LIFTING HARMONIC MORPHISMS II: TROPICAL CURVES AND METRIZED COMPLEXES

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ABSTRACT. In this paper we prove several lifting theorems for morphisms of tropical curves. We interpret the obstruction to lifting a finite harmonic morphism of augmented metric graphs to a morphism of algebraic curves as the non-vanishing of certain Hurwitz numbers, and we give various conditions under which this obstruction does vanish. In particular we show that any finite harmonic morphism of (non-augmented) metric graphs lifts. We also give various applications of these results. For example, we show that linear equivalence of divisors on a tropical curve C coincides with the equivalence relation generated by declaring that the fibers of every finite harmonic morphism from C to the tropical projective line are equivalent. We study liftability of metrized complexes equipped with a finite group action, and use this to classify all augmented metric graphs arising as the tropicalization of a hyperelliptic curve. We prove that there exists a d-gonal tropical curve that does not lift to a d-gonal algebraic curve.

This article is the second in a series of two.

Throughout this paper, unless explicitly stated otherwise, K denotes a complete algebraically closed non-Archimedean field with nontrivial valuation $\mathrm{val} : K \to \mathbf{R} \cup \{\infty\}$. Its valuation ring is denoted R, its maximal ideal is $\mathfrak{m}_R$, and the residue field is $k = R/\mathfrak{m}_R$. We denote the value group of K by $\Lambda = \mathrm{val}(K^\times) \subset \mathbf{R}$.

1. INTRODUCTION

This article is the second in a series of two. The first, entitled *Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta*, will be cited as [ABBR13]; references of the form "Theorem I.1.1" will refer to Theorem 1.1 in [ABBR13].

1.1. The basic motivation behind the investigations in this paper is to understand the relationship between tropical and algebraic curves. A fundamental problem along these lines is to understand which morphisms between tropical curves arise as tropicalizations[1] of morphisms of algebraic curves. More precisely, we are interested in the following question:

(Q) Given a curve X with tropicalization C, can we classify the branched covers of X in terms of (a suitable notion of) branched covers of C?

In addition to this lifting problem for morphisms of tropical curves, we also study questions such as "Which tropical curves arise as tropicalizations of hyperelliptic curves?". This naturally leads us to study group actions on tropical curves and how notions such as gonality change under tropicalization.

In this paper we will consider three different kinds of "tropical" objects which one can associate to a smooth, proper, connected algebraic curve X/K, each depending on the choice of a *triangulation* of X. Roughly speaking, a triangulation $(X, V \cup D)$ of X (with respect to a finite set of punctures $D \subset X(K)$) is a finite set V of points in the Berkovich analytification X^{an} of X whose removal partitions X^{an} into open balls and finitely many open annuli (with the punctures belonging to distinct open balls). Triangulations of (X, D) are naturally in one-to-one correspondence with semistable models $\mathfrak{X}$ of (X, D); see Section I.5. The skeleton of a triangulated curve is the dual graph of the

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[1] In the present paper tropicalization is defined via Berkovich's theory of analytic spaces (see also [Pay09], [BPR11], [CLD12]). Another framework for tropicalization has been proposed by Kontsevich-Soibelman [KS01] and Mikhalkin (see for example [Mik06]), where the link between tropical geometry and complex algebraic geometry is provided by real one-parameter families of complex varieties. For some conjectural relations between the two approaches see [KS01, KS06].

special fiber $\mathfrak{X}_k$ of the corresponding semistable model, with infinite rays for the punctures, equipped with its canonical metric.

To any triangulated curve, one may associate the three following "tropical" objects, at each step adding some additional structure:

(1) a *metric graph* Γ: this is the skeleton of the triangulated curve $(X, V \cup D)$;
(2) an *augmented metric graph* (Γ, g), i.e., a metric graph Γ enhanced with a genus function $g : \Gamma \to \mathbf{Z}_{\geq 0}$ which is non-zero only at finitely many points: this is the above metric graph together with the function g satisfying $g(p) = 0$ for $p \notin V$ and $g(p) = \mathrm{genus}(C_p)$ for $p \in V$, where C_p is the (normalization of the) irreducible component of $\mathfrak{X}_k$ corresponding to p;
(3) a *metrized complex of curves* $\mathcal{C}$, i.e., an augmented metric graph Γ equipped with a vertex set V and a punctured algebraic curve over k of genus $g(p)$ for each point $p \in V$, with the punctures in bijection with the tangent directions to p in Γ: this is the above metric graph, together with the curves C_p for $p \in V$ and punctures given by the singular points of $\mathfrak{X}_k$.

An (augmented) metric graph or metrized complex of curves arising from a triangulated curve by the above procedure is said to be *liftable*. If $(X, V \cup D)$ and $(X, V' \cup D')$ are triangulations of the same curve X, with $D' \subset D$ and $V' \subset V$, then the corresponding metric graphs are related by a so-called *tropical modification*. Tropical modifications generate an equivalence relation on the set of (augmented) metric graphs, and an equivalence class for this relation is called an *(augmented) tropical curve*. The (augmented) *tropicalization* of a K-curve X is by definition the (augmented) tropical curve C corresponding to any triangulation of X. Tropical curves and augmented tropical curves can be thought of as "purely combinatorial" objects, whereas metrized complexes are a mixture of combinatorial objects (which one thinks of as living over the value group Λ of K) and algebro-geometric objects over the residue field k of K.

There is a natural notion of *finite harmonic morphism* between metric graphs which induces a natural notion of *tropical morphism* between tropical curves. There is a corresponding notion of tropical morphism for *augmented* tropical curves, where in addition to the harmonicity condition one imposes a "Riemann–Hurwitz condition" that the ramification divisor is effective. There is also a natural notion of finite harmonic morphism for metrized complexes of curves. Each kind of object (metric graphs, tropical curves, augmented tropical curves, metrized complexes) forms a category with respect to the corresponding notion of morphism. The construction of an (augmented) tropical curve C (resp. metrized complex $\mathcal{C}$) out of a K-curve X (resp. triangulated K-curve $(X, V \cup D)$) is functorial, in the sense that a finite morphism of curves induces in a natural way a tropical morphism $C' \to C$ (resp. a finite harmonic morphism $\mathcal{C}' \to \mathcal{C}$).

1.2. Our original question (Q) now breaks up into the following two separate questions.

(Q1) Which finite harmonic morphisms $\mathcal{C}' \to \mathcal{C}$ of metrized complexes can be lifted to finite morphisms of triangulated curves (with a pre-specified lift X of $\mathcal{C}$)?
(Q2) Which tropical morphisms between augmented tropical curves can be lifted to finite harmonic morphisms of metrized complexes?

One can also forget the augmentation function $g : \Gamma \to \mathbf{Z}_{\geq 0}$ and ask the following variant of (Q2):

(Q2$'$) Which tropical morphisms between tropical curves can be lifted to finite harmonic morphisms of metrized complexes?

A consequence of the results of [ABBR13] is that the answer to question (Q1) is essentially "all", so the situation here is rather satisfactory; there is no obstruction to lifting a finite harmonic morphism $\mathcal{C}' \to \mathcal{C}$ to a branched cover of X, at least assuming everywhere tame ramification when k has characteristic $p > 0$. In particular, if $\mathrm{char}(k) = 0$ then there are no tameness issues, and we have the following result:

Theorem. *Assume* $\mathrm{char}(k) = 0$ *and let* $\varphi : \Sigma' \to \Sigma$ *be a finite harmonic morphism of* Λ*-metrized complexes of* k*-curves. Then there exists a finite morphism of triangulated punctured curves lifting* φ*.*

This follows immediately from Proposition I.7.15 and Theorem I.3.24. We stress that the genus and degree are automatically preserved by such lifts.

Essentially by definition, (Q2) reduces to an existence problem for ramified coverings $\varphi_{p'} : C'_{p'} \to C_p$ of a given degree with some prescribed ramification profiles. Hence the answer to (Q2) is intimately linked with the question of non-vanishing of Hurwitz numbers. See Proposition 3.3. In particular one can easily construct tropical morphisms between augmented tropical curves which cannot be promoted to a finite harmonic morphism of metrized complexes (and hence cannot be lifted to a finite morphism of smooth proper curves over K). The simplest example of such a tropical morphism is depicted in Figure 1, and corresponds to the classical fact that although it would not violate the Riemann–Hurwitz formula, there is no degree 4 map of smooth proper connected curves over **C** having ramification profile $\{(2,2),(2,2),(3,1)\}$; this is a consequence of the (easy part of the) Riemann Existence Theorem (see Example 3.4 below for more details).

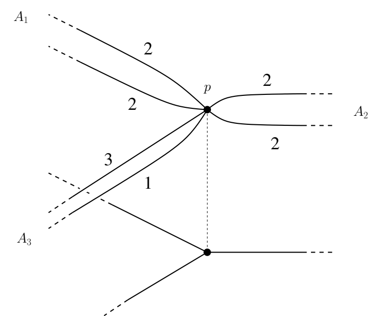


FIGURE 1. A tropical morphism of degree four which cannot be promoted to a degree 4 morphism of metrized complexes of curves. The labels on the edges are the "expansion factors" of the harmonic morphism. See Definition I.2.4.

Understanding when Hurwitz numbers vanish remains mysterious in general, so at present there is no satisfying "combinatorial" answer to question (Q2), in which we require that the genus of the objects in question be preserved by our lifts. However, if we drop the latter condition, i.e., if we consider instead question (Q2′), we will see that the answer to (Q2′) is also "all" (see Theorem 3.11):

Theorem. *Any finite harmonic morphism $\overline{\varphi} : \Gamma' \to \Gamma$ of Λ-metric graphs is liftable if $\mathrm{char}(k) = 0$.*

1.3. Applications. We prove a number of additional results which supplement and provide applications of the above results. Some of these results are as follows.

1.3.1. Tame group actions. Let $\mathcal{C}$ be a metrized complex and let H be a finite subgroup of $\mathrm{Aut}(\mathcal{C})$. We say the action of H on $\mathcal{C}$ is *tame* if for any vertex p of Γ, the stabilizer group H_p acts freely on a dense open subset of C_p, and for any point x of C_p, the stabilizer subgroup H_x of H is cyclic of the form $\mathbf{Z}/d\mathbf{Z}$ for some integer d, with $(d, p) = 1$ if $\mathrm{char}(k) = p > 0$ (see Remark 4.6 for further explanation of this condition). It follows from Theorem I.7.4 (in its strong form, i.e. using the calculation of the automorphism group of a lift) that we can lift $\mathcal{C}$ together with a tame action of H if and only if the quotient $\mathcal{C}/H$ exists in the category of metrized complexes. We characterize when such a quotient exists in Theorem 4.9, of which the following result is a special case:

Theorem. *Suppose that the action of H is tame and has no isolated fixed points on the underlying metric graph of $\mathcal{C}$. Then there exists a smooth, proper, and geometrically connected algebraic K-curve X lifting $\mathcal{C}$ which is equipped with an action of H commuting with the tropicalization map.*

In presence of isolated fixed points, there are additional hypothesis on the action of H to be liftable to a K-curve. As a concrete example, we prove the following characterization of all augmented tropical curves arising as the tropicalization of a hyperelliptic K-curve (see Corollary 4.15):

Theorem. *Let Γ be an augmented metric graph of genus $g \geq 2$ having no infinite vertices or degree one vertices of genus 0. Then there is a smooth proper hyperelliptic curve X over K of genus g having Γ as its minimal skeleton if and only if (a) there exists an involution s on Γ such that s fixes all the points $p \in \Gamma$ with $g(p) > 0$ and the quotient Γ/s is a metric tree, and (b) for every $p \in \Gamma$ the number of bridge edges adjacent to p is at most $2g(p) + 2$.*

1.3.2. Gonality of tropical curves. The *tropical projective line* is the augmented tropical curve $\mathbf{TP}^1$ represented by any tree with genus function identically zero. See Example 2.15. An augmented tropical curve C is called *d-gonal* if there exists a tropical morphism of degree d from C to $\mathbf{TP}^1$. By Corollary I.4.28, the gonality of an augmented tropical curve is always a lower bound for the gonality of any lift to a smooth proper curve over K. (See Remark 5.3 for a discussion of the various notions of gonality of tropical curves existing in the literature.) We prove in Section 5 that none of the lower bounds provided by tropical ranks and gonality are sharp. For example:

Theorem.

(1) *There exists an augmented tropical curve C of gonality 4 such that the gonality of any lifting of C is at least 5.*
(2) *There exists an effective divisor D on a tropical curve C such that D has tropical rank equal to 1, but any effective lifting of D has rank 0.*

The construction in (1) uses the vanishing of the degree 4 Hurwitz number $H^4_{0,0}((2,2),(2,2),(3,1))$. In fact we prove in Theorem 5.4 a much stronger statement: we exhibit an augmented (non-metric) graph G such that *none* of the augmented tropical curves with G as underlying augmented graph can be lifted to a 4-gonal K-curve. This means that there is a finite graph with *stable gonality* 4 (in the sense of [CKK]) which is not the (augmented) dual graph of any 4-gonal curve X/K.

The proof of (2) is based on our lifting results and an explicit example, due to Luo (see Example 5.13), of a degree 3 and rank 1 base-point free divisor D on a tropical curve C which does not appear as the fiber of any degree 3 tropical morphism from C to $\mathbf{TP}^1$.

1.3.3. Linear equivalence of divisors. When the target curve has genus zero, we investigate in (3.16) a variant of question (Q2′) in which the genus of the source curve may be prescribed, at the cost of losing control over the degree of the morphism. As an application, we show in Theorem 4.3 that linear equivalence of divisors on a tropical curve C coincides with the equivalence relation generated by declaring that the fibers of every tropical morphism from C to the tropical projective line $\mathbf{TP}^1$ are equivalent:

Theorem. *Let Γ be a metric graph. Linear equivalence of divisors on Γ is the additive equivalence relation generated by (the retraction to Γ of) fibers of finite harmonic morphisms from a tropical modification of Γ to a metric graph of genus zero.*

1.4. Organization of the paper. The paper is organized as follows. Precise definitions of tropical modifications and tropical curves are given in Section 2, along with various kinds of morphisms between these objects. In that section we also use results from [ABBR13] to define tropicalizations of morphisms of curves, and provide a number of examples. Lifting results for (augmented) metric graphs and tropical curves are proved in Section 3. Section 4 contains applications of our lifting results. For example, lifting results for metrized complexes equipped with a finite group action are discussed in (4.5). In (4.5) we also give a complete classification of all hyperelliptic augmented tropical curves which can be realized as the minimal skeleton of a hyperelliptic curve. Finally, in Section 5 we study tropical rank and gonality and related lifting questions.

1.5. Related work. The definition of effective harmonic morphisms of augmented metric graphs that we use is the same as in [BBM11]. The closely related, but slightly different, notion of an "indexed harmonic morphism" between weighted graphs was considered in [Cap]. The indexed pseudo-harmonic (resp. harmonic) morphisms in [Cap] are closely related to harmonic (resp. effective harmonic) morphisms in our sense when the vertex sets are fixed (see Definition I.2.4), and non-degenerate morphisms in the sense of [Cap] correspond to finite morphisms in our sense. One notable difference is that in [Cap], only the combinatorial type of the metric graphs are fixed; the choice of positive indices in an indexed pseudo-harmonic morphism determines the length of the edges in the source graph once the edge lengths in the target are fixed.

Tropical modifications and the "up-to-tropical-modification" point of view were introduced by Mikhalkin [Mik06].

In (5.1) we propose a definition for the stable gonality of a graph which coincides with the one used by Cornelissen et. al. in their recent preprint [CKK]. A slightly different notion of gonality for graphs was introduced by Caporaso in [Cap]. We also define the gonality of an augmented tropical curve, which strikes us as a more natural and perhaps more useful notion than the stable gonality of a graph (where the lengths of the edges in the source and target metric graphs are not pre-specified). We emphasize the importance of considering the dual graph of the special fiber of a semistable model of a smooth proper K-curve as an (augmented) *metric graph* and not just as a (vertex-weighted) graph. Keeping track of the natural edge lengths allows us to avoid pathological examples like Example 2.18 in [Cap] of a 3-gonal graph which is not divisorally 3-gonal.

The question of lifting effective harmonic morphisms of metric graphs also occurs naturally (in a related but different Archimedean framework) when one considers degenerating families of complex algebraic dynamical systems; see for example [DM08, Theorems 1.2 and 7.1] where DeMarco and McMullen prove a lifting theorem for polynomial-like endomorphisms of (locally finite) simplicial trees which has applications to studying dynamical compactifications of the moduli space of degree d polynomial maps. Our Theorem 3.15 was inspired by the results of DeMarco–McMullen.

2. Algebraic and tropical curves

In this section we introduce tropical curves and morphisms between them. We use the results of [ABBR13] to define functorial "intrinsic tropicalizations" of algebraic curves. We will freely use the definitions and notations in Section I.2. We reproduce some of them here for the convenience of the reader.

2.1. Metric graphs. A Λ-*metric graph* is a metric graph whose edge lengths belong to Λ. The length of an embedded segment e in a metric graph Γ is denoted $\ell(e)$. The set of tangent directions at a point p of Γ is denoted $T_p(\Gamma)$. To a harmonic morphism $\varphi : \Gamma' \to \Gamma$ of metric graphs are associated its degree $\deg(\varphi)$, its degree at a point $d_{p'}(\varphi)$, the degree of φ along an edge (also called the expansion factor) $d_{e'}(\varphi)$, the directional derivative of φ along a tangent direction at a vertex $d_{v'}(\varphi)$, and the induced map on tangent spaces $d\varphi(p')$ when $d_{p'}(\varphi) \neq 0$.

The group of divisors on a metric graph Γ is denoted $\mathrm{Div}(\Gamma)$. A harmonic morphism of metric graphs $\varphi : \Gamma' \to \Gamma$ gives rise to push-forward and pull-back homomorphisms $\varphi_* : \mathrm{Div}(\Gamma') \to \mathrm{Div}(\Gamma)$ and $\varphi^* : \mathrm{Div}(\Gamma) \to \mathrm{Div}(\Gamma')$ defined by

$$\varphi^*(p) = \sum_{p' \mapsto p} d_{p'}(\varphi)\,(p') \quad \text{and} \quad \varphi_*(p') = (\varphi(p'))$$

and extending linearly. It is clear that for $D \in \mathrm{Div}(\Gamma)$ we have $\deg(\varphi^*(D)) = \deg(\varphi) \cdot \deg(D)$ and $\deg(\varphi_*(D)) = \deg(D)$.

2.2. Augmented metric graphs. An augmented metric graph Γ has a genus function $g : \Gamma \to \mathbf{Z}_{\geq 0}$. We say that Γ is *totally degenerate* provided that g is identically zero. The genus of Γ is

$$g(\Gamma) = h_1(\Gamma) + \sum_{p \in \Gamma} g(p),$$

where $h_1(\Gamma)$ is the first Betti number of Γ. If $g(\Gamma) = 0$ then we say that Γ is *rational*. The canonical divisor of an augmented metric graph Γ is

$$K_\Gamma = \sum_{p\in\Gamma} (\mathrm{val}(p) + 2g(p) - 2)\,(p).$$

The degree of K_Γ is $\deg(K_\Gamma) = 2g(\Gamma) - 2$.

Let $\varphi : \Gamma' \to \Gamma$ be a harmonic morphism of augmented metric graphs. The *ramification divisor* of φ is the divisor $R = \sum R_{p'}(p')$, where for $p' \in \Gamma'$,

$$R_{p'} = d_{p'}(\varphi)\cdot\big(2 - 2g(\varphi(p'))\big) - \big(2 - 2g(p')\big) - \sum_{v'\in T_{p'}(\Gamma')} \big(d_{v'}(\varphi) - 1\big).$$

We have the Riemann–Hurwitz formula

$$K_{\Gamma'} = \varphi^*(K_\Gamma) + R.$$

We say that φ is *generically étale* if R is supported on the set of infinite vertices of Γ and is *étale* if $R = 0$.

2.3. Effective harmonic morphisms. We will use the following Riemann–Hurwitz condition in formulating lifting problems for harmonic morphisms of augmented metric graphs. Given a vertex $p' \in V(\Gamma')$ with $d_{p'}(\varphi) \neq 0$, we define the *ramification degree* of φ at p' to be

$$r_{p'} = R_{p'} - \#\{v' \in T_{p'}(\Gamma') \ :\ d_{v'}(\varphi) = 0\}.$$

Clearly $r_{p'} \le R_{p'}$, with $r_{p'} = R_{p'}$ if and only if $d_{v'}(\varphi) > 0$ for any $v' \in T_{p'}(\Gamma')$, i.e. the distinction between ramification divisors and ramification degrees only makes sense for non-finite harmonic morphisms. Our motivation not to restrict ourselves to finite harmonic morphisms is that non-finite harmonic morphisms show up naturally in many practical situations.

Definition 2.4. A harmonic morphism of augmented Λ-metric graphs $\varphi : \Gamma' \to \Gamma$ is said to be *effective* if $r_{p'} \ge 0$ for every finite vertex p' of Γ' with $d_{p'}(\varphi) \neq 0$.

The significance of the number $r_{p'}$ is given in Remark 2.7. In particular, only effective harmonic morphisms of augmented metric graphs have a chance to be liftable to a harmonic morphism of metrized complexes of curves, and possibly to a morphism of triangulated punctured K-curves. See Remark 2.10.

Note that a generically étale morphism of augmented metric graphs is effective.

Example 2.5. Consider the harmonic morphisms of graphs $\varphi : \Gamma' \to \Gamma$ represented in Figure 2. We use the following conventions in our pictures: black dots represent vertices of Γ' and Γ; we label an edge with its degree if and only if the degree is different from 0 and 1; we do not specify the lengths of edges of Γ' and Γ.

The morphisms in Figure 2(a,b,c) are effective provided that all the target graphs are totally degenerate. Suppose that all 1-valent vertices are infinite vertices in Figure 2 (d,e), and that $g(p) = 0$ in Figure 2(e) and $g(p) = 1$ in Figure 2(e). Then $r_{p'} = 2g(p') - 1$ and $r_{p'_i} = 2g(p'_i) - 2$, so the morphism depicted in 2(d) is effective if and only if $g(p') \ge 1$, and the morphism depicted in 2(e) is effective if and only if both vertices p'_1 and p'_2 have genus at least 1.

The morphism in Figure 1 is effective when both graphs are totally degenerate.

2.6. Metrized complexes of curves. Metrized complexes of curves and harmonic morphisms between them are defined in (I.2.16). We recall part of the definitions here. A *Λ-metrized complex of k-curves* $\mathcal{C}$ is the data of an underlying augmented Λ-metric graph Γ with a distinguished vertex set, and for each finite vertex $p \in \Gamma$ a smooth proper connected k-curve C_p of genus $g(p)$, called the *residue curve*, and an injective reduction map $\mathrm{red}_p : T_p(\Gamma) \hookrightarrow C_p(k)$. A *harmonic morphism* $\varphi : \mathcal{C}' \to \mathcal{C}$ is a harmonic morphism of underlying augmented metric graphs $\varphi : \Gamma' \to \Gamma$, taking finite vertices of Γ' to

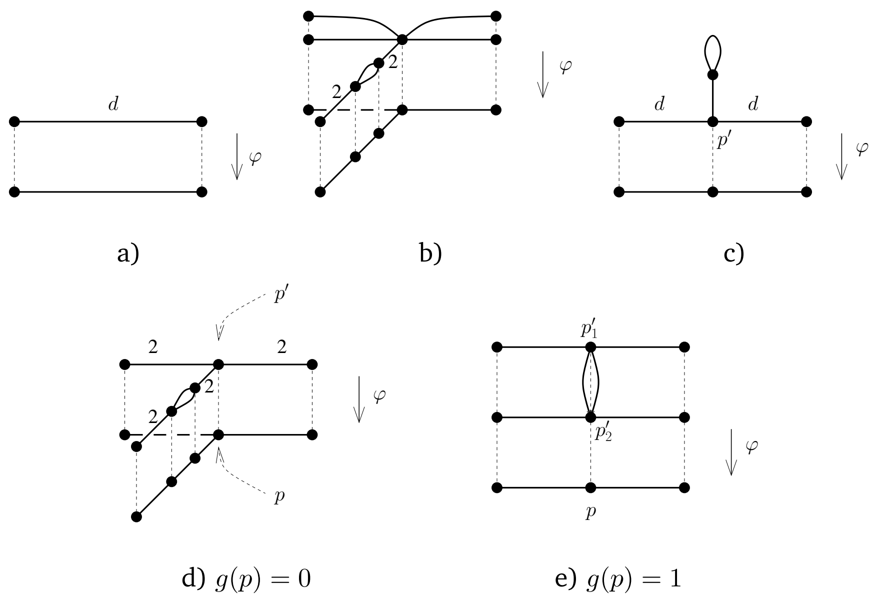


FIGURE 2

finite vertices of Γ, along with a finite morphism $\varphi : C_{p'} \to C_{\varphi(p')}$ for every finite vertex p' of Γ' such that $d_{p'}(\varphi) \neq 0$, satisfying the following compatibility conditions:

(1) For every finite vertex $p' \in V(\Gamma')$ and every tangent direction $v' \in T_{p'}(\Gamma')$ with $d_{v'}(\varphi) > 0$, we have $\varphi_{p'}(\mathrm{red}_{p'}(v')) = \mathrm{red}_{\varphi(p')}(d\varphi(p')(v'))$, and the ramification degree of $\varphi_{p'}$ at $\mathrm{red}_{p'}(v')$ is equal to $d_{v'}(\varphi)$.
(2) For every finite vertex $p' \in V(\Gamma')$ with $d_{p'}(\varphi) > 0$, every tangent direction $v \in T_{\varphi(p')}(\Gamma)$, and every point $x' \in \varphi_{p'}^{-1}(\mathrm{red}_{\varphi(p')}(v)) \subset C'_{p'}(k)$, there exists $v' \in T_{p'}(\Gamma')$ such that $\mathrm{red}_{p'}(v') = x'$.
(3) For every finite vertex $p' \in V(\Gamma')$ with $d_{p'}(\varphi) > 0$ we have $d_{p'}(\varphi) = \deg(\varphi_{p'})$.

Let $\varphi : \mathcal{C}' \to \mathcal{C}$ be a finite harmonic morphism of metrized complexes of curves. We say that φ is a *tame harmonic morphism* if $\varphi_{p'}$ is tamely ramified for all finite vertices $p' \in \Gamma'$. We call φ a *tame covering* if in addition it is a generically étale finite morphism of augmented metric graphs.

Remark 2.7. It follows from the Riemann–Hurwitz formula applied to the maps $\varphi_{p'} : C'_{p'} \to C_{\varphi(p')}$ that a harmonic morphism of metrized complexes of curves gives rise to an *effective* harmonic morphism of augmented metric graphs when each $\varphi_{p'}$ is a separable morphism of curves; the integer $r_{p'}$ is then the sum of ramification indices over all ramification points of $\varphi_{p'}$ not contained in $\mathrm{red}_{p'}(T_{p'}(\Gamma'))$. In particular, tame harmonic morphisms of metrized complexes of curves give rise to effective harmonic morphisms of augmented metric graphs.

2.8. Triangulated punctured curves and skeleta. Let X be a smooth, connected, proper algebraic K-curve and let $D \subset X(K)$ be a finite set of *punctures*. Recall from Definitions I.3.8 and I.3.9 that a *semistable vertex set of* (X, D) is a finite set V of type-2 points of X^{an} such that $X^{\mathrm{an}} \backslash (V \cup D)$ is a disjoint union of open balls and finitely many once-punctured open balls and open annuli. If V is a semistable vertex set of (X, D), then $(X, V \cup D)$ is called a *triangulated punctured curve*. The semistable vertex sets of (X, D) are in bijective correspondence with the semistable R-models of (X, D).

To a triangulated punctured curve $(X, V \cup D)$ one associates a canonical Λ-metrized complex of curves $\Sigma(X, V \cup D)$ called its *skeleton*. The genus of the underlying augmented metric graph Γ is equal to genus $g(X)$ of X. There is a canonical closed embedding $\Gamma \hookrightarrow X^{\mathrm{an}}$ and a retraction map $\tau : X^{\mathrm{an}} \to \Gamma$.

A *finite morphism* of triangulated punctured K-curves $\varphi : (X', V' \cup D') \to (X, V \cup D)$ consists of a finite morphsim $\varphi : X' \to X$ such that $\varphi^{-1}(V) = V'$, $\varphi^{-1}(D) = D'$, and $\varphi^{-1}(\Sigma(X, V \cup D)) = \Sigma(X', V' \cup D')$ as sets. Here we restate Corollary I.4.28:

Proposition. *Let $\varphi : (X', V' \cup D') \to (X, V \cup D)$ be a finite morphism of triangulated punctured curves. Then φ naturally induces a finite harmonic morphism of Λ-metrized complexes of curves*

$$\Sigma(X',\, V' \cup D') \longrightarrow \Sigma(X,\, V \cup D).$$

Definition 2.9. A finite harmonic morphism $\overline{\varphi} : \Gamma' \to \Gamma$ of metrized complexes of curves (resp. augmented metric graphs, resp. metric graphs) is said to be *liftable* provided that there exists a finite morphism of triangulated punctured K-curves $\varphi : (X', V' \cup D') \to (X, V \cup D)$ and an isomorphism of $\overline{\varphi}$ with the induced finite harmonic morphism of skeleta $\Sigma(X', V' \cup D') \to \Sigma(X, V \cup D)$ (resp. of augmented metric graphs underlying the skeleta, resp. of metric graphs underlying the skeleta).

Remark 2.10. Among all finite harmonic morphisms of augmented metric graphs, only the effective ones have a chance to be liftable to a tame finite morphism of triangulated punctured K-curves. Since the induced morphism of skeleta is a finite harmonic morphism of metrized complexes of curves, this follows from Remark 2.7.

2.11. Tropical modifications and tropical curves. Here we introduce an equivalence relation among metric graphs; an equivalence class for this relation will be called a *tropical curve*.

Definition 2.12. An *elementary tropical modification* of a Λ-metric graph Γ_0 is a Λ-metric graph $\Gamma = [0, +\infty] \cup \Gamma_0$ obtained from Γ_0 by attaching the segment $[0, +\infty]$ to Γ_0 in such a way that $0 \in [0, +\infty]$ gets identified with a finite Λ-point $p \in \Gamma_0$. If Γ_0 is augmented, then Γ naturally inherits a genus function from Γ_0 by declaring that every point of $(0, +\infty]$ has genus 0.

An (augmented) Λ-metric graph Γ obtained from an (augmented) Λ-metric graph Γ_0 by a finite sequence of elementary tropical modifications is called a *tropical modification* of Γ_0.

If Γ is a tropical modification of Γ_0, then there is a natural retraction map $\tau : \Gamma \to \Gamma_0$ which is the identity on Γ_0 and contracts each connected component of $\Gamma \setminus \Gamma_0$ to the unique point in Γ_0 lying in the topological closure of that component. The map τ is a (non-finite) harmonic morphism of (augmented) metric graphs.

Example 2.13. We depict an elementary tropical modification in Figure 3(a), and a tropical modification which is the sequence of two elementary tropical modifications in Figure 3(b).

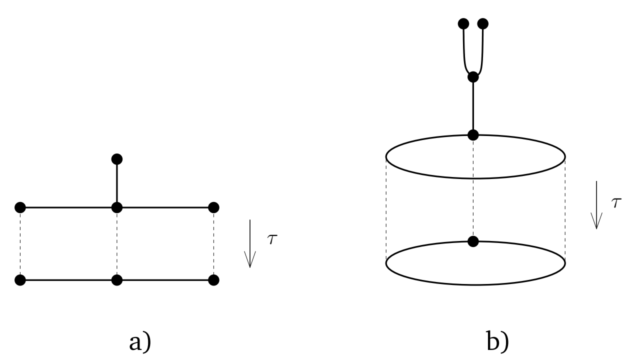


FIGURE 3. Two tropical modifications

Tropical modifications generate an equivalence relation $\sim$ on the set of (augmented) Λ-metric graphs.

Definition 2.14. A Λ-*tropical curve* (resp. an *augmented* Λ-*tropical curve*) is an equivalence class of Λ-metric graphs (resp. augmented Λ-metric graphs) with respect to $\sim$.

In other words, a Λ-tropical curve is a Λ-metric graph considered up to tropical modifications and their inverses (and similarly for augmented tropical curves). By abuse of terminology, we will often refer to a tropical curve in terms of one of its metric graph representatives.

Example 2.15. There exists a unique rational (augmented) tropical curve, which we denote by $\mathbf{TP}^1$. Any rational (augmented) metric graph whose 1-valent vertices are all infinite is obtained by a sequence of tropical modifications from the metric graph consisting of a unique finite vertex (of genus 0).

Example 2.16. Let Γ_0 be a Λ-metric graph, $p \in \Gamma_0$ a finite Λ-point, and $l \in \Lambda \setminus \{0\}$. We can construct a new Λ-metric graph Γ by attaching the segment $[0, l]$ to Γ_0 via the identification of $0 \in [0, l]$ with p. Then Γ_0 and Γ represent the same tropical curve, since the elementary tropical modification of Γ_0 at p and the elementary tropical modification of Γ at the right-hand endpoint of $[0, l]$ are the same metric graph.

Definition 2.17. Let Γ (resp. Γ') be a representative of a Λ-tropical curve C (resp. C'), and assume we are given a harmonic morphism of Λ-metric graphs $\varphi : \Gamma' \to \Gamma$.

An *elementary tropical modification of* φ is a harmonic morphism $\varphi_1 : \Gamma'_1 \to \Gamma_1$ of Λ-metric graphs, where $\tau : \Gamma_1 \to \Gamma$ is an elementary tropical modification, $\tau' : \Gamma'_1 \to \Gamma'$ is a tropical modification, and such that $\varphi \circ \tau' = \tau \circ \varphi_1$.

A *tropical modification of* φ is a finite sequence of elementary tropical modifications of φ.

Two harmonic morphisms φ_1 and φ_2 of Λ-metric graphs are said to be *tropically equivalent* if there exists a harmonic morphism which is a tropical modification of both φ_1 and φ_2.

A *tropical morphism of tropical curves* $\varphi : C' \to C$ is a harmonic morphism of Λ-metric graphs between some representatives of C' and C, considered up to (the equivalence relation generated by) tropical equivalence, and which has a finite representative.

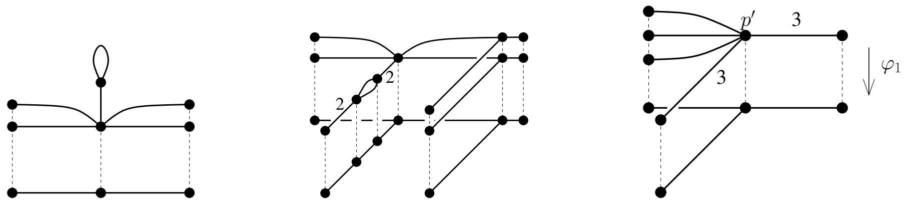
One makes similar definitions for morphisms of augmented tropical curves, with the additional condition that all harmonic morphisms should be effective.

Note that it might happen that two non-equivalent morphisms of augmented metric graphs represent the same tropical morphisms of non-augmented tropical curves.

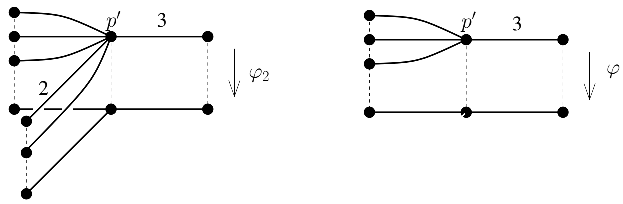
Remark 2.18. The collection of Λ-metric graphs (resp. augmented Λ-metric graphs), together with harmonic morphisms (resp. effective harmonic morphisms) between them, forms a category. Except for the condition of having a finite representative, one could try to think of tropical curves, together with tropical morphisms between them, as the localization of this category with respect to tropical modifications. However, there are some technical problems which arise when one tries to make this rigorous (at least if we demand that the localized category admit a calculus of fractions): as we will see in Example 2.19, tropical equivalence is *not* a transitive relation between morphisms of Λ-metric graphs. On the other hand, the restriction of tropical equivalence of morphisms (resp. of augmented morphisms) to the collection of finite morphisms (resp. of generically étale morphisms) *is* transitive (and hence an equivalence relation). This is one reason why we include the condition that φ has a finite representative in our definition of a morphism of tropical curves; another reason is that all morphisms of tropical curves which arise from algebraic geometry automatically satisfy this condition. See (2.21).

Example 2.19. The morphism of (totally degenerate augmented) metric graphs depicted in Figure 2(b) (resp. 4(b)) is an elementary tropical modification of the one depicted in 4(a) (resp. 2(b)).

The tropical morphisms φ_1 and φ_2 of totally degenerate augmented tropical curves depicted in Figure 4(c) and (d) are both elementary tropical modifications of the morphism φ depicted in Figure 4(e).



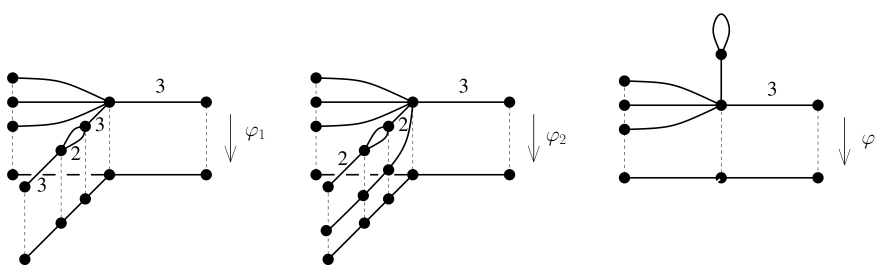
a) b) c) $g(p')=0$, $R_{p'}=0$



d) $g(p')=0$, $R_{p'}=1$ e) $g(p')=0$, $R_{p'}=2$

FIGURE 4

The tropical morphisms φ_1 and φ_2 depicted in Figure 5(a) and (b) are both elementary tropical modifications of the morphism φ depicted in Figure 5(c).



a) b) c)

FIGURE 5

On the other hand, the harmonic morphism $\varphi : \Gamma' \to \Gamma$ depicted in Figure 2(c) with $d = 1$ is not tropically equivalent to any finite morphism: since φ has degree 1, the cycle of the source graph will be contracted to a point by any harmonic morphism of metric graphs tropically equivalent to φ. In particular, φ does not give rise to a tropical morphism.

As mentioned above, tropical equivalence is not transitive among morphisms of metric graphs (resp. of augmented metric graphs). For example, the two morphisms φ_1 and φ_2 depicted in Figure 4(c) and (d) are not tropically equivalent as augmented morphisms: since $R_{p'} = 0$ in Figure 4(c), any edge appearing in a tropical modification of φ_1 will have degree 1.

Note that the preceding harmonic morphisms φ_1 and φ_2 *are* tropically equivalent as morphisms of metric graphs (i.e. forgetting the genus function). However, tropical equivalence is not transitive for tropical morphisms either, for essentially the same reason: the two tropical morphisms φ_1 and φ_2 depicted in Figure 5(a) and (b) are not tropically equivalent.

Nevertheless, the restriction of tropical equivalence of morphisms to the set of finite morphisms (resp. generically étale morphisms) is an equivalence relation. Hence a tropical morphism (resp. an augmented tropical morphism) can also be thought as an equivalence class of finite harmonic morphisms (resp. generically étale morphisms). In particular there exists a natural composition rule for tropical morhisms (resp. augmented tropical morphisms), turning tropical curves (resp. augmented tropical curves) equipped with tropical morphisms into a category.

Remark 2.20. In the definition of a tropical morphism of augmented tropical curves, in addition to the condition of being a harmonic morphism and the "up to tropical modifications" considerations, we imposed two rather strong conditions, namely being effective and having a finite representative. We already saw in Example 2.19 that the finiteness condition is non-trivial. The effectiveness condition is also non-trivial: for example, the harmonic morphism $\varphi : \Gamma' \to \Gamma$ of totally degenerate augmented metric graphs depicted in Figure 2(c) with $d = 2$ is not tropically equivalent to any finite effective morphism of totally degenerate augmented metric graphs. Indeed, for any tropical modification of φ which is effective, at most two edges adjacent to p' can have degree 2; since Γ' already has two such edges for φ, any tropical modification of φ which is finite and effective will contract the cycle of Γ' to a point.

We refer to [BM] for a general definition of a tropical morphism $\varphi : C \to X$ from an augmented tropical curve to a non-singular tropical variety, including Definition 2.17 as a particular case.

2.21. Algebraic and tropical curves. Restating Lemma I.3.15 and Remark I.3.16, we have:

Proposition. *Let $(X, V \cup D)$ be a triangulated punctured K-curve. Let $D' \subset X(K)$ be a finite set and let V' be a semistable vertex set of (X, D'), so $(X, V' \cup D')$ is another triangulated punctured K-curve with underlying curve X. Then the augmented metric graphs underlying $\Sigma(X, V' \cup D')$ and $\Sigma(X, V \cup D)$ represent the same tropical curve.*

The above Proposition implies that one can associate a canonical (augmented) tropical curve to any smooth proper connected K-curve X. This association is *functorial* by Corollary I.4.26:

Proposition. *Let $\varphi : X' \to X$ be a finite morphism of smooth proper connected K-curves, let $D \subset X(K)$ be a finite set, and let $D' = \varphi^{-1}(D)$. Then there exist semistable vertex sets V, V' of (X, D) and (X', D'), respectively, such that φ induces a finite morphism of triangulated punctured curves $\varphi : (X', V' \cup D') \to (X, V \cup D)$. In particular, φ induces a finite harmonic morphism on suitable choices of skeleta.*

Again we emphasize that a tropical morphism of tropical curves functorially induced by a finite morphism of algebraic curves is *effective* and has a *finite representative*.

Definition 2.22. We say that a tropical morphism of tropical curves $\overline{\varphi} : C' \to C$ is *liftable* provided that there exists a finite morphism of smooth proper connected K-curves $\varphi : X' \to X$ functorially inducing $\overline{\varphi}$ on skeleta in the above sense.

We will also make use in the text of the notion of tropical modifications of metrized complexes of curves.

Definition 2.23. Let $\mathcal{C}_0$ be a Λ-metrized complex of k-curves.

- A *refinement* of $\mathcal{C}_0$ is any Λ-metrized complex of k-curves $\mathcal{C}$ obtained from $\mathcal{C}_0$ by adding a finite set of Λ-points S of $\mathcal{C}_0 \setminus V(\mathcal{C}_0)$ to the set $V(\mathcal{C}_0)$ of vertices of $\mathcal{C}_0$ (see Definition I.2.17), setting $C_p = \mathbf{P}^1_k$ for all $p \in S$, and defining the map red_p by choosing any two distinct closed points of C_p.
- An *elementary tropical modification* of $\mathcal{C}_0$ is a Λ-metrized complex of k-curves $\mathcal{C}$ obtained from $\mathcal{C}_0$ by an elementary tropical modification of the underlying metric graph at a vertex p of $\mathcal{C}$, with the map red_p extended to e by choosing any closed point of C_p not in the image of the reduction map for $\mathcal{C}_0$.
- Any metrized complex of curves $\mathcal{C}$ obtained from a metrized complex of curves $\mathcal{C}_0$ by a finite sequence of refinements and elementary tropical modifications is called a *tropical modification* of $\mathcal{C}_0$.

3. Lifting harmonic morphisms of metric graphs to morphisms of metrized complexes

There is an obvious forgetful functor which sends metrized complexes of curves to (augmented) metric graphs, and harmonic morphisms of metrized complexes to harmonic morphisms of (augmented) metric graphs. A harmonic morphism of (augmented) metric graphs is said to be *liftable to a harmonic morphism of metrized complexes of k-curves* if it lies in the image of the forgetful functor.

We proved in Theorem I.7.7 that every tame covering of metrized complexes of curves can be lifted to a tame covering of algebraic curves. In this section we study the problem of lifting harmonic morphisms of (augmented) metric graphs to finite morphisms of metrized complexes (and thus to tame coverings of proper smooth curves, thanks to Proposition I.7.15).

3.1. Lifting finite augmented morphisms. Recall that k is an algebraically closed field of characteristic $p \geq 0$. A finite harmonic morphism φ of (augmented) metric graphs is called a *tame harmonic morphism* if either $p = 0$ or all the local degrees of φ along edges are prime to p. Lifting of tame harmonic morphisms of augmented metric graphs to tame harmonic morphisms of metrized complexes of k-curves is equivalent to the existence of tamely ramified covers of k-curves of given genus with some given prescribed ramification profile.

3.1.1. A partition μ of an integer d is a multiset of natural numbers $d_1, \ldots, d_l \geq 1$ with $\sum_i d_i = d$. The integer l, called the length of μ, will be denoted by $l(\mu)$.

Let $g', g \geq 0$ and $d > 0$ be integers, and let $M = \{\mu_1, \ldots, \mu_s\}$ be a collection of s partitions of d. Assume that the integer R defined by

$$R := d(2-2g) + 2g' - 2 - sd + \sum_{i=1}^{s} l(\mu_i) \qquad \textbf{(3.1.2)}$$

is non-negative. Denote by $\mathcal{A}^d_{g',g}(\mu_1, \ldots, \mu_s)$ the set of all tame coverings $\varphi : C' \to C$ of smooth proper curves over k, with the following properties:

(i) The curves C and C' are irreducible of genus g and g', respectively;
(ii) The degree of φ is equal to d;
(iii) The branch locus of φ contains (at least) s distinct points $x_1, \ldots, x_s \in C$, and the ramification profile of φ at the points $\varphi^{-1}(x_i)$ is given by μ_i, for $1 \leq i \leq s$.

As we will explain now, the lifting problem for morphisms of augmented metric graphs to morphisms of metrized complexes over a field k reduces to the emptiness or non-emptiness of certain sets $\mathcal{A}^d_{g',g}(\mu_1, \ldots, \mu_s)$. This latter problem is quite subtle, and no complete satisfactory answer is yet known (see also (3.3.1)). In some simple cases, however, one can ensure that $\mathcal{A}^d_{g',g}(\mu_1, \ldots, \mu_s)$ is non-empty. For example, if all the partitions μ_i are trivial (i.e., they each consist of d 1's), then $\mathcal{A}^d_{g',g}(\mu_1, \ldots, \mu_s)$ is non-empty. Here is another simple example.

Example 3.2. For an integer d prime to characteristic p of k, the set $\mathcal{A}^d_{0,0}((d),(d))$ is non-empty since it contains the map $z \mapsto z^d$. This is in fact the only map in $\mathcal{A}^d_{0,0}((d),(d))$ up to the action of the group $\mathrm{PGL}(2,k)$ on the target curve and $\mathbf{P}^1$-isomorphisms of coverings.

3.2.1. Let $\varphi : \Gamma' \to \Gamma$ be a finite harmonic morphism of augmented metric graphs. Using the definition of a harmonic morphism, one can associate to any point p' of Γ' a collection $\mu_1(p'), \ldots, \mu_s(p')$ of s partitions of the integer $d_{p'}(\varphi)$, where $s = \mathrm{val}(\varphi(p'))$, as follows: if $T_{\varphi(p)}(\Gamma) = \{v_1, \ldots, v_s\}$ denotes all the tangent directions to Γ at $\varphi(p')$, then $\mu_i(p')$ is the partition of $d_{p'}(\varphi)$ which consists of the various local degrees of φ in all tangent directions $v' \in T_{p'}(\Gamma')$ mapping to v_i.

The next proposition is an immediate consequence of the various definitions involved once we note that, by Example 3.2, there are only finitely points $p' \in \Gamma'$ for which the question of non-emptiness of the sets $\mathcal{A}^{d_{p'}(\varphi)}_{g(p'),g(\varphi(p'))}$ arises. It provides a "numerical criterion" for a tame harmonic morphism of augmented metric graphs to be liftable to a tame harmonic morphism of metrized complexes of curves.

Proposition 3.3. *Let $\varphi : \Gamma' \to \Gamma$ be a tame harmonic morphism of augmented metric graphs. Then φ can be lifted to a tame harmonic morphism of metrized complexes over k if and only if for every point p' in Γ', the set $\mathcal{A}^{d_{p'}(\varphi)}_{g(p'),g(\varphi(p'))}(\mu_1(p'),\dots,\mu_{\mathrm{val}(\varphi(p'))}(p'))$ is non-empty.*

3.3.1. In characteristic 0, the lifting problem for finite augmented morphisms of metric graphs can be further reduced to a vanishing question for certain Hurwitz numbers.

Fix an irreducible smooth proper curve C of genus g over k, and let $x_1,\dots,x_s,y_1,\dots,y_R$ be a set of distinct points on C. The Hurwitz set $\mathcal{H}^d_{g',g}(\mu_1,\dots,\mu_s)$ is the set of C-isomorphism classes of all coverings in $\mathcal{A}^d_{g',g}(\mu_1,\dots,\mu_s)$ satisfying (i), (ii), and (iii) in (3.1.1) for the curve C and the points $x_1,\dots,x_s$ that we have fixed, and which in addition satisfy:

(iv) The integer R is given by (3.1.2), and for each $1 \le i \le R$, φ has a unique simple ramification point y_i' lying above y_i.

Note that by the above condition, the branch locus of φ consists precisely of the points x_i, y_j. The Hurwitz number $H^d_{g',g}(\mu_1,\dots,\mu_s)$ is defined as

$$H^d_{g',g}(\mu_1,\dots,\mu_s) := \sum_{\varphi\in\mathcal{H}^d_{g',g}(\mu_1,\dots,\mu_s)} \frac{1}{|\mathrm{Aut}_C(\varphi)|},$$

and does not depend on the choice of C and the closed points $x_1,\dots,x_s,y_1,\dots,y_R$ in C.

Example 3.4. It is known, see for example [EKS84], that

$$H^2_{g,0} = \frac{1}{2}, \quad H^3_{g,0}((3),\dots(3)) > 0, \quad H^4_{0,0}((2,2),(2,2),(3,1)) = 0.$$

For the reader's convenience, and since we will use it several times in the sequel, we sketch a proof of the fact that $H^4_{0,0}((2,2),(2,2),(3,1)) = 0$. By the Riemann–Hurwitz formula and the Riemann Existence Theorem, $H^4_{0,0}((2,2),(2,2),(3,1)) \neq 0$ if and only if there exist elements $\sigma_1,\sigma_2,\sigma_3$ in the symmetric group $\mathfrak{S}_4$ having cycle decompositions of type $(2,2),(2,2),(3,1)$, respectively, such that $\sigma_1\sigma_2\sigma_3 = 1$ and such that the σ_i generate a transitive subgroup of $\mathfrak{S}_4$. However, elementary group theory shows that the product $\sigma_1\sigma_2$ cannot be of type $(3,1)$ (the transitivity condition does not intervene here). For a proof which works in any characteristic, see Lemma 5.10 below.

All Hurwitz numbers can be theoretically computed, for example using Frobenius Formula (see [LZ04, Theorem A.1.9]). Nevertheless, the problem of understanding their vanishing is wide open. The above example shows that Hurwitz numbers in degree at most three are all positive, which is not the case in degree four. Some families of (non-)vanishing Hurwitz numbers are known (see Example 3.5). However, in general one has to explicitly compute a given Hurwitz number to decide if this latter vanishes or not. We refer the reader to [EKS84], [PP06], and [PP08], along with the references therein, for an account of what is known about this subject. We will use the vanishing of $H^4_{0,0}((2,2),(2,2),(3,1))$ in Section 5 to construct a 4-gonal augmented graph (see Section 5 for the definition) which cannot be lifted to any 4-gonal proper smooth algebraic curve over K.

Example 3.5. Some partial results are known concerning the (non-)vanishing of Hurwitz numbers. For example, it is known that double Hurwitz numbers (i.e., when $s = 2$) are all positive (this can be seen for example from the presentation of the cut-join equation given in [CJM10]), as well as all the Hurwitz numbers $H^d_{g',g}(\mu_1,\dots,\mu_s)$ when $g \ge 1$ and $R \ge 0$ ([Hus62, EKS84]). On the other hand, it is proved in [PP08] that

$$H^d_{0,0}\left((d-2,2),(2,\dots,2),(\frac{d}{2}+1,1,\dots,1)\right) = 0 \quad \text{for all } d \ge 4 \text{ even.}$$

Example 3.6. As another example of non-vanishing Hurwitz numbers, one has $H^{d'}_{0,0}(\mu_1,\dots,\mu_s,(d')) > 0$ for all integers $d' \ge 1$ when the integer R defined in (3.1.2) is zero (i.e., if the combinatorial Riemann–Hurwitz formula holds); see [EKS84, Proposition 5.2] or [DM08, Proposition 7.2].

The non-emptiness of $\mathcal{A}^d_{g',g}(\mu_1,\dots,\mu_s)$ can be reduced to the non-emptiness of the Hurwitz set $\mathcal{H}^d_{g',g}(\mu_1,\dots,\mu_s)$.

Lemma 3.7. *Suppose that k has characteristic 0. Then $\mathcal{A}^d_{g',g}(\mu_1,\dots,\mu_s)$ is non-empty if and only if $H^d_{g',g}(\mu_1,\dots,\mu_s)\neq 0$.*

Proof. Since $\mathcal{H}^d_{g',g}(\mu_1,\dots,\mu_s)$ is a subset of $\mathcal{A}^d_{g',g}(\mu_1,\dots,\mu_s)$, obviously we only need to prove that if $\mathcal{A}^d_{g',g}(\mu_1,\dots,\mu_s)\neq\emptyset$, then the Hurwitz set is also non-empty. Let $\varphi: C'\to C$ be an element of $\mathcal{A}^d_{g',g}(\mu_1,\dots,\mu_s)$, branched over $x_i\in C$ with ramification profile μ_i for $i=1,\dots,s$, and let $z_1,\dots,z_t$ be all the other points in the branch locus of φ. Denote by ν_i the ramification profile of φ above the point z_i. Fix a closed point $\star$ of $C\setminus\{x_1,\dots,x_s,z_1,\dots,z_t\}$. The étale fundamental group $\pi_1(C\setminus\{x_1,\dots,x_s,z_1,\dots,z_t\},\star)$ is the profinite completion of the group generated by a system of generators $a_1,b_1,\dots,a_g,b_g,c_1,\dots,c_{s+t}$ satisying the relation $[a_1,b_1]\dots[a_g,b_g]c_1\dots c_{s+t}=1$, where $[a,b]=aba^{-1}b^{-1}$ (see [SGA1]). In addition, the data of φ is equivalent to the data of a surjective morphism ρ from $\pi_1(C\setminus\{x_1,\dots,x_s,z_1,\dots,z_t\},\star)$ to a transitive subgroup of the symmetric group $\mathfrak{S}_d$ of degree d such that the partition μ_i (resp. ν_i) of d corresponds to the lengths of the cyclic permutations in the decomposition of $\rho(c_i)$ (resp. $\rho(c_{s+i})$) in $\mathfrak{S}_d$ into products of cycles, for $1\le i\le s$ (resp. $1\le i\le t$). By Riemann–Hurwitz formula, we have $R=\sum_{i=1}^t(d-l(\nu_i))$.

Now note that each $\rho(c_{s+i})$ can be written as a product of $d-l(\nu_i)$ transpositions $\tau_i^1,\dots,\tau_i^{d-l(\nu_i)}$ in $\mathfrak{S}_d$, i.e., $\rho(c_{s+i})=\tau_i^1\dots\tau_i^{d-l(\nu_i)}$. Rename the set of R distinct points $y_1,\dots,y_R$ of $C\setminus\{x_1,\dots,x_s,\star\}$ as $z_i^1,\dots,z_i^{d-l(\nu_i)}$ for $1\le i\le t$.

The étale fundamental group $\pi_1(C\setminus\{x_1,\dots,x_s,z_1^1,\dots,z_1^{d-l(\nu_1)},\dots,z_t^{d-l(\nu_t)}\},\star)$ has, as a profinite group, a system of generators $a_1,b_1,\dots,a_g,b_g,c_1,\dots,c_s,c_{s+1}^1,\dots,c_{s+1}^{d-l(\nu_1)},\dots,c_{s+t}^{d-l(\nu_t)}$ verifying the relation

$$[a_1,b_1]\dots[a_g,b_g]c_1\dots c_s c_{s+1}^1\dots c_{s+1}^{d-l(\nu_1)}\dots c_{s+t}^1\dots c_{s+t}^{d-l(\nu_t)}=1,$$

and admits a surjective morphism to $\mathfrak{S}_d$ which coincides with ρ on $a_1,b_1,\dots,a_g,b_g$, and which sends c_{s+i}^j to τ_i^j for each $1\le i\le t$ and $1\le j\le d-l(\nu_i)$. The corresponding cover $C''\to C$ obviously belongs to $\mathcal{A}^d_{g',g}(\mu_1,\dots,\mu_s)$ and in addition has simple ramification profile (2) above each y_i, i.e., it verifies condition (iv) above. This shows that $\mathcal{H}^d_{g',g}(\mu_1,\dots,\mu_s)$ is non-empty. ∎

Corollary 3.8. *Suppose that k has characteristic 0. Let $\varphi:\Gamma'\to\Gamma$ be a finite morphism of augmented metric graphs, and let $\mathcal{C}$ be a metrized complex over k lifting Γ. There exists a lifting of φ to a finite harmonic morphism of metrized complexes $\mathcal{C}'\to\mathcal{C}$ over k (and thus to a morphism of smooth proper curves over K) if and only if*

$$\prod_{p'\in V(\Gamma')} H^{d_{p'}(\varphi)}_{g(p'),g(\varphi(p'))}\left(\mu_1(p'),\dots,\mu_{\mathrm{val}(\varphi(p'))}\right)\neq 0.$$

In particular, if φ is effective and $g(p)\ge 1$ for all the points of valency at least three in Γ, then φ lifts to a finite harmonic morphism of metrized complexes over k.

Remark 3.9. If k has positive characteristic $p>d$, then $\mathcal{A}^d_{g',g}(\mu_1,\dots,\mu_s)$ has the same cardinality as in characteristic zero. (This follows from [SGA1], which provides an isomorphism between the tame fundamental group in positive characteristic p and the prime-to-p part of the étale fundamental group in characteristic zero.) In particular, Lemma 3.7 also holds under the assumption that $p>d$.

3.10. Lifting finite harmonic morphisms. Now we turn to the lifting problem for finite morphisms of non-augmented metric graphs to morphisms of metrized complexes of k-curves. In this case there are no obstructions to the existence of such a lift.

Theorem 3.11. *Let $\varphi:\Gamma'\to\Gamma$ be a tame harmonic morphism of metric graphs, and suppose that Γ is augmented. There exists an enrichment of Γ' to an augmented metric graph (Γ',g') such that $\varphi:(\Gamma',g')\to(\Gamma,g)$ lifts to a tame harmonic morphism of metrized complexes of curves over k (and thus to a morphism of smooth proper curves over K).*

Theorem 3.11 is an immediate consequence of Proposition 3.3 and the following theorem. (For the statement, we say that a partition μ of d is *tame* if either $\mathrm{char}(k) = 0$ or all the integers appearing in μ are prime to p.)

Theorem 3.12. *Let $g \geq 0, d \geq 2, s \geq 1$ be integers. Let $\mu_1, \dots, \mu_s$ be a collection of s tame partitions of d. Then there exists a sufficiently large non-negative integer g' such that $\mathcal{A}^d_{g',g}(\mu_1, \dots, \mu_s)$ is non-empty.*

Proof. We first give a simple proof which works in characteristic zero, and more generally, in the case of a tame monodromy group. The proof in characteristic $p > 0$ is based on our lifting theorem and a deformation argument.

Suppose first that the characteristic of k is zero. By Lemma 3.7, we need to show that for large enough g' the set $\mathcal{H}^d_{g',g}(\mu_1, \dots, \mu_s)$ is non-empty.

If $g \geq 1$, for any large enough g' giving $R \geq 0$, we have $\mathcal{H}^d_{g',g}(\mu_1, \dots, \mu_s) \neq \emptyset$ [Hus62]. So suppose $g = 0$. Consider $s + R + 1$ distinct points $x_1, \dots, x_s, z_1, \dots, z_R, \star$ in C. The étale fundamental group $\pi_1(R) := \pi_1(C \setminus \{x_1, \dots, x_s, z_1, \dots, z_R\}, \star)$ has, as a profinite group, a system of generators $c_1, \dots, c_s, c_{s+1}, \dots, c_{s+R}$ verifying the relation

$$c_1 \dots c_r c_{s+1} \dots c_{s+R} = 1.$$

It will be enough to show that for a large enough R, there exists a surjective morphism ρ from $\pi_1(R)$ to $\mathfrak{S}_d$ so that $\rho(c_{s+i})$ is a transposition for any $i = 1, \dots, R$, and that for any $i = 1, \dots, s$, the partition of d given by the lengths of the cyclic permutations in the decomposition of $\rho(c_i)$ is equal to μ_i. In this case, the genus g' of the corresponding cover C' of C in $\mathcal{H}^d_{g',0}(\mu_1, \dots, \mu_s)$ will be given by

$$g' = 1 - d + \frac{1}{2}\big[sd + R - \sum_{i=1}^{s} l(\mu_i)\big].$$

Consider an arbitrary map ρ from $\{c_1, \dots, c_s\}$ to $\mathfrak{S}_d$ verifying the ramification profile condition for $\rho(c_1), \dots, \rho(c_r)$. Choose a system of d transpositions $\tau_1, \dots, \tau_d$ generating $\mathfrak{S}_d$, and consider a set of transpositions $\tau_{d+1}, \dots, \tau_R$ such that

$$\rho(c_1) \dots \rho(c_s)\, \tau_1 \dots \tau_d = \tau_R \dots \tau_{d+1}\,.$$

This proves Theorem 3.12 when k has characteristic 0.

Consider now the case of a base field k of positive characteristic $p > 0$. Note that since the prime to p part of the tame fundamental group has the same representation as in the case of characteristic zero, the group theoretic method we used in the previous case can be applied if the monodromy group is tame, i.e., has size prime to p. However, in general it is impossible to impose such a condition on the monodromy group. For example in the case when p divides d, the size of the monodromy group is always divisible by p.

We first describe how to reduce the proof of Theorem 3.12 to the case $s = 1$ and $g = 0$. Suppose that for each μ_i, $1 \leq i \leq s$, there exists a large enough g_i such that $\mathcal{A}^d_{g_i,0}(\mu_i)$ is non-empty, and consider a tame cover $\varphi_i : C_i \to \mathbf{P}^1_k$ in $\mathcal{A}_{g_i,0}(\mu_i)$ such that the ramification profile over $0 \in \mathbf{P}^1$ is given by μ_i, and choose two regular points $x_i, y_i \in \mathbf{P}^1$ (i.e. x_i, y_i are outside the branch locus of φ_i). Choose also a smooth proper curve C_0 of genus g which admits a tame cover $\varphi_0 : C_0' \to C_0$ of degree d from a smooth proper curve C_0' of large enough genus g_0'. (The existence of such a cover can be deduced by a similar trick as that discussed at the end of the proof below and depicted in Figure 7.) Let $y_0 \in C_0$ be a regular point of φ_0.

Let $\mathcal{C}_0$ be the metrized complex over k whose underlying metric graph is $[0, +\infty]$, with one finite vertex v_0 and one infinite vertex v_∞, equipped with the metric induced by $\mathbf{R}$, and with $C_{v_0} = C_0$ and $\mathrm{red}_{v_0}(\{v_0, v_\infty\}) = y_0$. Denote by $\mathcal{C}$ the modification of $\mathcal{C}_0$ obtained by taking a refinement at r distinct points $0 < v_1 < \cdots < v_s < \infty$, as depicted in Figure 6, and by setting $C_{v_i} = \mathbf{P}^1$ and $\mathrm{red}_{v_i}(\{v_i, v_{i-1}\}) = x_i$ and $\mathrm{red}_{v_i}(\{v_i, v_{i+1}\}) = y_i$ (here $v_{s+1} = v_\infty$), and by adding an infinite edge e_i to each v_i, and defining $\mathrm{red}_{v_i}(e_i) = 0 \in \mathbf{P}^1$. Denote by Γ the underlying metric graph of $\mathcal{C}$. See Figure 6.

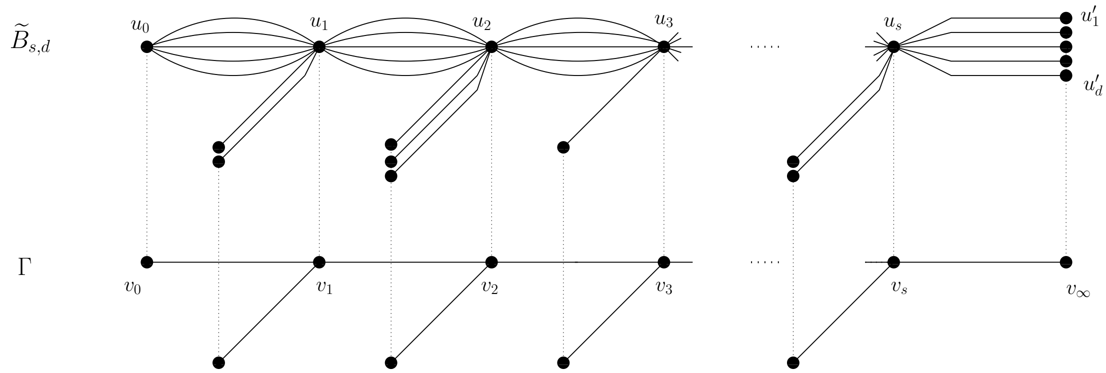


FIGURE 6

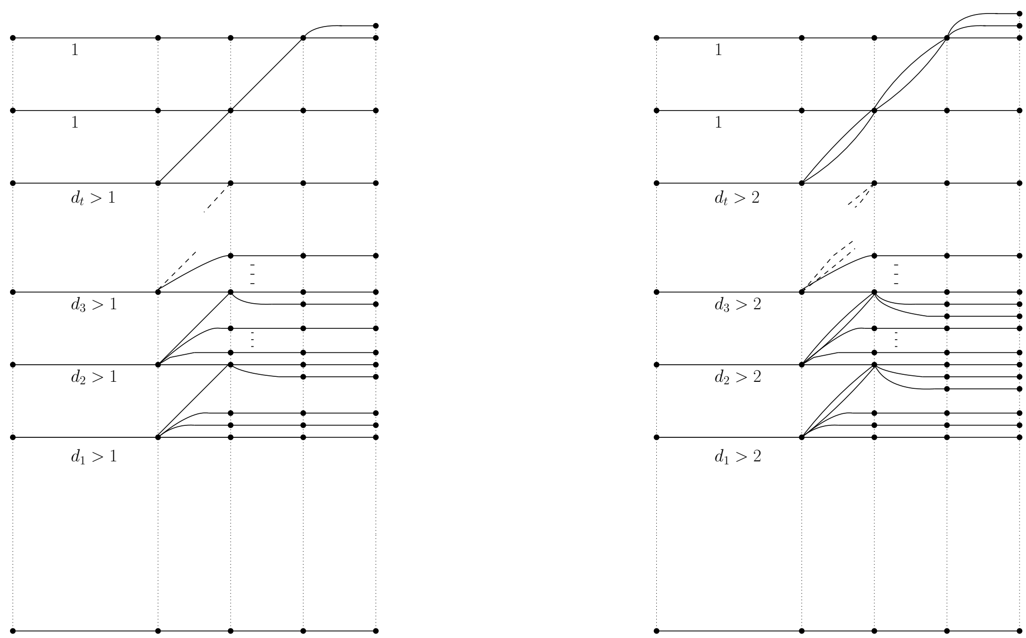
Define now the metric graph $B_{s,d}$ as the chain of s banana graphs of size d: $B_{s,d}$ has $s+1$ finite vertices $u_0, \ldots, u_s$ and $u'_1, \ldots, u'_d$ infinite vertices adjacent to u_s such that u_i is connected to u_{i+1} with d edges of length $\ell_\Gamma(\{v_{i+1} - v_i\})$. We denote by $\widetilde{B}_{s,d}$ the tropical modification of $B_{r,d}$ at $u_1, \ldots, u_s$, obtained by adding $l(\mu_i)$ infinite edges to u_i. Eventually we turn $\widetilde{B}_{s,d}$ into a metrized complex $\mathcal{C}_{s,d}$ over k by setting $C_{u_i} = C_i$, and defining red_{u_i} on the d edges between u_i and u_{i+1} by a bijection to the d points in $\varphi_i^{-1}(y_i)$, red_{u_i} on the edges between u_i and u_{i-1} by a bijection to the d points in $\varphi_i^{-1}(x_i)$, and red_{u_i} on the $l(\mu_i)$ infinite edges adjacent to u_i by a bijection to the $l(\mu_i)$ points in $\varphi_i^{-1}(0)$.

Obviously, there exists a degree d tame morphism $\varphi : \mathcal{C}_{s,d} \to \mathcal{C}$ of curve complexes over k which sends u_i to v_i, and has degrees given by integers in μ_i above the infinite edge of Γ adjacent to v_i, for $i = 1, \ldots, s$, and $\varphi_{u_i} = \varphi_i$ (see Figure 6). According to Proposition I.7.15, the map φ lifts to a tame morphism of smooth proper curves $\varphi_K : X \to X'$ over K the completion of the algebraic closure of $k[[t]]$. The map φ_K has partial ramification profile $\mu_1, \ldots, \mu_s$. To deduce now the non-emptiness of $\mathcal{A}^d_{g',g}(\mu_1, \ldots, \mu_s)$, we note that there exists a subring R of K finitely presented over k such that the map φ_K descends to a finite morphism $\varphi_R : \mathfrak{X} \to \mathfrak{X}'$ between smooth proper curves over $\mathrm{Spec}(R)$. In addition, over a non-empty open subset U of $\mathrm{Spec}(R)$, φ_R specializes to a tame cover with the same ramification profile as φ_K. Since U contains a k-rational point, we infer the existence of a large enough g' such that $\mathcal{A}^d_{g',g}(\mu_1, \ldots, \mu_s) \neq \emptyset$.

We are thus led to consider the case where $s = 1$, $g = 0$, $\mu = (d_1, d_2, \ldots, d_l)$ with $\sum_i d_i = d$, $d_1, \ldots, d_t > 1$, and $d_{t+1} = \cdots = d_l = 1$. Figure 7 shows that, just as in the previous reduction, one can reduce to the case where $s = 1$ and $\mu_1 = \{d\}$ with $(d, p) = 1$. (Note that in Figure 7(a) the degree of the morphism at some of the middle vertices is two; Figure 7(b) is arranged so that the degrees are all odd.) But this is just non-emptiness of $\mathcal{A}_{0,0}((d))$ (see Example 3.2). ■

Remark 3.13. As the above proof shows, when k has characteristic zero one can get an explicit upper bound on the smallest positive integer g' with $\mathcal{H}^d_{g',0}(\mu_1, \ldots, \mu_s) \neq \emptyset$. Indeed, the permutation $\rho(c_1) \ldots \rho(c_s)\tau_1 \ldots \tau_d$ can be written as the product of $d + \sum_{i=1}^s (d - l(\mu_i))$ transpositions. So without loss of generality we have $R - d = d + \sum_{i=1}^s (d - l(\mu_i))$, which means that one can take g' to be $1 + \sum_{i=1}^r (d - l(\mu_i))$. For $g \geq 1$, $\mathcal{H}_{g',g}$ is non-empty as soon as R is non-negative, which means in this case that one can take g' to be $1 + (g-1)d + \frac{1}{2}\sum_i (d - l(\mu_i))$.

3.14. Lifting polynomial-like harmonic morphisms of trees. There is a special case of Theorem 3.12 in which one does not need to increase the genus of the source curve. To state the result, we say (following [DM08]) that a degree d finite harmonic morphism $\overline{\varphi} : T' \to T$ of metric trees is *polynomial-like* if there exists an infinite vertex of T' with local degree equal to d.



a) Reduction in the case $p \neq 2$ (in this example, $d_1 = 4, d_2 = 4, d_3 = 3$, and $d_t = 2$).

b) Reduction in the case $p = 2$.

FIGURE 7

Theorem 3.15. *Assume that the residue characteristic of K is zero or bigger than d. Let $\overline{\varphi} : T' \to T$ be a generically étale polynomial-like harmonic morphism of metric trees. Then there exists a degree d polynomial map $\varphi : \mathbf{P}^1 \to \mathbf{P}^1$ over K lifting $\overline{\varphi}$.*

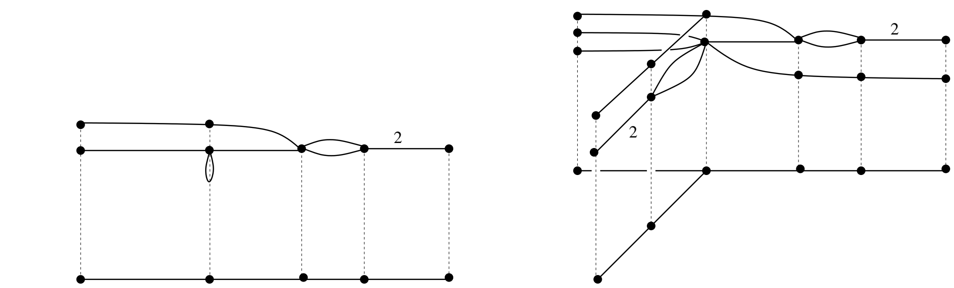
Proof. It suffices to prove that $\overline{\varphi}$ can be extended to a degree d harmonic morphism of genus zero metrized complexes of curves. By Theorem I.7.7, Proposition 3.3, and Remark 3.9, this reduces to showing that the Hurwitz numbers given by the ramification profiles around each finite vertex of T' are all non-zero. Fix an infinite vertex ∞ of T' with local degree d. Then it is easy to see that for any such vertex v', the local degree of $\overline{\varphi}$ at v' is equal to the local degree of $\overline{\varphi}$ in the tangent direction corresponding to the unique path from v' to ∞. (This is analogous to [DM08, Lemma 2.3].) The result now follows from Example 3.6. ■

3.16. Lifting of harmonic morphisms in the case the base has genus zero. We now consider the special case where Γ has genus zero and present more refined lifting results in this case. As explained in (2.11), a given harmonic morphism of (augmented) metric graphs does not necessarily have a tropical modification which is finite. We present below a weakened notion of finiteness of a harmonic morphism, and prove that any harmonic morphism from an (augmented) metric graph to an (augmented) rational metric graph satisfies this weak finiteness property. We discuss in Section 4 some consequences concerning linear equivalence of divisors on metric graphs.

Definition 3.17. A harmonic morphism $\varphi : \Gamma \to T$ from an augmented metric graph Γ to a metric tree T is said to admit a *weak resolution* if there exists a tropical modification $\tau : \widetilde{\Gamma} \to \Gamma$ and an augmented harmonic morphism $\widetilde{\varphi} : \widetilde{\Gamma} \to T$ such that the restriction $\widetilde{\varphi}_{|\Gamma}$ is equal to φ, and some tropical modification of $\widetilde{\varphi}$ is finite.

In other words, the morphism φ has a weak resolution if it can be extended, up to increasing the degree of φ using the modification τ, to a tropical morphism $\widetilde{\varphi} : \widetilde{\Gamma} \to T$.

Example 3.18. The harmonic morphism depicted in Figure 2(c) with $d = 1$ can be weakly resolved by the harmonic morphisms depicted in Figures 4(b) and 2(b). Another example of a weak resolution is depicted in Figure 8.



a) A harmonic morphism not tropically equivalent to any finite harmonic morphism

b) A weak resolution of the morphism in Figure 8(a)

FIGURE 8

Definition 3.19. Let $\varphi : \Gamma \to T$ be a harmonic morphism from a metric graph Γ to a metric tree T. A point $p \in \Gamma$ is *regular* if φ is non-constant on all neighborhoods of p.

The *contracted set* of φ, denoted by $\mathcal{E}(\varphi)$, is the set of all non-regular points of φ. A *contracted component* of φ is a connected component of $\mathcal{E}(\varphi)$.

The next proposition, together with Proposition I.7.15, allows us to conclude that any harmonic morphism from an augmented metric graph to a metric tree can be realized, up to weak resolutions, as the induced morphism on skeleta of a finite morphism of triangulated punctured curves. Recall that $\Lambda = \mathrm{val}(K^\times)$ is divisible since K is algebraically closed.

Proposition 3.20. (Weak resolution of contractions) *Let $\varphi : \Gamma \to T$ be a harmonic morphism of degree d from a metric graph Γ to a metric tree T.*

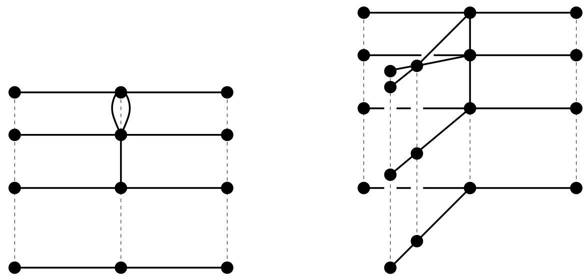
(1) *There exist tropical modifications $\tau : \widetilde{\Gamma} \to \Gamma$ and $\tau' : \widetilde{T} \to T$, and a harmonic morphism of metric graphs (of degree $\widetilde{d} \geq d$) $\widetilde{\varphi} : \widetilde{\Gamma} \to \widetilde{T}$, such that $\widetilde{\varphi}_{|\Gamma \setminus \mathcal{E}(\varphi)} = \varphi$, where $\mathcal{E}(\varphi)$ is the contracted part of Γ.*

(2) *Suppose in addition that Γ is augmented, and if $p > 0$ that all the non-zero degrees of φ along tangent directions at Γ are prime to p. Then there exist tropical modifications of Γ, T, and φ as above such that $\widetilde{\varphi}$ is tame and, in addition, there exists a tame harmonic morphism of metrized complexes of k-curves with $\widetilde{\varphi}$ as the underlying finite harmonic morphism of augmented metric graphs.*

Proof. Up to tropical modifications, we may assume that all 1-valent vertices of T are infinite vertices.

The proof of (1) goes by giving an algorithm to exhibit a weak resolution of φ. Note that this algorithm does not produce the weak resolutions presented in Example 3.18, since in these cases we could find simpler ones.

Let $V(\Gamma)$ be any vertex set of Γ with no loop edge. We denote by d the degree of φ, and by α the number of non-regular vertices of φ. Given v a finite non-regular vertex of Γ, we consider the tropical modification $\tau_v : \widetilde{\Gamma}_v \to \Gamma$ such that $\left(\widetilde{\Gamma}_v \setminus \Gamma\right) \cup \{v\}$ is isomorphic to T as a metric graph. Considering all those modifications for all non-regular vertices of φ, we obtain a modification $\tau : \widetilde{\Gamma} \to \Gamma$. We can naturally extend φ to a harmonic morphism $\widetilde{\varphi} : \widetilde{\Gamma} \to T$ of degree $d + \alpha$ such that $\widetilde{\varphi}_{|\Gamma} = \varphi$ and all degrees of $\widetilde{\varphi}$ on edges not in Γ are equal to 1 (see Figure 9(a) in the case of the harmonic morphism depicted in Figure 2(c) with $d = 1$).



a) The morphism $\widetilde{\varphi}$ b) The morphism ψ_e

FIGURE 9. The harmonic morphisms $\widetilde{\varphi}$ and ψ_e in the case of Figure 2(c) with $d = 1$

By construction, any contracted component of $\widetilde{\varphi}$ is an open edge of Γ, and this can be easily resolved. Indeed, if e is a finite contracted edge of $\widetilde{\varphi}$, we do the following (see Figure 9(b)):

- consider the tropical modification $\tau_T : \widetilde{T} \to T$ of T at $\widetilde{\varphi}(e)$; denote by e_1 the new end of $\widetilde{T}$;
- consider $\tau_e : \widetilde{\Gamma}_e \to \widetilde{\Gamma}$ the composition of two elementary tropical modifications of $\widetilde{\Gamma}$ at the middle of the edge e; denote by e_2 and e_3 the two new infinite edges of $\widetilde{\Gamma}_e$, and by e_4 and e_5 the two new finite edges of $\widetilde{\Gamma}_e$;
- subdivide e_1 into a finite edge e_1^0 of length equal to the lengths of e_4 and e_5, and an infinite edge e_1^∞;
- consider the morphism of metric graphs $\widetilde{\varphi}_e : \widetilde{\Gamma}_e \to \widetilde{T}$ defined by
  - $\widetilde{\varphi}_{e|\widetilde{\Gamma}\setminus\{e_2,e_3,e_4,e_5\}} = \widetilde{\varphi}$,
  - $\widetilde{\varphi}_e(e_2) = \widetilde{\varphi}_e(e_3) = e_1^\infty$, and $\widetilde{\varphi}_e(e_4) = \widetilde{\varphi}_e(e_5) = e_1^0$,
  - $d_{e_i}(\widetilde{\varphi}_e) = 1$ for $i = 2, 3, 4, 5$.
- extend $\widetilde{\varphi}_e$ to a harmonic morphism of metric graphs $\psi_e : \Gamma' \to \widetilde{T}$, where Γ' is a modification of $\widetilde{\Gamma}_e$ at regular vertices in $\widetilde{\varphi}_e^{-1}(\widetilde{\varphi}(e))$, with all degrees of $\widetilde{\varphi}$ on edges not in $\widetilde{\Gamma}_e$ equal to 1.

We resolve in the same way a contracted infinite end of $\widetilde{\Gamma}$. By applying this process to all contracted edges of $\widetilde{\varphi}$, we end up with a finite harmonic morphism of metric graphs which is a tropical modification of $\widetilde{\varphi}$.

Note that in the proof of (1) we increased some of the local degrees by one, but we could have increased these local degrees by any amount by inserting an arbitrary number of copies of T in the construction of $\widetilde{\Gamma}$. Based on this remark, the proof of (2) now follows the same steps as the proof of (1), using in addition the following claim:

Claim. Let $g' \geq 0$ and $d, s > 0$ be integers. Let $\mu_1, \ldots, \mu_s$ be a collection of s tame partitions of d. Then there exist arbitrarily large non-negative integers d' such that $\mathcal{A}^{d'}_{g',0}(\mu'_1, \ldots, \mu'_s)$ is non-empty, where μ'_i is the partition of d' obtained by adding a sequence of $d' - d$ numbers 1 to each partition μ_i.

Figure 10, Figure 7(a), our resolution procedure, and the argument used for the positive characteristic case of the proof of Theorem 3.12 reduce the proof of the claim to the case $s = 1$ and $\mu_1 = \{d\}$ with $(d, p) = 1$. But in this case, for any $g' \geq 0$, by the group theoretic method we used in the proof of Theorem 3.12, there exists a (tame) covering of $\mathbf{P}^1$ by a curve of genus g' having (tame) monodromy group the cyclic group $\mathbf{Z}/d\mathbf{Z}$, and with the property that the ramification profile above the point 1 of $\mathbf{P}^1$ is given by $\mu = \{d\}$. This finishes the proof of the claim, and the proposition follows. ■

4. APPLICATIONS

4.1. Linear equivalence of divisors. A *(tropical) rational function* on a metric graph Γ is a continuous piecewise affine function $F : \Gamma \to \mathbf{R}$ with integer slopes. If F is a rational function on Γ, $\mathrm{div}(F)$ is the divisor on Γ whose coefficient at a point x of Γ is given by $\sum_{v \in T_x} d_v F$, where the sum is over

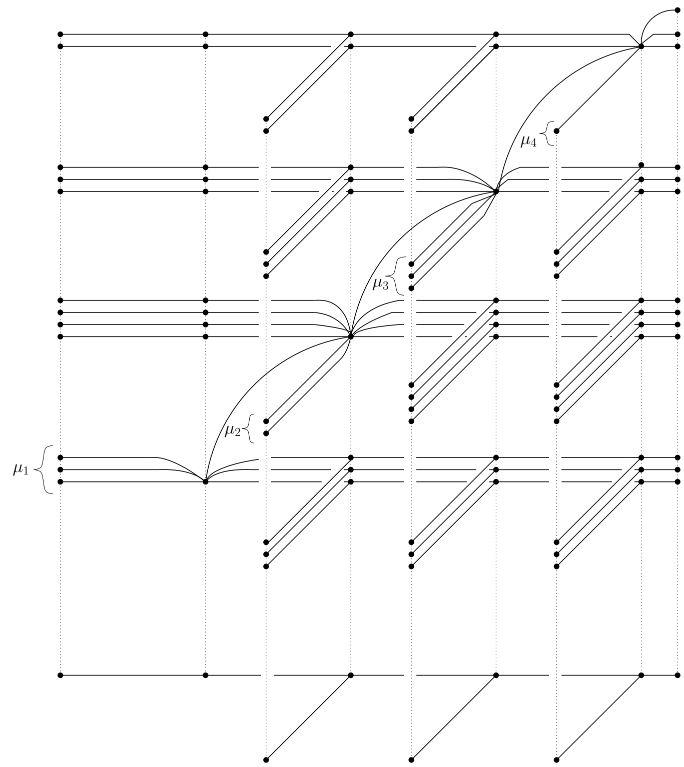


FIGURE 10. Reduction to the case $s = 1$ in the proof of (2) in Proposition 3.20. Degrees on (infinite) edges related to μ_i are exactly the integers appearing in μ_i. All the other degrees are one. Degrees over each infinite edge consist of a μ_i and precisely $(s-1)d$ numbers 1.

all tangent directions to Γ at x and $d_v F$ is the outgoing slope of F at x in the direction v. Two divisors D and D' on a metric graph Γ are called *linearly equivalent* if there exists a rational function F on Γ such that $D - D' = \operatorname{div}(F)$, in which case we write $D \sim D'$. For a divisor D on Γ, the *complete linear system of D*, denoted $|D|$, is the set of all effective divisors E linearly equivalent to D. The *rank* of a divisor $D \in \operatorname{Div}(\Gamma)$ is defined to be

$$r_\Gamma(D) := \min_{\substack{E\,:\,E\geq 0\\ |D-E|=\emptyset}} \deg(E) - 1.$$

Let $\varphi : \Gamma \to T$ be a finite harmonic morphism from Γ to a metric tree T of degree d. For any point $x \in T$, the (local degree of φ at the points of the) fiber $\varphi^{-1}(x)$ defines a divisor of degree d in $\operatorname{Div}(\Gamma)$ that we denote by $D_x(\varphi)$. We have

$$D_x(\varphi) := \sum_{y\in\varphi^{-1}(x)} d_y(\varphi)(y),$$

where $d_y(\varphi)$ denotes the local degree of φ at y.

Proposition 4.2. *Let $\varphi : \Gamma \to T$ be a finite harmonic morphism of degree d from Γ to a metric tree. Then for any two points x_1 and x_2 in T, we have $D_{x_1}(\varphi) \sim D_{x_2}(\varphi)$ in Γ. Moreover, for every $x \in T$ the rank of the divisor $D_x(\varphi)$ is at least one.*

Proof. Since T is connected, we may assume that x_1 and x_2 are sufficiently close; more precisely, we can suppose that x_2 lies on the same edge as x_1 with respect to some model G for Γ. Removing the open segment (x_1, x_2) from T leaves two connected components T_{x_1} and T_{x_2} which contain x_1

and x_2, respectively. Identifying the segment $[x_1, x_2]$ with the interval $[0, \ell]$ by a linear map (where $\ell = \ell([x_1, x_2])$ denotes the length in T of the segment $[x_1, x_2]$) gives a rational function $F : \Gamma \to [0, \ell]$ by sending $\varphi^{-1}(T_{x_1})$ and $\varphi^{-1}(T_{x_2})$ to 0 and ℓ, respectively. It is easy to verify that $D_{x_1}(\varphi) - D_{x_2}(\varphi) = \mathrm{div}(F)$, which establishes the first part.

The second part follows from the first, since y belongs to the support of the divisor $D_{\varphi(y)}(\varphi) \sim D_x(\varphi)$ for all $y \in \Gamma$, which shows that $r_\Gamma(D_x(\varphi)) \geq 1$. ∎

By Theorem 3.11, any finite morphism $\varphi : \Gamma \to T$ can be lifted to a morphism $\varphi : X \to \mathbf{P}^1$ of smooth proper curves, possibly with $g(X) > g(\Gamma)$. This shows that any effective divisor on Γ which appears as a fiber of a finite morphism to a metric tree can be lifted to a divisor of rank at least one on a smooth proper curve *of possibly higher genus*.

We are now going to show that the (additive) equivalence relation generated by fibers of "tropicalization" of finite morphisms $X \to \mathbf{P}^1$ coincides with tropical linear equivalence of divisors. To give a more precise statement, let Γ be a metric graph with first Betti number $h_1(\Gamma)$, and consider the family of all smooth proper curves of genus $h_1(\Gamma)$ over K which admit a semistable vertex set V and a finite set of K-points D such that the metric graph $\Sigma(X, V \cup D)$ is a modification of Γ. Given such a curve X and a finite morphism $\varphi : X \to \mathbf{P}^1$, there is a corresponding finite harmonic morphism $\varphi : \Sigma(X, V \cup D) \to T$ from a modification of Γ to a metric tree T. Two effective divisors D_0 and D_1 on Γ are called *strongly effectively linearly equivalent* if there exists a morphism $\varphi : \Sigma(X, V \cup D) \to T$ as above such that $D_0 = \tau_*(D_{x_0}(\varphi))$ and $D_1 = \tau_*(D_{x_1}(\varphi))$ for two points x_0 and x_1 in T. Here $\tau_* : \mathrm{Div}(\Sigma(X, V \cup D)) \to \mathrm{Div}(\Gamma)$ is the extension by linearity of the retraction map $\tau : \Sigma(X, V \cup D) \to \Gamma$. The equivalence relation on the abelian group $\mathrm{Div}(\Gamma)$ generated by this relation is called *effective linear equivalence* of divisors. In other words, two divisors D_0 and D_1 on Γ are effectively linearly equivalent if and only if there exists an effective divisor E on Γ such that $D_0 + E$ and $D_1 + E$ are strongly effectively linearly equivalent. This can be summarized as follows: D_0 and D_1 on Γ are effectively linearly equivalent if and only if there exists a lifting of Γ to a smooth proper curve X/K of genus $h_1(\Gamma)$, and a finite morphism $\varphi : X \to \mathbf{P}^1$ such that $\tau_*(\varphi^{-1}(0)) = D_0 + E$ and $\tau_*(\varphi^{-1}(\infty)) = D_1 + E$ for some effective divisor E, where τ_* is the natural retraction map from $\mathrm{Div}(X)$ to $\mathrm{Div}(\Gamma)$.

Theorem 4.3. *The two notions of linear equivalence and effective linear equivalence of divisors on a metric graph Γ coincide. As a consequence, linear equivalence of divisors is the additive equivalence relation generated by (the retraction to Γ of) fibers of finite harmonic morphisms from a tropical modification of Γ to a metric graph of genus zero.*

Proof. Consider two divisors D_0 and D_1 which are effectively linearly equivalent. There exists an effective divisor E and a finite harmonic morphism $\varphi : \widetilde{\Gamma} \to T$, from a tropical modification of Γ to a metric tree, such that $D_0 + E = D_{x_0}(\varphi)$ and $D_1 + E = D_{x_1}(\varphi)$ for two points $x_0, x_1 \in T$. By Proposition 4.2 we have $D_0 + E \sim D_1 + E$, which implies that D_0 and D_1 are linearly equivalent in in $\widetilde{\Gamma}$, and hence in Γ.

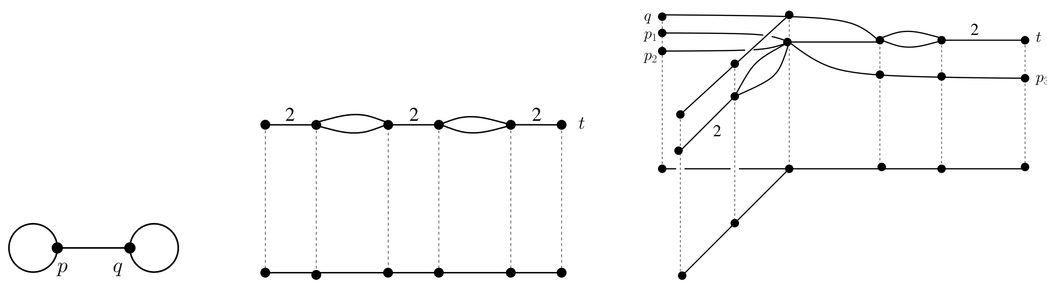
To prove the other direction, it will be enough to show that if D is linearly equivalent to zero, then there exists an effective divisor E such that $D + E$ and E are fibers of a finite harmonic morphism φ from a modification of Γ to a metric tree T, and such that φ can be lifted to a morphism $X \to \mathbf{P}^1$.

By assumption, there exists a rational function $f : \Gamma \to \mathbf{R} \cup \{\pm\infty\}$ such that $D + \mathrm{div}(f) = 0$. We claim that there is a tropical modification $\widetilde{\Gamma}$ of Γ together with an extension of f to a (not necessarily finite) harmonic morphism $\varphi_0 : \widetilde{\Gamma} \to \mathbf{R} \cup \{\pm\infty\}$. The tropical modification $\widetilde{\Gamma}$ is obtained from Γ by choosing a vertex set which contains all the points in the support of D, adding an infinite edge to any finite vertex in Γ with $\mathrm{ord}_v(f) \neq 0$, and extending f as an affine linear function of slope $-\mathrm{ord}_v(f)$ along this infinite edge. It is clear that the resulting map φ_0 is harmonic.

Consider now the retraction map $\tau : \widetilde{\Gamma} \to \Gamma$, and note that for the two divisors $D_{\pm\infty}(\varphi_0)$, we have $\tau_*\big(D_{\pm\infty}(\varphi_0)\big) = D_\pm$, where D_+ and D_- denote the positive and negative part of D, respectively. By Proposition 3.20, there exist tropical modifications $\overline{\Gamma}$ of $\widetilde{\Gamma}$ and T of $\mathbf{R} \cup \{\pm\infty\}$ such that φ_0 extends to a finite harmonic morphism $\varphi : \overline{\Gamma} \to T$ which can be lifted to a finite morphism $X \to \mathbf{P}^1$. If we denote (again) the retraction map $\overline{\Gamma} \to \Gamma$ by τ, then $\tau_*\big(D_{\pm\infty}(\varphi)\big) = D_\pm + E_0$ for some effective divisor E_0 in Γ. Setting $E = D_- + E_0$, the divisors $D + E$ and E are strongly effectively linearly equivalent, and the theorem follows. ■

Example 4.4. Here is an example which illustrates the distinction between the notions of (effective) linear equivalence and strongly effective linear equivalence of divisors, as introduced above.

Let Γ be the metric graph depicted in Figure 11(a), with arbitrary lengths, and $K_\Gamma = (p) + (q)$ the canonical divisor on Γ.



a) $K_\Gamma = (p) + (q)$ b) An effective lift of $2(t)$ c) A non-effective lift of K_Γ

FIGURE 11

We claim that K_Γ is not the specialization of any effective divisor of degree two representing the canonical class of a smooth proper curve of genus two over K. More precisely, we claim that for any triangulated punctured curve $(X, V \cup D)$ over K such that $\Sigma(X, V \cup D)$ is a tropical modification of Γ, and for any effective divisor $\mathcal{D}$ in $\mathrm{Div}(X)$ with $K_\Gamma = \tau_*(\mathcal{D})$, we must have $r_X(\mathcal{D}) = 0$. (Here τ_* denotes the specialization map from $\mathrm{Div}(X)$ to $\mathrm{Div}(\Gamma)$ and $r_X(D) = \dim_K(H^0(X, \mathcal{O}(\mathcal{D}))) - 1$.) Indeed, otherwise there would exist a degree 2 finite harmonic morphism $\pi : \widetilde{\Gamma} \to T$ from some tropical modification of Γ to a metric tree with the property that $\pi(p) = \pi(q)$. Restricting such a harmonic morphism to the preimage in $\widetilde{\Gamma}$ of the loop containing p would imply, by Proposition 4.2, that the divisor (p) has rank one in a genus-one metric graph, which is impossible. On the other hand, Figure 11(b) shows that the divisor $2(t) \sim (p) + (q)$ can be lifted to an effective representative of the canonical class K_X, where t is the middle point of the loop edge with vertex q. This shows that the two linearly equivalent divisors $D_0 = (p) + (q)$ and $D_1 = 2(t)$ are not strongly effectively linearly equivalent.

However, D_0 and D_1 are effectively linearly equivalent. Indeed, adding $E = (p)$ to D_0 and D_1, respectively, gives the two divisors $2(p) + (q)$ and $2(t) + (p)$ which are retractions of fibers of a degree 3 finite harmonic morphism from a tropical modification of Γ to a tree, as shown in Figure 11(c). Consequently, $D_0 + (p)$ and $D_1 + (p)$ can be lifted to linearly equivalent effective divisors on a smooth proper curve X.

Note also that Figure 11(c) shows that since $(p_1) + (p_2) + (q) - (p_3)$ can be lifted to a non-effective representative of the canonical class K_X, there exists a non-effective divisor $\mathcal{D}$ in the canonical class K_X of X such that $\tau_*(\mathcal{D}) = (p) + (q)$.

4.5. Tame actions and quotients. Let $\mathcal{C}$ be a metrized complex of k-curves, and denote by Γ the underlying metric graph of $\mathcal{C}$. An *automorphism* of $\mathcal{C}$ is a (degree one) finite harmonic morphism of metrized complexes $h : \mathcal{C} \to \mathcal{C}$ which has an inverse. The group of automorphisms of $\mathcal{C}$ is denoted by $\mathrm{Aut}(\mathcal{C})$.

Let H be a finite subgroup of $\mathrm{Aut}(\mathcal{C})$. The action of H on $\mathcal{C}$ is *generically free* if for any vertex v of Γ, the inertia (stabilizer) group H_v acts freely on an open subset of C_v. A finite subgroup H of $\mathrm{Aut}(\mathcal{C})$ is called *tame* if the action of H on $\mathcal{C}$ is generically free and all the inertia subgroups H_x for x

belonging to some C_v are cyclic of the form $\mathbf{Z}/d\mathbf{Z}$ for some positive integer d, with $(d,p)=1$ if $p>0$. In this case we say that the action of H on $\mathcal{C}$ is tame.

Remark 4.6. The stabilizer condition in the definition of tame actions is equivalent to requiring the cover $C_v \to C_v/H_v$ be tame, where H_v is the stablizer of the vertex v. To see that this latter condition implies all the stablizers of points on C_v are cyclic, consider a uniformizer π at a point x, and consider the map $H_x \to k^\times$ which sends an element $h \in H_x$ to $h(\pi)/\pi$. This is independent of the choice of the uniformizer, and embeds H_x in the subgroup of roots of unity in $k^\times$, from which the assertion follows. The other direction is clear from the definition.
Note that, more generally, one has a filtration of H_v with higher ramification groups $H_v \supseteq H_0 = H_x \supseteq H_1 \supseteq H_2 \supseteq \dots$, the quotient H_0/H_1 is a finite cyclic group of order prime to the characteristic p, and H_i/H_{i+1} are all p-groups. In the case of tame actions, H_1 is trivial.

In this section, we characterize tame group actions H on $\mathcal{C}$ which lift to an action of H on some smooth proper curve X/K lifting $\mathcal{C}$. The main problem to consider is whether there exists a refinement $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ and an extension of the action of H to $\widetilde{\mathcal{C}}$ such that the quotient $\widetilde{\mathcal{C}}/H$ can be defined, and such that the projection map $\pi : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}/H$ is a tame harmonic morphism. The lifting of the action of H to a smooth proper curve X as above will then be a consequence of our lifting theorem.

4.7. Let H be a tame group of automorphisms of a metrized complex $\mathcal{C}$. Let $W_H = W_H(\mathcal{C})$ be the set of all $w \in \Gamma$ lying in in the middle of an edge e such that there is an element $h \in H$ having w as an isolated fixed point. Denote by H_w the stabilizer of $w \in W_H$. It is easy to see that H_w consists of all elements h of H which restrict on e either to the identity or to the symmetry with center w. In particular, if $h|_e \neq \mathrm{id}$, then h permutes the two vertices p and q adjacent to e. For $w \in W_H$, the inertia group $H_{\mathrm{red}_p(e)} = H_{\mathrm{red}_q(e)} \cong \mathbf{Z}/d_e\mathbf{Z}$ (for some integer d_e) is a normal subgroup of index two in H_w:

$$0 \longrightarrow H_{\mathrm{red}_p(e)} \longrightarrow H_w \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

We make the following assumption on the groups H_w:

Definition 4.8. A tame group of automorphisms H of a metrized complex $\mathcal{C}$ satisfies the *dihedral condition* provided that, for all $w \in W_H$, the stabilizer group H_w is isomorphic to the dihedral group generated by two elements σ and ζ with the relations

$$\sigma^2 = 1, \quad \zeta^d = 1, \quad \text{and} \quad \sigma\zeta\sigma = \zeta^{-1}$$

for some integer d, such that $H_{\mathrm{red}_p(e)} = \langle \zeta \rangle$.

The dihedral condition means that the above short exact sequence splits, and the action of $\mathbf{Z}/2\mathbf{Z} \cong \{\pm 1\}$ on $H_{\mathrm{red}_p(e)}$ is given by $h \to h^{\pm 1}$ for $h \in H_{\mathrm{red}_p(e)}$.

We can now formulate our main theorem on lifting tame group actions:

Theorem 4.9. *Let H be a finite group with a tame action on a metrized complex $\mathcal{C}$.*

(1) *If $W_H \neq \emptyset$, then the dihedral condition and $\mathrm{char}(k) \neq 2$ are the necessary and sufficient conditions for the existence of a refinement $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ such that the action of H on $\mathcal{C}$ extends to a tame action on $\widetilde{\mathcal{C}}$ such that $W_H(\widetilde{\mathcal{C}}) = \emptyset$.*[2]
(2) *If $W_H = \emptyset$, then the quotient $\mathcal{C}/H$ exists in the category of metrized complexes. In addition, the action of H on $\mathcal{C}$ can be lifted to an action of H on a triangulated punctured K-curve $(X, V \cup D)$ such that $\Sigma(X, V \cup D_0) \cong \mathcal{C}$ with $D_0 \subset D$, the action of H on $X \setminus D$ is étale, and the inertia group H_x for $x \in D$ coincides with the inertia group $H_{\tau(x)}$ of the point $\tau(x) \in \Sigma(X, V \cup D_0) = \mathcal{C}$.*

Proof. Suppose that $W_H \neq \emptyset$, that the dihedral condition holds, and that $\mathrm{char}(k) \neq 2$. Fix an orientation of the edges of Γ, and for an oriented edge e, denote by p_0 and p_∞ the two vertices of Γ which form the tail and the head of e, respectively. Let w be a point lying in the middle of an oriented edge $e = (p_0, p_\infty)$ of Γ which is an isolated fixed point of some elements of H. Take the refinement $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ obtained by adding all such points w to the vertex set of Γ and by setting $C_w = \mathbf{P}^1_k$,

[2]See [Ray99, §2.3] for a related discussion, including remarks on the situation in characteristic 2.

$\mathrm{red}_e(\{w, p_0\}) = 0$, and $\mathrm{red}(\{w, p_\infty\}) = \infty$. To see that the action of H on $\mathcal{C}$ extends to $\widetilde{\mathcal{C}}$, first note that one can define a generically free action of H_w on $\mathbf{P}^1_k$ (equivalently, one can embed H_w in $\mathrm{Aut}(\mathbf{P}^1_k)$) in a way compatible with the action of H_w on Γ, i.e., such that all the elements of $H_{\mathrm{red}_{p_0}(e)} = H_{\mathrm{red}_{p_\infty}(e)}$ fix the two points 0 and ∞ of $\mathbf{P}^1_k$, and such that the other elements of H_w permutes the two points $0, \infty \in \mathbf{P}^1_k$. Indeed, the dihedral condition is the necessary and sufficient condition for the existence of such an action. Under this condition and upon a choice of a $d_e = |H_{\mathrm{red}_{p_0}(e)}|$-th root of unity $\zeta_{d_e} \in k$, and upon the choice of the point $1 \in \mathbf{P}^1_k$ as a fixed point of σ, the actions of the two generators σ and ζ of H_w on $\mathbf{P}^1$ are given by $\sigma(z) = 1/z$ and $\zeta(z) = \zeta_{d_e} z$, respectively.

Fix once for all a d-th root of unity $\zeta_d \in k$ for each positive integer d (with $(d, p) = 1$ in the case $p > 0$). Given $h \in H$, we extend the action of h on $\mathcal{C}$ to an action on $\widetilde{\mathcal{C}}$ in the following way. Let $w \in W_H(\mathcal{C})$ and let e be the edge containing w, with the orientation chosen above. If $h(w) \neq w$, we define $h_w : C_w \to C_{h(w)}$ by $h_w = \mathrm{id}_{\mathbf{P}^1}$ if h is compatible with the orientations of e and $h(e)$, and we set $h_w(z) = z^{-1}$ otherwise. If $h \in H_w$, we define the action of h on C_w as above. This defines a generically free action of H on $\widetilde{\mathcal{C}}$. The inertia groups of the points $0, \infty$, and ± 1 in C_w are $\mathbf{Z}/d_e\mathbf{Z}$, $\mathbf{Z}/d_e\mathbf{Z}$, and $\mathbf{Z}/2\mathbf{Z}$, respectively. Since $p \neq 2$, this shows that the action of H on $\widetilde{\mathcal{C}}$ is tame. By construction we have $W_H(\widetilde{\mathcal{C}}) = \emptyset$.

Working backward, one recovers the necessity of the dihedral condition and $\mathrm{char}(k) \neq 2$. Indeed, any $\widetilde{\mathcal{C}}$ satisfying the conditions of the theorem must contain each $w \in W_H(\mathcal{C})$ as a vertex. Since H_w acts on $\mathbf{P}^1_k$ in the manner described above, it must be a dihedral group; since its action on C_w has stabilizers of order ± 2, we must have $\mathrm{char}(k) \neq 2$.

Now we assume that the action of H on $\mathcal{C}$ is tame and that no element of H has an isolated fixed point in the middle of an edge. We will define the quotient metrized complex $\mathcal{C}/H$. The metric graph underlying $\mathcal{C}/H$ is the quotient graph Γ/H equipped with the following metric: given an edge e of Γ of length ℓ and stabilizer H_e, we define the length of its projection in Γ/H to be $\ell \cdot |H_e|$. The projection map $\Gamma \to \Gamma/H$ is a tame finite harmonic morphism.

For any vertex p of Γ, the k-curve associated to its image in $\mathcal{C}/H$ is C_p/H_p. The marked points of C_p/H_p are the different orbits of the marked points of C_p, and are naturally in bijection with the edges of Γ/H adjacent to the projection of p. The projection map $\mathcal{C} \to \mathcal{C}/H$ is a tame harmonic morphism of metrized complexes.

To see the second part, let $\mathcal{C}'$ be the (tropical) modification of $\mathcal{C}$ obtained as follows: for any closed point $x \in C_p$ with a non-trivial inertia group and which is not the reduction $\mathrm{red}_p(e)$ of any edge e adjacent to p, consider the elementary tropical modification of $\mathcal{C}$ at x. Extend the action of H to a tame action on $\mathcal{C}'$ by defining $h_x : e_x \to e_{h(x)}$ to be affine with slope one for any such point. Let $\pi : \mathcal{C}' \to \mathcal{C}'/H$ be the projection map. Let $(X', V' \cup D')$ be a triangulated punctured K-curve such that $\mathcal{C}(X', V' \cup D') \cong \mathcal{C}'/H$. By Theorem I.7.4, the tame harmonic morphism π lifts to a morphism of triangulated punctured K-curves $(X, V \cup D) \to (X', V' \cup D')$. By Remark I.7.5, we have an injection $\iota : \mathrm{Aut}_{X'}(X) \hookrightarrow \mathrm{Aut}_{\mathcal{C}'/H}(\mathcal{C}')$. By the construction given in the proof of Theorem I.7.4, it is easy to see that every $h \in H$ lies in the image of ι, and thus $H \subset \mathrm{Aut}_{X'}(X)$. The last part follows formally from the definition of the modification $\mathcal{C}'$ and the choice of X as the lifting of $\pi : \mathcal{C}' \to \mathcal{C}'/H$. ■

Remark 4.10. (Compare with Remark 4.6) If the characteristic of k is positive, the lifting of the action of a finite group on a metrized complexes cannot be guaranteed in general without further assumptions. Indeed, even in the smooth case, i.e., where the metrized complex consists of a single vertex v and a single curve C_v, there are obstructions to the lifting [Oor87, OSS89, GM98, BM00], e.g., due to the fact that the automorphism group of a smooth proper curve in positive characteristic does not respect the Hurwitz upper bound $84(g-1)$. However, Pop's recent proof of the Oort conjecture [Pop14], based on the results of Obus and Wewer [OW14], shows that in the smooth case, the action can be lifted under the assumption that the stablizers of points are all cyclic. A natural question is then to see whether our theorem can be extended by only requiring all the stablizers of points to be cyclic (without the tameness assumption).

4.11. Characterization of liftable hyperelliptic augmented metric graphs. Let Γ be an augmented metric graph and denote by $r^{\#}$ the weighted rank function on divisors introduced in [AC13]. Recall that this is the rank function on the non-augmented metric graph $\Gamma^{\#}$ obtained from Γ by attaching $g(p)$ cycles, called *virtual cycles*, of (arbitrary) positive lengths to each $p \in \Gamma$ with $g(p) > 0$. We say that an augmented metric graph Γ is *hyperelliptic* if $g(\Gamma) \geq 2$ and there exists a divisor D in Γ of degree two such that $r^{\#}_{\Gamma}(D) = 1$. An augmented metric graph is said to be *minimal* if it contains neither infinite vertices nor 1-valent vertices of genus 0. Every augmented metric graph Γ is tropically equivalent to a minimal augmented metric graph Γ', which is furthermore unique if $g(\Gamma) \geq 2$. Since the tropical rank and weighted rank functions are invariant under tropical modifications, an augmented metric graph Γ is hyperelliptic if and only if Γ' is. Hence we restrict in this section to the case of minimal augmented metric graphs.

The following proposition is a refinement of a result from [Cha12] on vertex-weighted metric graphs (itself a strengthening of results from [BN09]):

Proposition 4.12. *For a minimal augmented metric graph Γ of genus at least two, the following assertions are equivalent:*

(1) *Γ is hyperelliptic;*
(2) *There exists an involution s on Γ such that:*
   (a) *s fixes all the points $p \in \Gamma$ with $g(p) > 0$;*
   (b) *the quotient Γ/s is a metric tree;*
(3) *There exists an effective finite harmonic morphism of degree two $\varphi : \Gamma \to T$ from Γ to a metric tree T such that the local degree at any point $p \in \Gamma$ with $g(p) > 0$ is two.*

Furthermore if the involution s exists, then it is unique.

Proof. The implication $(2) \Rightarrow (3)$ is obtained by taking $T = \Gamma/s$ and letting φ be the natural quotient map.

To prove $(3) \Rightarrow (1)$, we observe that a finite harmonic morphism of degree two $\varphi : \Gamma \to T$ with local degree two at each vertex p with $g(p) > 0$ naturally extends to an effective finite harmonic morphism of degree two from a tropical modification Γ' of $\Gamma^{\#}$ to a tropical modification T' of T as follows: Γ' is obtained by modifying $\Gamma^{\#}$ once at the midpoint of each of its virtual cycles, and T' is obtained by modifying T precisely $g(p)$ times at each point $\varphi(p)$ with $g(p) > 0$. The map φ extends uniquely to an effective finite degree two harmonic morphism $\varphi' : \Gamma' \to T'$, since φ has local degree two at p whenever $g(p) > 0$. By Proposition 4.2, the linearly equivalent degree two divisors $D_x(\varphi')$ have rank one in Γ' as x varies over all points of T', which shows that Γ is hyperelliptic.

It remains to prove $(1) \Rightarrow (2)$. A *bridge edge* of Γ is an edge e such that $\Gamma \setminus e$ is not connected. Let Γ' be the augmented metric graph obtained by removing all bridge edges from Γ. Since Γ is minimal, any connected component of Γ' has positive genus. In particular the involution s, if exists, has to restrict to an involution on each such connected component. This implies that s has to fix pointwise any bridge edge. Hence we may now assume without loss of generality that Γ has no bridge edge. In this case s has the following simple definition: for any point $p \in \Gamma$, since $r_{\Gamma^{\#}}(D) = 1$ and Γ is two-edge connected, there exists a unique point $q = s(p)$ such that $D \sim (p) + (q)$. This also proves the uniqueness of the involution. ∎

From now until the end of the section we assume that $\mathrm{char}(k) \neq 2$. An *involution* on a metrized complex $\mathcal{C}$ is a finite harmonic morphism $s : \mathcal{C} \to \mathcal{C}$ with $s^2 = \mathrm{id}_{\mathcal{C}}$. An involution is called *tame* if the action of the group generated by $\langle s \rangle \cong \mathbf{Z}/2\mathbf{Z}$ on $\mathcal{C}$ is tame.

If X/K is a (smooth proper) hyperelliptic curve, then the augmented metric graph Γ associated to stable model of X is hyperelliptic. Indeed if s_X is an involution on X, then the quotient map $X \to X/s$ tropicalizes to an effective tropical morphism $\varphi : \Gamma \to T$ of degree 2. The condition that φ has local degree 2 at each point p with $g(p) > 0$ comes from the fact that any non-constant algebraic map from a positive genus curve to $\mathbf{P}^1$ has degree at least two. The next theorem, combined with Proposition 4.12, provides a complete characterization of hyperelliptic augmented metric graphs which can be realized as the skeleton of a hyperelliptic curve over K.

Theorem 4.13. *Let Γ be a minimal hyperelliptic augmented metric graph, and let $s : \Gamma \to \Gamma$ be the involution given by Proposition 4.12 (2). Then the following assertions are equivalent:*

(1) *There exists a hyperelliptic smooth proper curve X over K and an involution $s_X : X \to X$ such that Γ is the minimal skelton of X, and s coincides with the reduction of s_X to Γ.*
(2) *For every $p \in \Gamma$ we have*
$$2g(p) \geq \kappa(p) - 2,$$
where $\kappa(p)$ denotes the number of tangent directions at p which are fixed by s.

Proof. Consider the finite harmonic morphism $\pi : \Gamma \to \Gamma/s$. We note that the tangent directions at p which are fixed by s are exactly those along which π has local degree two. Thus the condition $2g(p) \geq \kappa(p) - 2$ is equivalent to the ramification index R_p being non-negative: see Section 2. This proves $(1) \Rightarrow (2)$.

To prove $(2) \Rightarrow (1)$, we use Proposition I.7.15 and Theorem 4.9. According to these results, it suffices to prove that the involution $s : \Gamma \to \Gamma$ lifts to an involution $\overline{s} : \mathcal{C} \to \mathcal{C}$ for some metrized complex $\mathcal{C}$ with underlying augmented metric graph Γ such that $\mathcal{C}/\overline{s}$ has genus zero. The existence of such a lift follows from the observation that Hurwitz numbers of degree two are all positive (see Example 3.4). ■

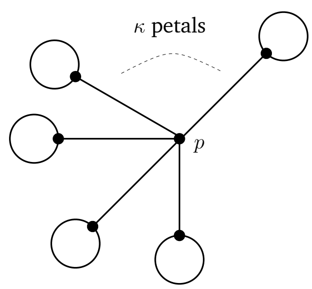


FIGURE 12

Example 4.14. Let Γ be the augmented metric graph of genus g depicted in Figure 12 with arbitrary positive lengths. It is clearly hyperelliptic, and since the involution s restricts to the identity on each bridge edge, it fixes all tangent directions at p. Then one can lift Γ as a hyperellitptic curve of genus g if and only $2g(p) \geq \kappa - 2$. In particular, if $g(p) = 0$ then this metric graph cannot be realized as the skeleton of a hyperelliptic curve as soon as $\kappa \geq 3$.

Since the hyperelliptic involution is unique for both curves and minimal augmented metric graphs, and since the tangent directions fixed by the hyperelliptic involution on an augmented metric graph correspond to bridge edges, we can reformulate Theorem 4.13 as follows, obtaining a metric strengthening of [Cap, Theorem 4.8]:

Corollary 4.15. *Let Γ be a minimal augmented metric graph of genus $g \geq 2$. Then there is a smooth proper hyperelliptic curve X over K of genus g having Γ as its minimal skeleton if and only if Γ is hyperelliptic and for every $p \in \Gamma$ the number of bridge edges adjacent to p is at most $2g(p) + 2$.*

5. GONALITY AND RANK

A fundamental (if vaguely formulated) question in tropical geometry is the following: If X is an algebraic variety and $\mathbf{T}X$ is a tropicalization of X (whatever it means), which properties of X can be read off from $\mathbf{T}X$? In this section, we discuss more precisely (for curves) the relation between the classical and tropical notions of gonality, and of the rank of a divisor. It is not difficult to prove that the gonality of a tropical curve (resp. the rank of a tropical divisor) provides a lower bound for the gonality (resp. an upper bound for the rank) of any lift (this is a consequence, for example, of Corollary I.4.28). Here we address the question of sharpness for these inequalities:

(1) Can a d-gonal (augmented or non-augmented) tropical curve C always be lifted to a d-gonal algebraic curve?
(2) Can a divisor D on an (augmented or non-augmented) tropical curve C always be lifted to divisor of the same rank on an algebraic curve lifting C?

It follows immediately from Theorem 3.11 that the answer to Question (1) is *yes* if C is not augmented, i.e., if we are allowed to arbitrarily increase the genus of finitely many points in C. On the other hand, we prove in this section that the answer to Question (1) in the case C is augmented, and the answer to Question (2) in both cases, is *no*.

We refer to [BN07, MZ08, AC13, AB12] for the basic definitions concerning ranks of divisors on metric graphs, augmented metric graphs, and metrized complexes of curves.

5.1. Gonality of augmented graphs versus gonality of algebraic curves. An augmented tropical curve C is said to have an augmented (non-metric) graph G as its *combinatorial type* if C admits a representative whose underlying augmented graph is G. Given an augmented graph G, we denote by $\mathcal{M}(G)$ the set of all augmented metric graphs which define a tropical curve C with combinatorial type G. In other words, $\mathcal{M}(G)$ consists of all augmented metric graphs which can be obtained by a finite sequence of tropical modifications (and their inverses) from an augmented metric graph Γ with underlying augmented graph G. When no confusion is possible, we identify an (augmented) tropical curve with any of its representatives as an (augmented) metric graph: in what follows, we deliberately write $C \in \mathcal{M}(G)$ for a tropical curve C with combinatorial type G. Note that the spaces $\mathcal{M}(G)$ appear naturally in the stratification of the moduli space of tropical curves of genus $g(G)$, see for example [Cap].

Definition 5.2. An augmented tropical curve C is called *d-gonal* if there exists a tropical morphism $C \to \mathbf{TP}^1$ of degree d.

An augmented graph G is called *stably d-gonal* if there exists a d-gonal augmented tropical curve C whose combinatorial type is G.

In other words, an augmented graph G is stably d-gonal if and only if there is an augmented metric graph $\Gamma \in \mathcal{M}(G)$ which admits an effective finite harmonic morphism of degree d to a metric tree.

Remark 5.3. Our definition of the stable gonality of a graph is equivalent to the one given in [CKK]. See Appendix A of *loc. cit.* for a detailed discussion of the relationship between stable gonality and other tropical or graph-theoretic notions of gonality in the literature, e.g. Caporaso's definition in [Cap].

In this section we prove the following theorem, which is an immediate consequence of Corollary I.4.28 and Propositions 5.5 and 5.6 below.

Theorem 5.4. *There exists an augmented stably d-gonal graph G such that for any augmented metric graph $\Gamma \in \mathcal{M}(G)$ and any smooth proper connected K-curve X lifting Γ, the gonality of X is strictly larger than d.*

Let G_{27} be the graph depicted in Figure 13, which we promote to a totally degenerate augmented graph by taking the genus function to be identically zero. Note that $g(G_{27}) = 27$, and that $G_{27} \setminus \{p\}$ has three connected components, which we denote by A_1, A_2, and A_3 according to Figure 13.

Given an element $\Gamma \in \mathcal{M}(G_{27})$ and a tropical morphism $\varphi : C \to \mathbf{TP}^1$ from the tropical curve represented by Γ to $\mathbf{TP}^1$, we denote by φ_i the restriction of φ to (the metric subgraph in Γ which corresponds to) A_i, and by φ_p the restriction of φ to a small neighborhood of the point p.

Proposition 5.5. *The graph G_{27} depicted in Figure 13 is stably 4-gonal.*

Proof. We need to show the existence of a suitable tropical curve C with combinatorial type G_{27} which admits a tropical morphism of degree four to $\mathbf{TP}^1$. For a suitable choice of edge lengths on G_{27},

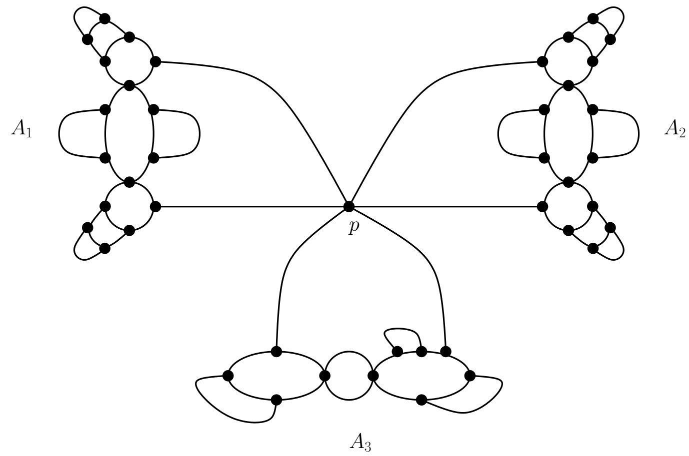


FIGURE 13. The graph G_{27}

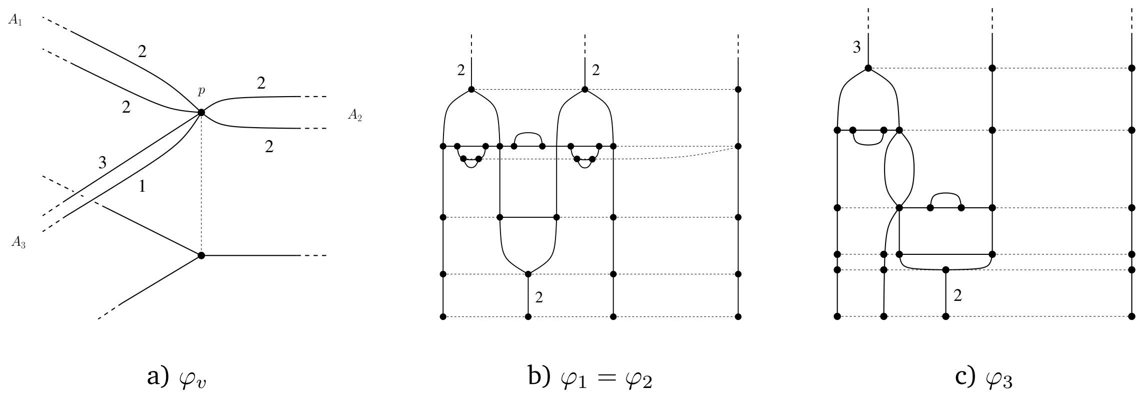
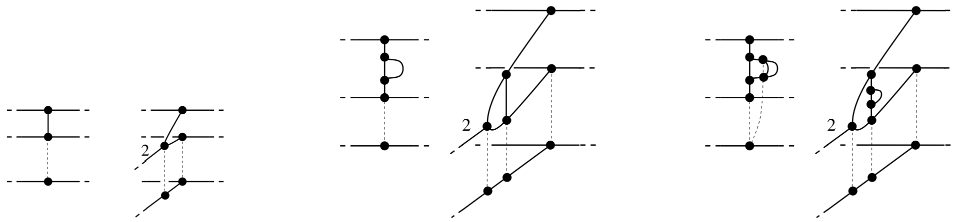


FIGURE 14. A tropical morphism of degree four.

we get an element $\Gamma \in \mathcal{M}(G_{27})$ such that there exists a harmonic morphism from Γ to a star-shaped genus zero augmented metric graph with three infinite edges, which has restrictions $\varphi_1, \varphi_2, \varphi_3, \varphi_v$ to A_1, A_2, A_3, and a small neighborhood of p, respectively, given as in Figure 14. We claim that φ induces a tropical morphism, i.e., that there exists a tropical modification of φ which is finite and effective.

Note that each of the morphisms φ_1 and φ_2 contains a fiber of genus five, while the morphism φ_3 has two different fibers of genus one. All the other fibers of $\varphi_1, \varphi_2,$ and φ_3 are either finite or connected of genus zero. We depict in Figure 15 a few patterns which show how to resolve contractions of φ, turning φ into an augmented tropical morphism. Figure 15(a) shows how to resolve a contracted segment (resolving contracted fibers of genus zero). Figure 15(b) shows how to resolve a contracted cycle (resolving the contracted cycles in φ_3 and the middle contracted cycle in φ_1 and φ_2): the idea is to reduce to the case of a contracted segment, in which case one can use the resolution given in Figure 15(a) to finish. And finally, Figure 15(c) shows how to resolve the two contracted double-cycles in φ_1 and φ_2 by reducing to the case already treated in Figure 15(b). Note that performing these tropical modifications impose conditions on the length of the contracted edges in Γ, e.g., in Figure 15(b), the two edges adjacent to the contracted cycle should have the same length. Nevertheless, by appropriately choosing the edge lengths, we get the existence of a metric graph $\Gamma \in \mathcal{M}(G_{27})$ which admits a finite morphism of degree four to a metric tree. It is easily seen that this morphism is effective; thus we get a tropical curve C with combinatorial type G_{27} and a tropical morphism of degree four to $\mathbf{TP}^1$, finishing the proof of the proposition. ∎



a) Resolution in one step
b) Resolution in two steps (combined with case a)
c) Resolution in three steps (combined with case b)

FIGURE 15. Patterns to resolve contractions in the harmonic morphisms φ_1, φ_2, and φ_3.

To conclude the proof of Theorem 5.4, we now show the following:

Proposition 5.6. *There is no metrized complex of k-curves with underlying augmented metric graph in $\mathcal{M}(G_{27})$ and admitting a finite morphism of degree four to a metrized complex of k-curves of genus zero.*

We emphasize that the statement holds for any (algebraically closed) field k. The proof of Proposition 5.6 relies on some technical lemmas that we are now going to state.

We first recall a formula given in [AB12] for the rank of divisors on a metric graph $\Gamma = \Gamma_1 \vee \Gamma_2$ which is obtained as a wedge sum of two metric graphs Γ_1 and Γ_2. Recall that given two metric graphs Γ_1 and Γ_2 and distinguished points $t_1 \in \Gamma_1$ and $t_2 \in \Gamma_2$, the *wedge sum* or *direct sum* of (Γ_i, t_i), denoted $\Gamma = \Gamma_1 \vee \Gamma_2$, is the metric graph obtained by identifying the points t_1 and t_2 in the disjoint union of Γ_1 and Γ_2. Denoting by $t \in \Gamma$ the image of t_1 and t_2 in Γ, one refers to $t \in \Gamma$ as a *cut-vertex* and to $\Gamma = \Gamma_1 \vee \Gamma_2$ as the decomposition corresponding to the cut-vertex t. (By abuse of notation, we will use t to denote both t_1 in Γ_1 and t_2 in Γ_2.) There is an addition map $\mathrm{Div}(\Gamma_1) \oplus \mathrm{Div}(\Gamma_2) \to \mathrm{Div}(\Gamma)$ which sends a pair of divisors D_1 and D_2 in $\mathrm{Div}(\Gamma_1)$ and $\mathrm{Div}(\Gamma_2)$ to the divisor $D_1 + D_2$ on Γ defined by pointwise addition of the coefficients in D_1 and D_2.

Let $D_1 \in \mathrm{Div}(\Gamma_1)$ and $D_2 \in \mathrm{Div}(\Gamma_2)$. For any non-negative m, define $\eta_{\Gamma_1, D_1}(m)$ as minimum integer h such that $r_{\Gamma_1}(D_1 + h(t_1)) = m$. Then

(5.6.1)
$$r_\Gamma(D) = \min_{m \geq 0} \big\{ m + r_{\Gamma_2}(D_2 - \eta_{\Gamma_1, D_1}(m)(t_2)) \big\}.$$

(see [AB12] for details).

In what follows, equation (5.6.1) will be applied to a metric graph $\Gamma \in \mathcal{M}(A_1) = \mathcal{M}(A_2)$ (see Figure 16(a) and Lemma 5.7), to a metric graph $\Gamma \in \mathcal{M}(A_3)$ (see Figure 16(b) and Lemma 5.9), and to $\Gamma_{27} \in \mathcal{M}(G_{27})$ with cut-vertex p in the proof of Proposition 5.6.

Lemma 5.7. *Let Γ be a metric graph in $\mathcal{M}(A_1) = \mathcal{M}(A_2)$ as depicted in Figure 16(a). For any non-negative integers $a \leq 3$ and $b \leq 1$, the divisors $a(p) + b(q)$ and $b(p) + a(q)$ have rank zero in Γ.*

Proof. By symmetry it is enough to prove the lemma for the divisor $D = 3(p) + (q)$. Consider the decomposition $\Gamma = \Gamma_p \vee \Gamma_q$ associated to the cut-vertex t in Γ, where Γ_p and Γ_q denote the closure in Γ of the the two connected components of $\Gamma \setminus \{t\}$ which contain the points p and q, respectively.

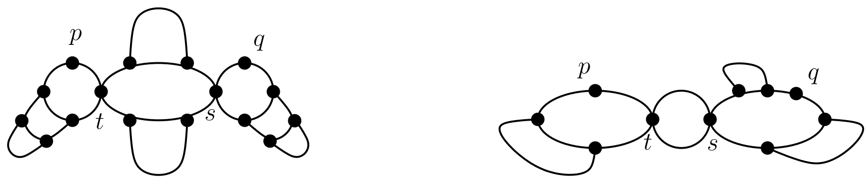
We claim that $\eta_{\Gamma_q,(q)}(1) = 3$. Assume for the moment that this is true. Then by (5.6.1), we have

$$0 \leq r_\Gamma(3(p) + (q)) \leq 1 + r_{\Gamma_p}(3(p) - 3(t)).$$

By Lemma 5.8 below, in Γ_p we have $r_{\Gamma_p}(3(p) - 3(t)) = -1$. We thus infer that $r_\Gamma(3(p) + (q)) = 0$.

It remains to prove that $\eta_{\Gamma_q,(q)}(1) = 3$. In other words, we need to show that in Γ_q we have $r_{\Gamma_q}(2(t) + (q)) = 0$. For this, consider the decomposition $\Gamma_q = \Gamma_q^t \vee \Gamma_q^q$ corresponding to the cut-vertex s in Γ_q, where Γ_q^t and Γ_q^q denote the components which contain t and q, respectively. We claim that $\eta_{\Gamma_q^t, 2(t)}(1) = 1$. Assuming the claim, we have $0 \leq r_{\Gamma_q}(2(t) + (q)) \leq 1 + r_{\Gamma_q^q}((q) - (s)) = 0$ (since q and

s are not linearly equivalent in Γ_q^q; see Lemma 5.8). So it remains to prove that $\eta_{\Gamma_q^t, 2(t)}(1) = 1$. This is equivalent to $r_{\Gamma_q^t}(2(t)) = 0$, which is obviously the case. ■



a) A metric graph Γ in $\mathcal{M}(A_1) = \mathcal{M}(A_2)$ b) A metric graph Γ in $\mathcal{M}(A_3)$

FIGURE 16

Lemma 5.8. *Let Γ be any metric graph in $\mathcal{M}(G_3)$, where G_3 is the totally degenerate graph depicted in Figure 17(a). Then the two divisors $3(p)$ and $3(t)$ are not linearly equivalent in Γ.*

Proof. By symmetry we can assume that the length of the edge $\{u, p\}$ is less than or equal to the length of the edge $\{t, w\}$. Then there exists a point t' in the segment $[t, w]$ so that $3(p) - 3(t) \sim 3(u) - 3(t')$ — see Figure 17(b) — and we are led to prove that $D = 3(u) - 3(t')$ is not linearly equivalent to zero. Consider the unique t'-reduced divisor $D_{t'}$ linearly equivalent to D in Γ (see e.g. [Ami13, BN07] for the definition and basic properties of reduced divisors). It will be enough to show that $D_{t'} \neq 0$. Three cases can occur, depending on the lengths ℓ_z, ℓ_w, and $\ell_{t'}$ in Γ of the edges $\{u, z\}, \{u, w\}$, and the segment $\{u, t'\}$, respectively:

- If $\ell_z = \min\Big\{\ell_z, \ell_u, \ell_{t'}\Big\}$, then there are two points w' and t'' on the segments $\{u, w\}$ and $\{u, t'\}$, respectively, such that $D_{t'} = (z) + (w') + (t'') - 3(t')$.
- If $\ell_u = \min\Big\{\ell_z, \ell_u, \ell_{t'}\Big\}$, then there are two points z' and t'' on the segments $\{u, z\}$ and $\{u, t'\}$, respectively, such that $D_{t'} = (z') + (w) + (t'') - 3(t')$.
- If $\ell_{t'} = \min\Big\{\ell_z, \ell_u, \ell_{t'}\Big\}$, then there are two points z' and w' on the segments $\{u, z\}$ and $\{u, w\}$, respectively, such that $D_{t'} = (z') + (w') - 2(t')$.

In all the three cases, we have $D_{t'} \neq 0$, which shows that D cannot be equivalent to zero in Γ. ■

a) the divisor $3(p) - 3(t)$ is not rationally equivalent to zero. b) $3(p) - 3(t) \sim 3(u) - 3(t')$

FIGURE 17

Lemma 5.9. *Let $\Gamma \in \mathcal{M}(A_3)$ be a metric graph as depicted in Figure 16(b). For any $a, b \leq 2$, the divisor $a(p) + b(q)$ has rank zero on Γ.*

Proof. The arguments are similar to the ones used in the proof of Lemma 5.7. Consider the cut-vertex t in Γ and denote by Γ_p and Γ_q the corresponding components containing p and q, respectively. We claim that $\eta_{\Gamma_q, 2(q)}(1) = 2$. This obviously implies the lemma. Indeed, $r_{\Gamma_p}(2(p) - 2(t)) = -1$ (which can be verified by an analogue of Lemma 5.8 in Γ_p), and thus (5.6.1) implies that $r_\Gamma(2(p)+2(q)) \leq 1 + r_{\Gamma_p}(2(p)-2(t)) = 0$.

To show that $\eta_{\Gamma_q,2(q)}(1) = 2$, it will be enough to show that $r_{\Gamma_q}(2(q)+(t)) = 0$. This can be done in exactly the same way by considering the other cut-vertex s adjacent to t in Γ_q. ∎

Lemma 5.10. *Let x_1, x_2, and x_3 be distinct points in $\mathbf{P}^1(k)$. Then there does not exist a morphism $f : \mathbf{P}^1 \to \mathbf{P}^1$ of degree four branched over x_1, x_2 and x_3 and having ramification profile $(2,2)$, $(2,2)$, and $(3,1)$ at these three points.*

Proof. Suppose that such a rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ exists. The monodromy group of f is a subgroup of $\mathfrak{S}_4$, so its cardinality is of the form 2^a3^b. In particular, if the characteristic of k is neither 2 nor 3, then f has a tame monodromy group and the non-existence of f then comes from the fact that $H^4_{0,0}\left((2,2),(2,2),(3,1)\right) = 0$ (see Example 3.4).

Hence it remains to check the lemma for $\mathrm{char}(k) = 2, 3$. Note that the same technique we use in this case works in any characteristic, but the computations are a bit more tedious in characteristic different from 2 and 3.

Up to the action of $\mathrm{GL}(2,k)$ on $\mathbf{P}^1$ via automorphisms, we may assume that $x_1 = 0$, $x_2 = \infty$, and $x_3 = 1$, and that

$$f(X) = a\frac{X^2(X+1)^2}{(X+b)^2}$$

with $a \neq 0$ and $b \neq 0, -1$. Hence the condition on the ramification profile of x_3 translates as

$$aX^2(X+1)^2 - (X+b)^2 = c(X-d)^3(X-e)$$

with $c \neq 0$, $d \neq 0, -1, b$, and $e \neq 0, -1, b, d$. Looking at the coefficients of the two polynomials, we obtain the following five equations

$$(E_1):\ a = c, \qquad (E_2):\ ec = -2a - 3cd, \qquad (E_3):\ a - 1 = 3cd(d+e),$$
$$(E_4):\ 2b = cd^2(d+3e), \qquad (E_5):\ -b^2 = cd^3e.$$

If k has characteristic 2, then (E_2) becomes $ec = cd$ which contradicts the fact that $e \neq d$.

If k has characteristic 3, then these five equations become

$$(E_1):\ a = c, \quad (E_2):\ ec = a, \quad (E_3):\ a = 1, \quad (E_4):\ -b = cd^3, \quad (E_5):\ -b^2 = cd^3e.$$

Equations $(E_1), (E_2), (E_3)$ imply $a = c = e = 1$. Then (E_4) and (E_5) become $-b = d^3 = -b^2$; hence $b = 1 = e$, which contradicts our assumptions. ∎

We can now give the promised proof of Proposition 5.6.

Proof. (Proof of Proposition 5.6) Suppose that there exists a metrized complex of k-curves $\mathcal{C}_{27}$ of genus 27 with underlying augmented metric graph Γ_{27} in $\mathcal{M}(G_{27})$, and admitting a finite harmonic morphism of metrized complexes of degree four $\varphi : \mathcal{C}_{27} \to \mathcal{T}$, for $\mathcal{T}$ of genus zero with underlying metric tree denoted by T. Without loss of generality, we may assume that T has no infinite vertex $q \in V_\infty(T)$ such that any infinite edge e' adjacent to an infinite vertex $q' \in \varphi^{-1}(q)$ has $d_{e'}(\varphi) = 1$.

We are going to prove below that the local degree at p is 4. Assuming that this is the case, we show how the proposition follows. Denote by Γ_1, Γ_2, and Γ_3 the three components of $\Gamma_{27} \setminus \{p\}$ which contain A_1, A_2, and A_3, respectively. Since the degree of φ at p is four, we have $\varphi^{-1}(\varphi(p)) = \{p\}$. Therefore, by the connectivity of Γ_i, the images of Γ_i under φ are pairwise disjoint in T. This shows that for x sufficiently close to $\varphi(p)$ in T, the support of the divisor $D_x(\varphi)$ lives entirely in one of the Γ_i for $i \in \{1,2,3\}$. Choose x_i sufficiently close to $\varphi(p)$ such that the support of $D_{x_i}(\varphi)$ is contained in Γ_i. Applying Proposition 4.2, we see that each divisor $D_{x_i}(\varphi)$ has rank one in Γ_i. Now, according to Lemma 5.7, the degree-four divisor $D_{x_1}(\varphi)$ (resp. $D_{x_2}(\varphi)$) must be of the form $2(a)+2(b)$ for two points a and b sufficiently close to p and lying on the two different branches of Γ_1 (resp. Γ_2) adjacent to p. Similarly, by Lemma 5.9, the divisor $D_{x_3}(\varphi)$ has to be of the form $3(a)+(b)$ for two points a and b sufficiently close to p and lying on the two different branches of Γ_3 adjacent to p. This shows that the map φ_p, the restriction of φ to a sufficiently small neighborhood of p in Γ_{27}, coincides with the map depicted in Figure14(a). The proposition now follows from Lemma 5.10.

It remains to prove that $d_p(\varphi) = 4$. We first claim that φ maps one of the components Γ_i, for $i = 1, 2, 3$, onto a connected component of $T \setminus \{\varphi(p)\}$. Otherwise, for the sake of contradiction, suppose that $\varphi^{-1}(\varphi(p))$ consists of p and one point p_i in each of the components Γ_i for $i = 1, 2, 3$. Then φ has local degree one at each of the points p_i. By Proposition 4.2, $D_{\varphi(p)}(\varphi) = (p) + (p_1) + (p_2) + (p_3)$ has rank one in Γ. By equation (5.6.1) applied to the cut-vertex p in Γ_{27}, we infer that the divisor $(p) + (p_i)$ has rank one in the metric graph $\overline{\Gamma}_i$, the closure of Γ_i in Γ_{27}. In other words, the metric graphs $\overline{\Gamma}_i$ are hyperelliptic, which is clearly not the case. This gives a contradiction and the claim follows.

Summarizing, there must exist at least one Γ_i such that φ maps Γ_i onto one of the connected components of $T \setminus \{\varphi(p)\}$. Reasoning again as in the first part of the proof, it follows from Proposition 4.2 and Lemmas 5.7 and 5.9 that the restriction of φ to Γ_i has degree four, which implies that $d_p(\varphi) = 4$. ■

5.11. Lifting divisors of given rank. First, recall that to a smooth proper curve X over K together with a semistable vertex set V and a subset D_0 of $X(K)$ compatible with V, we can naturally associate a metrized complex of curves $\mathcal{C} = \Sigma(X, V \cup D_0)$ with underlying augmented metric graph Γ. As in [AB12], there are natural specialization maps on divisors, which we denote for simplicity by the same letter τ_*:

$$\tau_* : \mathrm{Div}(X) \to \mathrm{Div}(\mathcal{C}), \quad \text{and} \quad \tau_* : \mathrm{Div}(\mathcal{C}) \to \mathrm{Div}(\Gamma).$$

Since this discussion is pointless in the case of rational curves, we may assume that X (equivalently, $\mathcal{C}$ or the augmented metric graph Γ) has positive genus. We will also assume that Γ does not have any infinite vertices, i.e., that D_0 is empty, which does not lead to any real loss of generality and which makes various statements easier to write down and understand. We may also assume without loss of generality that V is a strongly semistable vertex set of X.

According to the Specialization Inequality [Bak08, AC13, AB12]), for any divisor D in $\mathrm{Div}(X)$ one has

$$r_X(D) \le r_{\mathcal{C}}(\tau_*(D)) \le r_\Gamma^{\#}(\tau_*(D)) \le r_\Gamma(\tau_*(D)), \tag{5.11.1}$$

where r_X, $r_{\mathcal{C}}$ and r_Γ denote rank of divisors on X, $\mathcal{C}$ and (unaugmented) Γ, respectively, and $r_\Gamma^{\#}$ denotes the weighted rank in the augmented metric graph (Γ, g) (see (4.11)).

We spend the rest of this section discussing the sharpness of the inequalities appearing in (5.11.1).

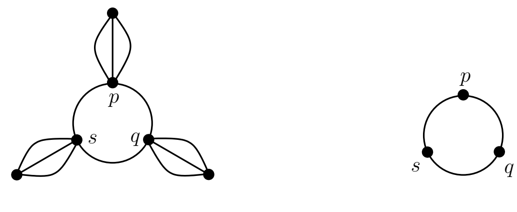
Definition 5.12. Let $\mathcal{C}$ be a metrized complex of curves whose underlying metric graph Γ has no infinite leaves, and let $\mathcal{D}$ be a Λ-rational divisor in $\mathrm{Div}_\Lambda(\mathcal{C})$. A *lifting* of the pair $(\mathcal{C}, \mathcal{D})$ consists of a triple $(X, V; D_X)$ where X is a smooth proper curve over K, V is a strongly semistable vertex set for which $\mathcal{C} = \Sigma(X, V)$, and D_X is a divisor in $\mathrm{Div}(X)$ with $\mathcal{D} \sim \tau_*(D_X)$. We say that the inequality $r_X \le r_{\mathcal{C}}$ is *sharp* if for any metrized complex of curves $\mathcal{C}$ and any divisor $\mathcal{D} \in \mathrm{Div}(\mathcal{C})$, there exists a lifting $(X, V; D_X)$ of $(\mathcal{C}, \mathcal{D})$ such that $r_X(D_X) = r_{\mathcal{C}}(\mathcal{D})$.

We can define in a similar way what it means to lift a divisor on an (augmented) metric graph to a divisor on a metrized complex of curves or to a smooth proper curve over K, and what it means for the corresponding specialization inequalities to be sharp.

It is easy to see that the inequality $r_\Gamma^{\#} \le r_\Gamma$ is not sharp (see [AB12] for a precise formula relating the two rank functions).

The following example is due to Ye Luo (unpublished); we thank him for his permission to include it here. Together with Corollary I.4.28, it implies that the inequality $r_X \le r_\Gamma$ is not sharp.

Example 5.13. (Luo) Let Γ be a metric graph in $\mathcal{M}(G_7)$, where G_7 is the graph of genus seven depicted in Figure 18(a), such that all edge lengths in Γ are equal, and let $D = (p) + (q) + (s) \in \mathrm{Div}(\Gamma)$. Then $r_\Gamma(D) = 1$, however there does not exist any finite harmonic morphism of metric graphs $\varphi : \Gamma' \to T$ of degree three to a metric tree for any $\Gamma' \in \mathcal{M}(G_7)$. In particular, this shows that the stable gonality of an augmented graph can be greater than its divisorial gonality.



a) Graph G_7 b) Metric graph $\Gamma'_1 \subset \Gamma'$

FIGURE 18

We briefly sketch a proof. Suppose that such a finite harmonic morphism $\varphi : \Gamma' \to T$ exists. Since Γ' is not hyperelliptic, one easily verifies that $D_{\varphi(p)}(\varphi) = 3(p)$, $D_{\varphi(q)}(\varphi) = 3(q)$, and $D_{\varphi(s)}(\varphi) = 3(s)$. This shows the existence of a finite morphism $\varphi' : \Gamma'_1 \to T'$ of degree 3 to a metric tree T' where Γ'_1 is depicted in Figure 18(b), so that $D_{\varphi'(p)}(\varphi') = 3(p)$, $D_{\varphi'(q)}(\varphi') = 3(q)$, and $D_{\varphi(s)}(\varphi') = 3(s)$. But it is easy to verify by hand that such a morphism φ' does not exist.

Proposition 5.14. *Neither of the inequalties $r_X \leq r_{\mathcal{C}}$ and $r_{\mathcal{C}} \leq r^{\#}_{\Gamma}$ is sharp.*

Proof. To show the non-sharpness of the inequality $r_X \leq r_{\mathcal{C}}$, let $\mathcal{C}$ be a metrized complex of curves whose underlying metric graph Γ belongs to the family depicted in Figure 12, with first Betti number κ, and whose genus function is positive at each vertex. Consider the divisor $\mathcal{D}_d = d(p) \oplus d(x)$ in $\mathcal{C}$ for a closed point x in C_p and d a positive integer. If d is sufficiently large compared to the genera of the vertices, then $r_{\mathcal{C}}(\mathcal{D}_d) \geq 1$. If the pair $(\mathcal{C}, \mathcal{D}_d)$ lifted to a triple $(X, V; D_X)$ with $\tau_*(D_X) \sim \mathcal{D}_d$, then there would exist a finite harmonic morphism $\varphi : \widetilde{\mathcal{C}} \to \mathcal{T}$ from a modification of $\mathcal{C}$ to a metrized complex of curves of genus zero. But this would imply the existence of a degree d morphism $\varphi_p : C_p \to \mathbf{P}^1$ such that the image of red_p (on edges adjacent to p in Γ) is contained in the set of critical values of φ_p. By the Riemann–Hurwitz formula, this is impossible for κ large enough compared to d.

To show the non-sharpness of the inequality $r_{\mathcal{C}} \leq r^{\#}$, let again (Γ, g) be an augmented metric graph with underlying graph depicted in Figure 12 with $\kappa \geq 3$ and $2 \leq 2g(p) < \kappa - 2$, and let $D = 2(p)$. One easily computes that $r^{\#}_{\Gamma}(D) = 1$. An algebraic curve of genus $g(p) \geq 1$ contains at most $2g(p) + 2$ distinct points p such that $2(p)$ is in a given linear system of degree two, which implies that (Γ, g) cannot be lifted to a hyperelliptic metrized complex of curves. This shows that the inequality $r_{\mathcal{C}} \leq r^{\#}_{\Gamma}$ is not sharp. ■

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