{C}H^n$, où $\langle r_{\mathbb{C}} \rangle^{\perp}$ est l'ensemble $\{x \in \mathbb{C}H^n \mid h(x, r) = 0\}$. Supposons que la condition suivante soit satisfaite :

Condition (voir [ACT02b]). *Si $r, t \in \Lambda$ sont des éléments de norme un, et si $h(r, t) \neq 0$, alors soit $\langle r_{\mathbb{C}} \rangle^{\perp} = \langle t_{\mathbb{C}} \rangle^{\perp} \subset \mathbb{C}H^n$, soit $\langle r_{\mathbb{C}} \rangle^{\perp} \cap \langle t_{\mathbb{C}} \rangle^{\perp} = \emptyset \subset \mathbb{C}H^n$.*

Cette condition est toujours vérifiée lorsque le corps K vérifie certaines hypothèses (voir les Conditions 4.49), satisfaites lorsque K est quadratique imaginaire ou égal à $\mathbb{Q}(\zeta_p)$ pour un nombre premier $p > 2$ (voir le Théorème 4.50 et le Lemme 4.52).

Nous fournissons un moyen canonique de recoller les différentes copies $\mathbb{R}H^n_\alpha := (\mathbb{C}H^n)^\alpha \subset \mathbb{C}H^n, \alpha \in P\mathscr{A}$ de l'espace hyperbolique réel $\mathbb{R}H^n$ le long de l'arrangement d'hyperplans $\mathscr{H}$. Cela donne un espace topologique que nous désignons par Y, sur lequel agit $P\Gamma$. Définissons $P\Gamma_\alpha \subset P\Gamma$ comme étant le stabilisateur de $\mathbb{R}H^n_\alpha$.

Le but du Chapitre 4 est de démontrer le théorème suivant.

Théorème 9.5. *L'espace topologique $P\Gamma \setminus Y$ porte une structure canonique d'orbifold hyperbolique réelle, pour laquelle il existe un plongement ouvert d'orbifolds*

$$\coprod_{\alpha \in P\Gamma \setminus P\mathscr{A}} [P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})] \hookrightarrow P\Gamma \setminus Y.$$

De plus, pour chaque composante connexe $C \subset P\Gamma \setminus Y$ il existe un réseau $P\Gamma_C \subset \mathrm{PO}(n, 1)$ ainsi qu'un isomorphisme d'orbifolds hyperboliques réels $C \cong [P\Gamma_C \setminus \mathbb{R}H^n]$.

9.5 RÉSUMÉ DU CHAPITRE 5: MODULES DES QUINTIQUES BINAIRES RÉELLES

Soit $X \cong \mathbb{A}^6_{\mathbb{R}}$ l'espace affine paramétrant les polynômes homogènes $F \in \mathbb{R}[x,y]$ de degré cinq. Considérons les sous-variétés $X_0 \subset X_s \subset X$ paramétrant les polynômes à racines distinctes, resp. les polynômes à racines de multiplicité au plus deux (c'est-à-dire stables au sens de la théorie des invariants géométriques).

Le but du Chapitre 5 est d'étudier les espaces de modules des *quintiques binaires réelles* stables et lisses

$$\mathcal{M}_s(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \setminus X_s(\mathbb{R}) \supset \mathrm{GL}_2(\mathbb{R}) \setminus X_0(\mathbb{R}) = \mathcal{M}_0(\mathbb{R}).$$

Si $P_s \subset \mathbb{P}^1(\mathbb{C})^5$ est l'espace des cinq-uples $(x_1, \ldots, x_5)$ dans $\mathbb{P}^1(\mathbb{C})$ tels que pour i, j, k deux à deux distincts on n'ait pas $x_i = x_j = x_k$ (c.f. [MS72]), et $P_0 \subset P_s$ le sous-espace des cinq-uples dont toutes les coordonnées sont distinctes, alors

$$\mathcal{M}_0(\mathbb{R}) \cong \mathrm{PGL}_2(\mathbb{R}) \setminus (P_0/\mathfrak{S}_5)(\mathbb{R}) \qquad \text{et} \qquad \mathcal{M}_s(\mathbb{R}) \cong \mathrm{PGL}_2(\mathbb{R}) \setminus (P_s/\mathfrak{S}_5)(\mathbb{R}).$$

Pour $i = 0, 1, 2$, définissons $\mathscr{M}_i$ comme la composante connexe de $\mathcal{M}_0(\mathbb{R})$ paramétrant les $S \subset \mathbb{P}^1(\mathbb{C})$ avec exactement $5 - 2i$ points réels.

Il existe une application de périodes naturelle qui définit un isomorphisme entre $\mathcal{M}_s(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) \setminus X_s(\mathbb{C})$ et un certain quotient de boule arithmétique $P\Gamma \setminus \mathbb{C}H^2$. Sous cet isomorphisme, les quintiques strictement stables correspondent à des points dans un arrangement d'hyperplans $\mathscr{H} \subset \mathbb{C}H^2$ (Théorème 5.14 et Proposition 5.15). En étudiant l'équivariance de l'application des périodes sous des involutions anti-holomorphes $\mathbb{C}H^2 \to \mathbb{C}H^2$ adéquates, nous obtenons le résultat suivant :

Théorème 9.6. *Pour chaque $i \in \{0, 1, 2\}$, l'application des périodes induit un isomorphisme d'orbifolds analytiques réels $\mathscr{M}_i \cong P\Gamma_i \setminus \left(\mathbb{R}H^2 - \mathscr{H}_i\right)$. Ici, $\mathbb{R}H^2$ est le plan hyperbolique réel, $\mathscr{H}_i$ une union de sous-espaces géodésiques dans $\mathbb{R}H^2$ et $P\Gamma_i$ un réseau arithmétique dans $\mathrm{PO}(2,1)$ attaché à une forme quadratique sur $\mathbb{Z}[\zeta_5 + \zeta_5^{-1}]$.*

Le Théorème 9.6 dote chaque composante $\mathscr{M}_i$ d'une métrique hyperbolique naturelle. Puisque l'on peut déformer le type topologique d'une partie à cinq éléments de $\mathbb{P}^1(\mathbb{C})$, stable sous $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, en permettant à deux points d'être recollés, la compactification $\mathcal{M}_s(\mathbb{R}) \supset \mathcal{M}_0(\mathbb{R})$ est connexe. On peut donc se demander si les métriques sur les $\mathscr{M}_i$ s'étendent en une métrique sur $\mathcal{M}_s(\mathbb{R})$ et, le cas écheant, à quoi ressemble l'espace résultant à la frontière. Le résultat principal du Chapitre 4 est le suivant, qui est une application des Théorèmes 9.5 et 9.6 ci-dessus.

Théorème 9.7. *Il existe une métrique hyperbolique complète sur $\mathcal{M}_s(\mathbb{R})$ dont la restriction à $\mathscr{M}_i$ est la métrique induite par le Théorème 9.6. Muni de cette métrique, $\mathcal{M}_s(\mathbb{R})$ est isométrique au triangle hyperbolique d'angles $\pi/3, \pi/5, \pi/10$ (voir Figure 5.1). De là, si*

$$P\Gamma_{3,5,10} = \langle \alpha_1, \alpha_2, \alpha_3 | \alpha_i^2 = (\alpha_1\alpha_2)^3 = (\alpha_1\alpha_3)^5 = (\alpha_2\alpha_3)^{10} = 1 \rangle,$$

il existe un homéomorphisme

$$\mathcal{M}_s(\mathbb{R}) \cong P\Gamma_{3,5,10} \setminus \mathbb{R}H^2 \quad \textit{et un plongement} \quad \coprod_i P\Gamma_i \setminus \left(\mathbb{R}H^2 - \mathscr{H}_i\right) \subset P\Gamma_{3,5,10} \setminus \mathbb{R}H^2$$

compatibles avec les isomorphismes $\mathscr{M}_i \cong P\Gamma_i \setminus \left(\mathbb{R}H^2 - \mathscr{H}_i\right)$ du Théorème 9.6.

9.6 RÉSUMÉ DU CHAPITRE 6: TRANSFORMATIONS DE FOURIER ENTIÈRES

(d'après un travail en commun avec Thorsten Beckmann)

Soit g un entier positif et A une variété abélienne de dimension g sur un corps k. *Les transformations de Fourier* sont des correspondances attachées au faisceau de Poincaré $\mathcal{P}_A$ sur $A \times \widehat{A}$ entre les catégories dérivées, les groupes de Chow rationnels et la cohomologie de A et de $\widehat{A}$ [Muk81, Bea82, Huy06]. La transformation de Fourier sur les groupes de Chow rationnels

$$\mathcal{F}_A \colon \mathrm{CH}(A)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}(\widehat{A})_{\mathbb{Q}} \tag{9.3}$$

fournit un outil puissant d'étude des cycles algébriques sur A. Au niveau de la cohomologie, la transformation de Fourier préserve la cohomologie étale ℓ-adique entière lorsque k est séparablement clos, et la cohomologie de Betti entière lorsque $k = \mathbb{C}$. Il est donc naturel de se demander si la transformation de Fourier (9.3) se relève en un homomorphisme entre groupes de Chow entiers. Cette question a été soulevée par Moonen–Polishchuk dans [MP10] et par Totaro dans [Tot21], et sera explorée dans le Chapitre 6. Considérons le résultat suivant :

Théorème (Moonen–Polishchuk). *1. Soit k un corps et C une courbe hyperelliptique sur k telle que l'un des points de Weierstrass de C est défini sur k. Pour la jacobienne*

$A = J(C)$ de C, la transformation de Fourier (9.3) se relève en un homomorphisme motivique $\mathcal{F}\colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$.

2. *Soit $g \in \mathbb{Z}_{\geq 2}$ et considérons le champ de modules $\mathcal{M}_{g,1}$. Soit K son corps de fonctions et (C, p_0) sont point générique. Pour la jacobienne $A = J(C)$ de C, la transformation de Fourier (9.3) se ne se relève pas en un homomorphisme $\mathcal{F}\colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$.*

Nous établissons le résultat suivant :

Théorème 9.8. *Soit k un corps, A une variété abélienne sur k de dimension $g \geq 1$, et $\Theta \in \mathrm{CH}^1(A)$ la classe d'un diviseur induisant une polarisation principale sur A.*

1. *Si le cycle rationnel $\Theta^{g-1}/(g-1)! \in \mathrm{CH}_1(A)_{\mathbb{Q}}$ se relève en un cycle entier $\Gamma \in \mathrm{CH}_1(A)$, alors la transformation de Fourier (9.3) se relève en un homomorphisme motivique $\mathcal{F}\colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$.*

2. *Si $(-1)^*\Theta = \Theta \in \mathrm{CH}^1(A)$, l'inverse de l'assertion 1 est également vrai.*

9.7 RÉSUMÉ DU CHAPITRE 7: CYCLES ALGÉBRIQUES SUR LES JACOBIENNES

(d'après un travail en commun avec Thorsten Beckmann)

Savoir si la transformation de Fourier (9.3) se relève aux groupes de Chow entiers a des conséquences importantes pour l'image de l'application classe de cycle. Rappelons que, bien que la conjecture de Hodge entière échoue en général [AH62, tre92, Tot97], elle reste une question ouverte pour les variétés abéliennes. En nous appuyant sur les résultats du Chapitre 6, nous prouvons ce qui suit au Chapitre 7 :

Théorème 9.9. *Soit A une variété abélienne complexe de dimension $g \geq 1$ avec fibré de Poincaré $\mathcal{P}_A$. Les trois assertions suivantes sont équivalentes :*

1. *La classe de cohomologie $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ est algébrique.*

2. *Le caractère de Chern $\mathrm{ch}(\mathcal{P}_A) = \exp(c_1(\mathcal{P}_A)) \in \mathrm{H}^\bullet(A \times \widehat{A}, \mathbb{Z})$ est algébrique.*

3. *La conjecture de Hodge entière pour les courbes est valable pour $A \times \widehat{A}$.*

L'une quelconque de ces assertions implique la conjecture entière de Hodge pour les courbes sur A. De plus, si A est principalement polarisée par $\theta \in \mathrm{Hdg}^2(A, \mathbb{Z})$, les énoncés 1 - 3 sont équivalents à l'algébricité de la classe $\theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$.

Nous obtenons comme corollaire le résultat suivant :

Théorème 9.10. *Soient $C_1, \dots, C_n$ des courbes projectives lisses sur $\mathbb{C}$. La conjecture de Hodge entière pour les courbes vaut pour le produit des jacobiennes $J(C_1) \times \cdots \times J(C_n)$.*

De plus, à l'aide du Théorème 9.9, nous démontrons :

Théorème 9.11. *Soit $g \in \mathbb{Z}_{\geq 2}$. Il existe une union dénombrable $X \subset \mathcal{A}_g(\mathbb{C})$ de sous-variétés algébriques de dimension au moins $3g-3$, satisfaisant les conditions suivantes: X est dense pour la topologie euclidienne dans $\mathcal{A}_g(\mathbb{C})$ et la conjecture de Hodge entière pour les courbes vaut pour les variétés abéliennes définissant un point dans X.*

La généralité de la théorie des transformations de Fourier entières que nous avons développée (voir la Section 9.6) permet d'obtenir des résultats similaires en caractéristique non nulle. Étant donnée k, la clôture séparable d'un corps finiment engendré, une variété projective lisse X de dimension n sur k satisfait la *conjecture de Tate entière pour les courbes sur k* si pour tout nombre premier $\ell \neq \mathrm{char}(k)$ et tout corps de définition finiment engendré $k_0 \subset k$ de X, l'application classe de cycle

$$cl\colon \mathrm{CH}_1(X)_{\mathbb{Z}_\ell} = \mathrm{CH}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \bigcup_U \mathrm{H}^{2n-2}_{\text{ét}}(X, \mathbb{Z}_\ell(n-1))^U \tag{9.4}$$

est surjective, où U parcourt les sous-groupes ouverts de $\mathrm{Gal}(k/k_0)$.

Théorème 9.12. *Soit (A, λ) une variété abélienne principalement polarisée sur k, de dimension g et de classe de polarisation $\theta_\ell \in \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1))$. L'application classe de cycle (9.4) est surjective si $\theta_\ell^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}_{\text{ét}}(A, \mathbb{Z}_\ell(g-1))$ est dans son image.*

Observer que la condition dans le Théorème 9.12 est toujours vérifiée si $\ell > (g-1)!$. Nous obtenons comme corollaire le résultat suivant :

Théorème 9.13. *Soient $C_1, \dots, C_n$ des courbes projectives lisses sur k. La conjecture de Tate entière pour les courbes sur k vaut pour le produit des jacobiennes $J(C_1) \times \cdots \times J(C_n)$.*

Soit k la clôture algébrique d'un corps finiment engendré de caractéristique non nulle, et soit X une variété projective lisse sur k. La *conjecture entière de Tate pour les courbes sur k* est l'analogue de la propriété introduite ci-dessus.

Théorème 9.14. *Soit A_g l'espace de modules grossier des variétés abéliennes principalement polarisées de dimension g sur k. Les points de $\mathsf{A}_g(k)$ correspondant à des variétés abéliennes satisfaisant la conjecture de Tate entière pour les courbes sur k sont Zariski denses dans A_g.*

9.8 RÉSUMÉ DU CHAPITRE 8: COURBES SUR LES SOLIDES ABÉLIENS RÉELS

Posons $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Pour toute variété projective lisse X sur $\mathbb{R}$, Benoist et Wittenberg [BW20a] ont défini un sous-groupe $\mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0$ du groupe de cohomologie équivariant $\mathrm{H}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$ et ont étudié l'application classe de cycle

$$\mathrm{CH}_i(X) \to \mathrm{Hdg}^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))_0 \qquad (i + k = \dim(X))\,. \tag{9.5}$$

La *conjecture de Hodge entière réelle pour les i-cycles* questionne la surjectivité de (9.5). Comme dans la situation complexe, la surjectivité de (9.5) est valable pour toute variété projective lisse X sur $\mathbb{R}$ si $i \in \{\dim(X), \dim(X) - 1, 0\}$ [Kra91, MvH98, Ham97, BW20a], mais peut échouer pour d'autres valeurs de $i \in \{0, \dots, \dim(X)\}$.

En utilisant la théorie du Chapitre 6, nous fournissons dans le Chapitre 8 des résultats soutenant la conjecture de Hodge entière réelle pour les solides abéliens.

Théorème 9.15. *Tout solide abélien A sur $\mathbb{R}$ satisfait la conjecture de Hodge entière réelle modulo la torsion, au sens où l'homomorphisme suivant est surjectif :*

$$\mathrm{CH}_1(A) \ \to \mathrm{Hdg}^4_G(A(\mathbb{C}), \mathbb{Z}(2))_0/(\mathrm{torsion}) = \mathrm{Hdg}^4(A(\mathbb{C}), \mathbb{Z}(2))^G.$$

Nous obtenons comme corollaire le résultat suivant :

Théorème 9.16. *Soit A un solide abélien sur $\mathbb{R}$ tel que $A(\mathbb{R})$ est connexe. Alors A satisfait la conjecture de Hodge entière réelle.*

En ce qui concerne la conjecture de Hodge entière réelle pour les solides abéliens principalement polarisés, nous démontrons le théorème de réduction suivant :

Théorème 9.17. *Soit $\mathcal{A}_3(\mathbb{R})^+$ une composante connexe de $\mathcal{A}_3(\mathbb{R})$, l'espace de modules des solides abéliens réels principalement polarisés. Supposons que la conjecture de Hodge entière réelle vaut pour toute jacobienne $J(C)$ tel que $[J(C)] \in \mathcal{A}_3(\mathbb{R})^+$ et $C(\mathbb{R}) \neq \emptyset$. Alors la conjecture de Hodge entière réelle vaut pour toute variété abélienne dans $\mathcal{A}_3(\mathbb{R})^+$.*

L'espace de modules $\mathcal{A}_3(\mathbb{R})$ a quatre composantes connexes, correspondant aux types topologiques des lieux réels des solides abéliens réels principalement polarisés. Dans trois de ces quatre composantes, il n'y a pas d'obstruction topologique à la conjecture de Hodge entière réelle, comme suit de notre dernier résultat :

Théorème 9.18. *Soit B une surface abélienne réelle, et E une courbe elliptique avec $E(\mathbb{R})$ connexe. Le solide abélien réel $A = B \times E$ satisfait la conjecture de Hodge entière réelle.*

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ABSTRACT

This thesis intends to make a contribution to the theories of algebraic cycles and moduli spaces over the real numbers. In the study of the subvarieties of a projective algebraic variety, smooth over the field of real numbers, the cycle class map between the Chow ring and the equivariant cohomology ring plays an important role. The image of the cycle class map remains difficult to describe in general; we study this group in detail in the case of real abelian varieties. To do so, we construct integral Fourier transforms on Chow rings of abelian varieties over any field. They allow us to prove the integral Hodge conjecture for one-cycles on complex Jacobian varieties, and the real integral Hodge conjecture modulo torsion for real abelian threefolds.

For the theory of real algebraic cycles, and for several other purposes in real algebraic geometry, it is useful to have moduli spaces of real varieties to our disposal. Insight in the topology of a real moduli space provides insight in the geometry of a real variety that defines a point in it, and the other way around. In the moduli space of real abelian varieties, as well as in the Torelli locus contained in it, we prove density of the set of moduli points attached to abelian varieties containing an abelian subvariety of fixed dimension. Moreover, we provide the moduli space of stable real binary quintics with a hyperbolic orbifold structure, compatible with the period map on the locus of smooth quintics. This structure identifies the moduli space of stable real binary quintics with a non-arithmetic ball quotient.

KEYWORDS

Moduli spaces, algebraic cycles, real algebraic geometry, Hodge theory.

RÉSUMÉ

À travers ce manuscrit, nous souhaitons apporter une contribution aux théories des cycles algébriques et des espaces de modules sur les nombres réels. Dans l'étude des sous-variétés d'une variété algébrique projective lisse sur le corps des nombres réels, l'application classe de cycle entre l'anneau de Chow et l'anneau de cohomologie équivariante joue un rôle important. L'image de l'application classe de cycle reste difficile à décrire en général ; nous étudions ce groupe en détail dans le cas des variétés abéliennes réelles. Pour ce faire, nous construisons des transformations de Fourier entières sur les anneaux de Chow des variétés abéliennes sur un corps quelconque. Elles nous permettent de prouver la conjecture de Hodge entière pour les courbes dans le cas des variétés jacobiennes complexes, ainsi que la conjecture de Hodge entière réelle modulo torsion pour les solides abéliens réels.

Pour l'étude des cycles algébriques réels, de même que pour plusieurs autres questions en géométrie algébrique réelle, il est utile de disposer d'espaces de modules de variétés réelles. La connaissance de la topologie d'un espace de modules réel permet de comprendre la géométrie d'une variété réelle qui définit un point dans cet espace et inversement. Dans l'espace de modules des variétés abéliennes réelles, ainsi que dans le lieu de Torelli qu'il contient, nous prouvons la densité de l'ensemble des points attachés aux variétés abéliennes contenant une sous-variété abélienne de dimension fixe. De plus, nous munissons l'espace de modules des quintiques binaires réelles stables d'une structure d'orbifold hyperbolique, compatible à l'application des périodes sur le lieu des quintiques binaires lisses. Cette structure identifie l'espace de modules des quintiques binaires réelles stables avec un quotient non arithmétique de la boule réelle.

MOTS CLÉS

Espaces de modules, cycles algébriques, géométrie algébrique réelle, théorie de Hodge.