THE EXCHANGE GRAPH AND VARIATIONS OF THE RATIO OF THE TWO SYMANZIK POLYNOMIALS

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Abstract. Correlation functions in quantum field theory are calculated using Feynman amplitudes, which are finite dimensional integrals associated to graphs. The integrand is the exponential of the ratio of the first and second Symanzik polynomials associated to the Feynman graph, which are described in terms of the spanning trees and spanning 2-forests of the graph, respectively.

In a previous paper with Bloch, Burgos and Fresán, we related this ratio to the asymptotic of the Archimedean height pairing between degree zero divisors on degenerating families of Riemann surfaces. Motivated by this, we consider in this paper the variation of the ratio of the two Symanzik polynomials under bounded perturbations of the geometry of the graph. This is a natural problem in connection with the theory of nilpotent and SL2 orbits in Hodge theory.

Our main result is the boundedness of variation of the ratio. For this we define the exchange graph of a given graph which encodes the exchange properties between spanning trees and spanning 2-forests in the graph. We provide a complete description of the connected components of this graph, and use this to prove our result on boundedness of the variations.

1. Introduction

Feynman amplitudes in quantum field theory are described as finite dimensional integrals associated to graphs. A Feynman graph $(G, \mathbf{p})$ consists of a finite connected graph $G = (V, E)$, with vertex and edge sets V and E, respectively, together with a collection of external momenta $\underline{\mathbf{p}} = (\mathbf{p}_v)_{v\in V}$, $\mathbf{p}_v \in \mathbb{R}^D$ which satisfy the conservation law

$$\sum_{v\in V} \mathbf{p}_v = 0. \tag{1.1}$$

Here $\mathbb{R}^D$ is the space-time endowed with a Minkowski bilinear form.

One associates to a Feynman graph $(G, \underline{\mathbf{p}})$ two polynomials in the variables $\underline{Y} = (Y_e)_{e\in E}$. Denote by $\mathcal{ST}$ the set of all the spanning trees of the graph G. (Recall that a spanning tree of a connected graph is a maximal subgraph which does not contain any cycle. It has precisely $|V| - 1$ edges.) The first Symanzik ψ_G, which depends only on the graph G, is given by the following sum over the spanning trees of G:

$$\psi_G(\underline{Y}) := \sum_{T\in\mathcal{ST}} \prod_{e\notin T} Y_e.$$

A spanning 2-forest in a connected graph G is a maximal subgraph of G without any cycle and with precisely two connected components. Such a subgraph has precisely $|V| - 2$ edges. Denote by $\mathcal{SF}_2$ the set of all the spanning 2-forests of G. The second Symanzik polynomial

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ϕ_G, which depends on the external momenta as well, is defined by

$$\phi_G(\underline{\mathbf{p}}, \underline{Y}) := \sum_{F \in \mathcal{SF}_2} q(F) \prod_{e \notin F} Y_e.$$

Here F runs through the set of spanning 2-forests of G, and for F_1 and F_2 the two connected components of F, $q(F)$ is the real number $-\langle \mathbf{p}_{F_1}, \mathbf{p}_{F_2} \rangle$, where $\mathbf{p}_{F_1}$ and $\mathbf{p}_{F_2}$ denote the total momentum entering the two connected components F_1 and F_2 of F, i.e.,

$$\mathbf{p}_{F_1} := \sum_{v \in V(F_1)} \mathbf{p}_v \qquad\qquad \mathbf{p}_{F_2} := \sum_{u \in V(F_2)} \mathbf{p}_u.$$

The Feynman amplitude associated to $(G, \underline{\mathbf{p}})$ is a path integral on the space of metrics (i.e., edge lengths) on G with the action given by ϕ_G/ψ_G. It is given by

$$I_G(\underline{\mathbf{p}}) = C \int_{[0,\infty]^E} \exp(-i\, \phi_G/\psi_G)\; d\pi_G,$$

for a constant C, and the volume form $d\pi_G = \psi_G^{-D/2} \prod_E dY_e$ on $\mathbb{R}_+^E$, c.f. [6, Equation (6-89)].

Motivated by the question of describing Feynman amplitudes as the infinite tension limit of bosonic string theory, in [1] we proved results describing the ratio of the two Symanzik polynomials in the Feynman amplitude as asymptotic behaviour of the Archimedean height pairing between degree zero divisors in degenerating families of Riemann surfaces. A natural problem arising from [1] is to consider the variation of ϕ_G/ψ_G obtained by perturbation of the geometry of the graph, in a sense that we describe below. In order the state the theorem, we need to recall the determinantal representation of the two Symanzik polynomials. We refer to [1] where the discussion below appears in more detail.

1.1. **Determinantal representation of the Symanzik polynomials.** Let $G = (V, E)$ be a finite connected graph on the set of vertices V of size n and with the set of edges $E = \{e_1, \dots, e_m\}$ of size m. Denote by h the genus of G, which is by definition the integer $h = m - n + 1$.

Let R be a ring of coefficients (that we will later assume to be either $\mathbb{R}$ or $\mathbb{Z}$), and consider the free R-module $R^E \simeq R^m = \left\{ \sum_{i=1}^m a_i e_i \mid a_i \in R \right\}$ of rank m generated by the elements of E. For any element $a \in R^E$, we denote by a_i the coefficient of e_i in a.

Any edge e_i in E gives a bilinear form of rank one $\langle\, .\, ,. \rangle_i$ on R^m by the formula

$$\langle a, b \rangle_i := a_i b_i.$$

Let $\underline{y} = \{y_i\}_{e_i \in E}$ be a collection of elements of R indexed by E, and consider the symmetric bilinear form $\alpha = \langle\, .\, ,. \rangle_{\underline{y}} := \sum_{e_i \in E} y_i \langle\, .\, ,. \rangle_i$. In the standard basis $\{e_i\}$ of R^E, α is the diagonal matrix with y_i in the i-th entry, for $i = 1, \dots, m$. We denote by $Y := \mathrm{diag}(y_1, \dots, y_m)$ this diagonal matrix.

Let $H \subseteq R^E$ be a free R-submodule of rank r. The bilinear form α restricts to a bilinear form $\alpha_{|H}$ on H. Fixing a basis $B = \{\gamma_1, \dots, \gamma_r\}$ of H over R, and denoting by M the $r \times m$ matrix with row vectors γ_i written in the standard basis $\{e_i\}$ of R^E, the restriction $\alpha_{|H}$ can be identified with the symmetric $r \times r$ matrix MYM^τ so that for two vectors $c, d \in R^r \simeq H$ with $a = \sum_{j=1}^r c_j \gamma_j$ and $b = \sum_{j=1}^r d_j \gamma_j$, we have

$$\alpha(a, b) = cMYM^\tau d^\tau.$$

The Symanzik polynomial $\psi(H, \underline{y})$ associated to the free R-submodule $H \subseteq R^E$ is defined as

$$\psi(H, \underline{y}) := \det(MYM^\tau).$$

Note that since the coordinates of MYM^τ are linear forms in $y_1, \dots, y_m$, $\psi(H, \underline{y})$ is a homogeneous polynomial of degree r in variables y_i.

For a different choice of basis $B' = \{\gamma'_1, \dots, \gamma'_r\}$ of H over R, the matrix M is replaced by PM where P is the $r \times r$ invertible matrix over R transforming one basis into the other. So the matrix of $\alpha_{|H}$ in the new basis is given by $PMYM^\tau P^\tau$, , and the determinant gets multiplied by an element of $R^{\times 2}$. It follows that $\psi(H, \underline{y})$ is well-defined up to an invertible element in $R^{\times 2}$. In particular, if $R = \mathbb{Z}$, the quantity $\psi(H, \underline{y})$ is independent of the choice of the basis and is therefore well-defined.

From now on, we fix an orientation on the edges of the graph. We have a boundary map $\partial : R^E \to R^V,\ e \mapsto \partial^+(e) - \partial^-(e)$, where ∂^+ and ∂^- denote the head and the tail of e, respectively. The homology of G is defined via the exact sequence

$$0 \to H_1(G, R) \to R^E \xrightarrow{\partial} R^V \to R \to 0. \tag{1.2}$$

The homology group $H = H_1(G, R)$ is a submodule of $R^E \simeq R^m$ free of rank h, the genus of the graph G, for any ring R. In particular, by the preceding discussion, fixing a basis B of $H_1(G, \mathbb{Z})$, the polynomial

$$\psi_G(\underline{y}) := \psi(H, \underline{y})$$

is independent of the choice of B. Writing M for the $h \times m$ matrix of the basis B in the standard basis $\{e_i\}$ of R^E, one sees that

$$\psi_G(\underline{y}) = \det(MYM^\tau).$$

It follows from the Kirchhoff's matrix-tree theorem [7] that

$$\psi_G(\underline{Y}) = \sum_{T \in \mathcal{ST}} \prod_{e \notin T} Y_e,$$

which is the form of the first Symanzik polynomial given at the beginning of this section.

The exact sequence (1.2) yields an isomorphism

$$R^E / H \simeq R^{V,0},$$

where $R^{V,0}$ consists of those $x \in R^V$ whose coordinates sum up to zero.

Let now $\mathbf{p} \in R^{V,0}$ be a non-zero element, and let ω be any element in $\partial^{-1}(\mathbf{p})$. Denote by $H_\omega = \partial^{-1}(R.\mathbf{p}) = H + R.\omega$, and note that H_ω is a free R-module of rank $h+1$ which comes with the basis $B_\omega = B \sqcup \{\omega\}$.

The second Symanzik polynomial of $(G, \underline{\mathbf{p}})$ is

$$\phi_G(\underline{\mathbf{p}}, \underline{y}) = \psi(H_\omega, \underline{y}).$$

The polynomial $\phi_G(\underline{\mathbf{p}}, \underline{y})$ is homogeneous of degree $h+1$ in variables y_i, which is as noted in [1], independent of the choice of the element $\omega \in \partial^{-1}(\mathbf{p})$. Writing N for the $(h+1) \times m$ matrix for the the basis B_ω in the standard basis of R^E, we see that

$$\phi_G(\underline{\mathbf{p}}, \underline{y}) = \det(NYN^\tau).$$

The definition can be extended to $\mathbf{p} \in \mathbb{R}^D$ using the Minkowski bilinear form on $\mathbb{R}^D$, as discussed in [1].

We have the following expression for the second Symanzik polynomial, see e.g. to [3] or Section 3,

$$\phi_G(\underline{\mathbf{p}}, \underline{y}) = \sum_{F\in\mathcal{SF}_2} q(F) \prod_{e\notin E(F)} y_e\,,$$

which is precisely the form of the second Symanzik polynomial given at the beginning of this introduction.

1.2. **Statement of the main theorem.** Let U be a topological space and let $y_1, \dots, y_m : U \to \mathbb{R}_{>0}$ be m continuous functions. Let $\mathbf{p} \in (\mathbb{R})^{V,0}$ be a fixed vector, and consider the two functions $\psi_G(\underline{y}) : U \to \mathbb{R}_{>0}$ and $\phi_G(\mathbf{p}, \underline{y}) : U \to \mathbb{R}_{>0}$ be the real-valued functions on U defined by the first and second Symanzik polynomials.

Denote by Y the matrix values function on U defined by $Y(s) := \mathrm{diag}\big(y_1(s), \dots, y_m(s)\big)$ for any $s \in U$.

Notation. We introduce the following terminology which will be convenient for what follows. For two real-valued functions F_1 and F_2 defined on a topological space U, we write $F_1 = O_{\underline{y}}(F_2)$ if there exist constants $c, C > 0$ such that $|F_1(s)| \leq c|F_2(s)|$ at all points s in U which verify $y_j(s) \geq C$ for all $j = 1, \dots, m$.

Let $A : U \to \mathrm{Mat}_{m\times m}(\mathbb{R})$ be a matrix-valued map taking at $s \in U$ the value $A(s)$. Assume that A verifies the following two properties

(i) A is a bounded function, i.e., all the entries $A_{i,j}$ of A take values in a bounded interval $[-C, C]$ of $\mathbb{R}$, for some positive constant $C > 0$.
(ii) The two matrices $M(Y + A)M^\tau$ and $N(Y + A)N^\tau$ are invertible.

One might view the contribution of A as a perturbation of the standard scalar product on the edges of the graph given by the (length) functions $y_1, \dots, y_m$, which can be further regarded as changing the geometry of the graph, seen as a discrete metric space. The main result of this paper is the following.

Theorem 1.1. *Assume $A : U \to \mathrm{Mat}_{m\times m}(\mathbb{R})$ verifies the condition (i) and (ii) above. The difference $\frac{\det(N(Y+A)N^\tau)}{\det(M(Y+A)M^\tau)} - \frac{\det(NYN^\tau)}{\det(MYM^\tau)}$ is $O_{\underline{y}}(1)$.*

This result might appear somehow surprising, given that the rational functions which appear in the expression above are of degree one. Moreover, simple examples of rational functions in several variables such as y_1^a/y_2^b, for natural numbers a and b, show that depending on the relative size of the different parameters, the behaviour at infinity can be very irregular. E.g., in the example y_1^2/y_2, if y_2 grows at any rate slower than y_1, then the ratio is unbounded at infinity. The content of the theorem is thus a strong stability theorem at infinity for the ratio of the two Symanzik polynomials.

The proof of the above theorem is rather unexpectedly linked to a combinatorial result about the exchange properties between spanning forests of a given graph. Exchange properties between spanning trees in graphs are well-known and form a part of the axiomatic definition of more general matroids. On the other hand, exchange properties between spanning forests are less studied, and this is what we do here in order to obtain the theorem.

To prove Theorem 1.1, using Cauchy-Binet formula and some preliminary observations, we are led to introduce a graph which encodes the exchange properties between the edge set of spanning trees and the edge set of spanning 2-forests in the graph that we call the exchange graph of G, see Definition 2.3. As our first result, we give in Theorem 2.13 a classification of the connected components of the exchange graph. This classification theorem combined with further combinatorial arguments are then used in Section 3 to prove Theorem 1.1.

We note that a similar result to our theorem above has been proved using different tools in a recent paper of Burgos, de Jong and Holmes [2] in the setting of what is called *normlike functions*. The perturbations in [2] are nevertheless required to be symmetric for the method to work, though, strictly speaking, the result in [2] is more general and goes beyond the case of graphs. In comparison, the methods in this paper are purely combinatorial, the results on the exchange graph should be of independent interest, and the approach taken here applies in much more generality.

Indeed, since the first appearance of this paper, Matthieu Piquerez has obtained a generalisation of Theorem 1.1 to the setting of higher dimensional simplicial complexes and matroids. The strategy of the proof is very similar to our strategy here: the exchange graph we use here is replaced by a similar exchange graph for the matroid, and the arguments of Section 3 can be applied to this general setting in order to obtain the above mentioned generalisation of Theorem 1.1. We refer to [8] for more details.

We now explain an application of Theorem 1.1 from [1], c.f. Theorem 1.2 below, discussed in more detail in Section 4.

1.3. **Boundedness of variation of the Archimedean height pairing.** Let C_0 be a stable curve of genus g over $\mathbb{C}$, and with dual graph $G = (V, E)$ which has genus $h = |E| - |V| + 1$, $h \leq g$.

Consider the versal analytic deformation $\pi : \mathcal{C} \to S$ of C_0, where S is a polydisc of dimension $3g - 3$. The total space $\mathcal{C}$ is regular and we let $D_e \subset S$ denote the divisor parametrising those deformations in which the point associated to e remains singular. The divisor $D = \bigcup_{e \in E} D_e$ is a normal crossings divisor whose complement $U = S \setminus D$ can be identified with $(\Delta^*)^E \times \Delta^{3g-3-|E|}$. Assume that two collections of sections of π are given, which we denote by $\sigma_1 = (\sigma_{\ell,1})_{\ell=1,\dots,n}$ and $\sigma_2 = (\sigma_{\ell,2})_{\ell=1,\dots,n}$. Since $\mathcal{C}$ is regular, the points $\sigma_{l,i}(0)$ lie on the smooth locus of C_0. Consider two fixed vectors $\underline{\mathbf{p}}_1 = (\mathbf{p}_{l,1})_{l=1}^n$ and $\underline{\mathbf{p}}_2 = (\mathbf{p}_{l,2})_{l=1}^n$ with $\mathbf{p}_{l,i} \in \mathbb{R}^D$ which each satisfy the conservation of momentum (1.1). We obtain a pair of relative degree zero $\mathbb{R}^D$-valued divisors

$$\mathfrak{A}_s = \sum_{l=1}^{n} \mathbf{p}_{l,1}\sigma_{l,1}, \qquad \mathfrak{B}_s = \sum_{l=1}^{n} \mathbf{p}_{l,2}\sigma_{l,2}.$$

Assume further that σ_1 and σ_2 are disjoint on each fiber of π. To any pair $\mathfrak{A}$, $\mathfrak{B}$ of degree zero (integer-valued) divisors with disjoint support on a smooth projective complex curve C, one associates a real number, the Archimedean height

$$\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathrm{Re}\Big(\int_{\gamma_{\mathfrak{B}}} \omega_{\mathfrak{A}}\Big),$$

by integrating a canonical logarithmic differential $\omega_{\mathfrak{A}}$ with residue $\mathfrak{A}$ along any 1-chain $\gamma_{\mathfrak{B}}$ supported on $C \setminus |\mathfrak{A}|$ and having boundary $\mathfrak{B}$. Coupling with the Minkowski bilinear form on $\mathbb{R}^D$, the definition extends to $\mathbb{R}^D$-valued divisors [1]. We thus get a real-valued function

$$s \mapsto \langle \mathfrak{A}_s, \mathfrak{B}_s \rangle,$$

defined on U.

For any point $s \in U$, and an edge $e \in E$, we denote by $s_e \in \Delta^*$ the e-th coordinate of s when U is identified with $\Delta^{*,E} \times \Delta^{3g-3-|E|}$. For any point $s \in U$ and an edge $e \in E$, define $y_e := \frac{-1}{2\pi} \log |s_e|$ and put $\underline{y} = \underline{y}(s) = (y_e)_{e \in E}$. We have shown in [1] that after shrinking U, if necessary, the asymptotic of the height pairing is given by the following theorem. Here $\phi_G(\underline{\mathbf{p}}, \underline{\mathbf{p}}', \underline{Y})$ denotes the bilinear form associated to ϕ_G (which is a quadratic form in $\underline{\mathbf{p}}$).

Theorem 1.2 (Amini, Bloch, Burgos, Fresán [1])**.** *Notations as above, there exists a bounded function $h\colon U \to \mathbb{R}$ such that*

$$\langle \mathfrak{A}_s, \mathfrak{B}_s \rangle = 2\pi \frac{\phi_G(\underline{\mathbf{p}}_1^G, \underline{\mathbf{p}}_2^G, \underline{y})}{\psi_G(\underline{y})} + h(s).$$

In Section 4, we will show how to deduce this theorem from Theorem 1.1 and the explicit formula obtained in [1] by means of the nilpotent orbit theorem in Hodge theory for the variation of the Archimedean height pairing, c.f. Proposition 4.2.

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2. Exchange graph

Let $G = (V, E)$ be a connected multigraph with vertex set V and edge set E. By a spanning subgraph of G we mean a subgraph H of G with $V(H) = V$. For an integer $k \geq 1$, a spanning k-forest in G is a subgraph of G with vertex set V without any cycle which has precisely k-connected components; a spanning k-forest has precisely $|V| - k$ edges. For $k = 1$, a spanning 1-forest is precisely a spanning tree of G. We are particularly interested in the "exchange properties" between spanning 2-forest and spanning trees in a graph G. To make this precise, we will define a new graph $\mathscr{H}$ that we call the exchange graph of G. First we need to define an equivalence relation on the set of spanning 2-forests of G.

Definition 2.1.
- For a spanning 2-forest F of a graph G, we denote by $\mathcal{P}(F) = \{X, Y\}$ the partition $V = X \sqcup Y$ of the vertices into the vertex sets X and Y of the two connected components of F.
- For any partition $\mathcal{P}$ of V, we denote by $E(\mathcal{P})$ the set of all edges in G which connect two vertices lying in two different elements of $\mathcal{P}$.
- Two 2-forests F and F' are called *(vertex) equivalent*, and we write $F \sim_v F'$, if $\mathcal{P}(F) = \mathcal{P}(F')$.

The following proposition is straightforward.

Proposition 2.2. *The following statements are equivalent for $F, F' \in \mathcal{SF}_2$:*
1. *F and F' are not (vertex) equivalent.*
2. *there exists an edge $e \in F'$ such that $F \cup \{e\}$ is a tree.*

Notation. In what follows, for a spanning subgraph G' of $G = (V, E)$ and $e \in E \setminus E(G')$, we simply write $G' + e$ to denote the spanning subgraph of G with the edge set $E(G') \cup \{e\}$.

For an edge $e \in E(G')$, we write $G' - e$ for the spanning subgraph of G with the edge set $E(G') \setminus \{e\}$.

Definition 2.3. The *exchange graph* $\mathscr{H} = \mathscr{H}_G = (\mathscr{V}, \mathscr{E})$ of G is defined as follows. The vertex set $\mathscr{V}$ of $\mathscr{H}$ is the disjoint union of two sets $\mathscr{V}_1$ and $\mathscr{V}_2$, where

$$\mathscr{V}_1 := \Big\{ (F,T) \,\big|\, F \in \mathcal{SF}_2(G), T \in \mathcal{ST}(G)\,,\, E(F) \cap E(T) = \emptyset \Big\},$$

and

$$\mathscr{V}_2 := \Big\{ (T,F) \,\big|\, T \in \mathcal{ST}(G), F \in \mathcal{SF}_2(G)\,,\, E(F) \cap E(T) = \emptyset \Big\}.$$

There is an edge in $\mathscr{E}$ connecting $(F,T) \in \mathscr{V}_1$ to $(T',F') \in \mathscr{V}_2$ if there is an edge $e \in E(T)$ such that $F' = T - e$ and $T' = F + e$.

Definition 2.4. If (T,F) and (F',T') are adjacent in $\mathscr{H}$ and $F' = T - e$, we say (F',T') is obtained from (T,F) by *pivoting* involving the edge e.

Our aim in this section is to describe the connected components of $\mathscr{H}$.

First note that there is no isolated vertex in $\mathscr{H}$: consider a spanning tree T and a spanning 2-forest F of G with disjoint sets of edges. Let $\mathcal{P}(F) = \{X, Y\}$, be the vertex sets of the two connected components of F. By connectivity of T, there is an edge e of T which joins a vertex of X to a vertex of Y. It follows that $T' = F + e$ and $F' = T - e$ are spanning tree and 2-forest in G, respectively, and $(F,T) \in \mathscr{V}_1$ is connected to $(T',F') \in \mathscr{V}_2$.

Let now $\mathscr{H}_0 = (\mathscr{V}_0, \mathscr{E}_0)$ be a connected component of $\mathscr{H}$. Write $\mathscr{V}_0 = \mathscr{V}_{0,1} \sqcup \mathscr{V}_{0,2}$ with $\mathscr{V}_{0,i} \subset \mathscr{V}_i$, for $i = 1, 2$. Note that both $\mathscr{V}_{0,1}$ and $\mathscr{V}_{0,1}$ are non-empty. Let $(F,T) \in \mathscr{V}_{0,i}$. Let $G_0 = (V, E_0)$ be the spanning subgraph of G having the edge set $E_0 = E(T) \cup E(F)$. By definition of the edges in $\mathscr{H}$, and connectivity of $\mathscr{H}_0$, we have for all $(A,B) \in \mathscr{V}_0$, $E(A) \cup E(B) = E(G_0)$. We refer to G_0 as the spanning subgraph of G associated to the connected component $\mathscr{H}_0$ of $\mathscr{H}$.

Notation. For a subset $X \subset V$ of the vertices of a (multi)graph $G = (V,E)$, we denote by $G[X]$ the induced graph on X: it has vertex set X and edge set all the edge of E with both end-points lying both in X.

Note that for any subset $X \subset V$, the induced subgraph $G_0[X]$ has at most $2|X| - 2$ edges. The following natural definition thus distinguishes the subsets for which the equality holds.

Definition 2.5 (Saturated sets and components)**.** A subset X of vertices of G_0 is called *saturated* (with respect to G_0) if the induced subgraph $G_0[X]$ has precisely $2|X| - 2$ edges. A *saturated component* X of G_0 is a saturated subset of vertices which is maximal for inclusion.

2.1. **Partition of the vertex set induced by saturated components.** Let $\mathscr{H}_0$ be a connected component of $\mathscr{H}$ with associated spanning subgraph G_0. We will show in a moment that the saturated components of G_0 form a partition of its vertex set.

Lemma 2.6. *Let X be a saturated subset of G_0. Then for all vertices $(A,B) \in \mathscr{V}_0$, X is connected in both A and B, i.e., the induced graphs $A[X]$ and $B[X]$ are disjoint trees on the vertex set X.*

Proof. Both $A[X]$ and $B[X]$ are free of cycles. Since $G_0[X]$ has precisely $2|X| - 2$ edges, and $A[X]$ and $B[X]$ are disjoint, both $A[X]$ and $B[X]$ are trees on vertex set X. □

Let now X be saturated subset of G_0, and (A,B) a vertex of $\mathscr{V}_0$. Since the induced graphs $A[X]$ and $B[X]$ are both trees, for any edge e of A with both end-points in X, the graph $B+e$ has a cycle. Similarly, for any edge e of B which lie in X, the graph $A+e$ has a cycle. It follows that pivoting in G_0 do not involve any edge in X, and by connectivity of $\mathscr{H}_0$, we thus have for any pair $(A',B') \in \mathscr{V}_0$ that $A'[X] = A[X]$ and $B'[X] = B[X]$.

Proposition 2.7. *Saturated components of G_0 form a partition of $V(G) = V(G_0)$.*

Proof. Let X and X' be two distinct saturated components of G_0. We need to show that $X \cap X' = \emptyset$. Let F and T be any spanning 2-forest and spanning tree of G, respectively, such that $E_0 = E(F) \sqcup E(T)$. Since X and X' are saturated with respect to G_0, it follows from the previous lemma that all the induced subgraphs $F[X]$, $T[X]$, $F[X']$, and $T[X']$ are connected. For the sake of a contradiction, suppose X and X' have a non-empty intersection. It follows that the induced subgraphs $F[X \cup X']$ and $T[X \cup X']$ are both connected, which implies that the set $X \cup X'$ is saturated. By the maximality of X and X' and distinct, this is impossible, and the proposition follows. □

Denote by $X_1, \dots, X_r$ all the different saturated components of G_0, thus we get a partition of $V = X_1 \sqcup \dots \sqcup X_r$.

Note that, by definition, there exist for any $j = 1, \dots, r$, two disjoint trees $T_{j,1}$ and $T_{j,2}$ with vertex set X_j so that for any pair $(A,B) \in \mathscr{V}_0$, we have $A[X_j] = T_{j,1}$ and $B[X_j] = T_{j,2}$.

2.2. **Alternative characterisation of saturated components.** We now give another characterisation of the saturated components of G_0. This will be in terms of the connected component $\mathscr{H}_0 = (\mathscr{V}_0, \mathscr{E}_0)$ of $\mathscr{H}$ and involves the definition of two equivalence relations $\simeq_1$ and $\simeq_2$ on the set of vertices, defined as follows. Note that the vertex set $\mathscr{V}_0$ is partitioned into sets $\mathscr{V}_{0,1}$ and $\mathscr{V}_{0,2}$.

Definition 2.8. For any pair of vertices $u, v \in V$,

- we say $u \simeq_1 v$ if for any $(F,T) \in \mathscr{V}_{0,1}$, both vertices u and v lie in the same connected component of $T \setminus E(\mathcal{P}(F))$. Similarly,
- we say $u \simeq_2 v$ if for any $(T,F) \in \mathscr{V}_{0,2}$, both vertices u and v lie in the same connected component of $T \setminus E(\mathcal{P}(F))$.

It is straightforward to show that $\simeq_1$ and $\simeq_2$ induce an equivalence relation on the set of vertices V. We actually show that the two equivalence relations above are in fact identical. We need the following basic lemma.

Lemma 2.9. *Let F and T be a spanning 2-forest and a spanning tree of G_0, respectively. Let $u, v \in V$ be a pair of vertices lying in two different connected components of $T \setminus E(\mathcal{P}(F))$. There exists an edge $e \in E(\mathcal{P}(F)) \cap E(T)$ such that u and v are not connected in $T - e$.*

Proof. Denote by S_u and S_v the two connected components of $T \setminus E(\mathcal{P}(F))$ which contain u and v, respectively. There is a unique path in T joining S_u to S_v. Since $S_u \neq S_v$, it contains an edge $e \in E(\mathcal{P}(F))$. For this edge e, clearly u and v are not connected in $T - e$. □

The previous lemma allows to prove the following proposition.

Proposition 2.10. *The two equivalence relations $\simeq_1$ and $\simeq_2$ are the same.*

Proof. Let $u, v \in V$ be two vertices. By symmetry, it will be enough to show that if $u \not\simeq_2 v$, then $u \not\simeq_1 v$. Since $u \not\simeq_2 v$, by definition, there must exist a pair $(T, F) \in \mathscr{V}_{0,2}$ such that u, v belong to two different connected components of $T \setminus E(\mathcal{P}(F))$. Applying the previous lemma, we infer the existence of an edge $e \in E(T) \cap E(\mathcal{P}(F))$ such that u and v are not connected in $T - e$. Pivoting involving e gives a pair $(F', T') \in \mathscr{V}_{0,1}$ such that u and v lie in two different connected components of F'. It follows that $u \not\simeq_1 v$. □

Since the two equivalence relations are identical, we drop the indices and denote by $\simeq$ both $\simeq_i$. We have actually proved the following

Proposition 2.11. *The following properties are equivalent for any pair $u, v \in V$:*
(1) we have $u \not\simeq v$.
(2) there exists $(F, T) \in \mathscr{V}_{0,1}$ such that u and v lie in different connected components of F.
(3) there exists $(T', F') \in \mathscr{V}_{0,2}$ such that u, v lie in two different connected components of F'.

Denote by $\mathcal{P}_\simeq$ the partition of V induced by the equivalence classes of $\simeq$. We have

Proposition 2.12. *The partition $\mathcal{P}_\simeq$ coincides with the partition of V into saturated components of G_0.*

Proof. Let u and v be two vertices in $V = V(G_0)$. If u and v lie in a saturated component X of G_0, then for any pair $(F, T) \in \mathscr{V}_{0,1}$, the induced graph $F[X]$ is connected. This shows u and v are in the same connected component of F, and so u and v are equivalent for $\simeq$. This shows the partition into saturated components is a refinement of $\mathcal{P}_\simeq$.

In order to prove the proposition, it will be thus enough to show that each element in $\mathcal{P}_\simeq$ is saturated with respect to G_0. Let $X \subset V$ be an element of $\mathcal{P}_\simeq$, and consider two vertices $a, b \in X$. Let $(F, T) \in \mathscr{V}_0$ be a vertex of $\mathscr{H}_0$, and let P be the unique path in T joining a and b. We claim that P is contained in X. To see this, note that there is no edge $e \in E(\mathcal{P}(F))$ in the path P: otherwise, the pair $(F + e, T - e)$ would be a vertex of $\mathscr{H}_0$, and the two vertices a and b would lie in two different connected components of the 2-forest $T - e$, which would be clearly in contradiction with Proposition 2.11.

By definition of the edges in $\mathscr{H}$, and by connectivity of $\mathscr{H}_0$, this shows that for any $(F_1, T_1), (T_2, F_2) \in \mathscr{V}_0$, the path P is included in T_1 and F_2. By the definition of the equivalence relation $\simeq$ and Proposition 2.11, we infer that X contains all the vertices of the path P. This shows that $T[X]$ is connected.

A similar argument shows that the induced graph $F[X]$ is connected. Since the sets $E(F)$ and $E(T)$ are disjoint, we infer that X is a saturated set with respect to G_0. □

2.3. **Classification of the components of the exchange graph $\mathscr{H}$.** We can now state the main result of this section.

Theorem 2.13. *Let G be a multigraph.*

(1) *The exchange graph $\mathscr{H}$ is connected if and only if the following two conditions hold:*
  (i) *the edge set of G can be partitioned as $E(G) = E(T) \sqcup E(F)$ for a spanning tree T and a spanning 2-forest F of G; and*
  (ii) *any non-empty subset X of V saturated with respect to G consists of a single vertex.*

(2) *More generally, there is a bijection between the connected components $\mathscr{H}_0$ of $\mathscr{H}$ and the pair $(G_0; \{T_{1,1}, T_{1,2}, \dots, T_{r,1}, T_{r,2}\})$ where*

(i) *G_0 is a spanning subgraph of G which is a disjoint union of a spanning tree T and a spanning 2-forest F of G;*

(ii) *denoting the maximal subsets of V saturated with respect to G_0 by $X_1, \dots, X_r$, then $T_{j,1}$ and T_{j_2} are two disjoint spanning trees on the vertex set X_j, and $E(G_0[X_j]) = E(T_{j,1}) \sqcup E(T_{j,2})$, for $j = 1, \dots, r$.*

Under this correspondence, the vertex set of $\mathscr{H}_0$ consists of all the vertices $(A, B) \in \mathscr{V}$ which verify $E(A) \cup E(B) = E(G_0)$, and for all $j = 1, \dots, r$, $A[X_j] = T_{j,1}$ and $B[X_j] = T_{j,2}$.

Figure 1. Example of a graph G, on the left, which is a disjoint union of a spanning tree and a spanning 2-forest, in which all saturated components are singletons. Note that G contains a spanning tree T, given on the right, with a complement which is not a spanning 2-forest.

The rest of this section is devoted to the proof of this theorem. In the following, we will use the well-known exchange property for the spanning trees of G: it asserts that for a pair of spanning trees T and T', and for any edge $e \in E(T) \setminus E(T')$, there exists an edge $e' \in E(T') \setminus E(T)$ such that $T - e + e'$ is a spanning tree of G. (In other words, spanning trees of G form the basis of a matroid on the ground set E. Such matroids are called graphic.)

Before giving the proof of this theorem, we make the following remark.

Remark 2.14. Let G be a graph whose edge set is a disjoint union of the edges of a spanning tree and a spanning 2-forest, and with the property that there is no saturated subset of size larger than two. The graph G might contain spanning trees T with the property that $G \setminus E(T)$ is not a spanning 2-forest. An example is given in Figure 2.3. In a sense, Theorem 1.2 concerns smaller number of spanning trees of G, and the theorem does not seem to follow from the well-known connectivity property of edge-exchanges for spanning trees.

Proof of Theorem 2.13(1). We first show the necessity of (i) and (ii).

So suppose that the exchange graph $\mathscr{H}$ is connected. We show $E(G) = E(T) \sqcup E(F)$, which proves (i). For the sake of a contradiction, suppose this is not the case, and let e be an edge of G which is neither in T nor in F. There exists an edge e' in T so that $T' = T - e' + e$ is a spanning tree of G. The pair (T', F) is then a vertex of $\mathscr{H}$ which obviously cannot be in the same connected component as (T, F) by the very definition of the edges in the exchange graph. This contradicts the assumption on the connectivity of $\mathscr{H}$, and proves (i).

To prove (ii), let $X_1, \dots, X_r$ be all the different saturated components of G, and assume for the sake of a contradiction, and without loss of generality, that $|X_1| > 1$. Let $T_{j,1}$ and $T_{j,2}$ be the two edge-disjoint trees on X_j associated to $\mathscr{H}$. Recall that this means we have $T_{j,1} = A[X_j]$ and $T_{j,2} = B[X_j]$ for any vertex (A, B) of $\mathscr{H}$ (the connectivity of $\mathscr{H}$ implies the definition is independent of the choice of the vertex (A, B)).

Let $(A, B) \in \mathscr{V}$ be a vertex of $\mathscr{H}$. Define the pair (A', B') by $A' = A - E(T_{1,1}) + E(T_{1,2})$ and $B' = B - E(T_{1,2}) + E(T_{1,1})$. Note that (A', B') is a vertex of $\mathscr{H}$ since A' and B' have the same number of edges as A and B, respectively, both are without cycles, and $E(G) = E(A) \cup E(B) = E(A') \cup E(B')$. On the other hand, since pivoting only involves edges which are neither in $T_{1,1}$ nor in $T_{1,2}$, this shows that (A', B') cannot be connected to (A, B), which is a contradiction with the assumption on the connectivity of $\mathscr{H}$.

We now prove the sufficiency of (i) in (ii). So suppose that both (i) and (ii) in (1) hold, we show that the exchange graph $\mathscr{H}$ is connected.

Since any vertex (F, T) in $\mathscr{V}_1$ is connected to a vertex of $\mathscr{V}_2$, it will be enough to prove that any two vertices $(T, F), (T', F') \in \mathscr{V}_2$ are connected by a path in $\mathscr{H}$.

We prove this proceeding by induction on the integer number

$$r = \mathrm{diff}(T, T') := \big|E(T) \setminus E(T')\big|.$$

• For the base of our induction, If $r = 0$, then $T = T'$, and so by (i), we must have $F = F'$, and the claim trivially holds.

• Assuming the assertion holds for $r \in \mathbb{N} \cup \{0\}$, we prove it holds for $r+1$. So let $\mathfrak{v} = (T, F)$, $\mathfrak{v}' = (T', F') \in \mathscr{V}_2$ be two vertices with $\big|\, E(T) \setminus E(T')\,\big| = r+1$. For the sake of a contradiction, assume that $\mathfrak{v}$ and $\mathfrak{v}'$ are not connected in $\mathscr{H}$. Denote by $\mathscr{H}_0$ the connected component of $\mathscr{H}$ which contains $\mathfrak{v}$. A contradiction will be achieved through a set of claims (I) - (V).

We claim

(I) *There is no edge e in $E(T) \setminus E(T')$ with $F + e \in \mathcal{ST}(G)$.* (Similarly, there is no edge e in $E(T') \setminus E(T)$ with $F' + e \in \mathcal{ST}(G)$.)

Otherwise, suppose $e \in E(T) \setminus E(T')$ be an edge such that $F + e$ is a spanning tree of G. There exists $e' \in E(T') \setminus E(T)$ such that $T'' := T - e + e'$ is a spanning tree of G. The complement of T'' in G is $F'' := F + e - e'$. Since $F + e$ is a spanning tree of G, and $e' \in F$, the subgraph F'' is a spanning 2-forest of G, and thus $\mathfrak{v}'' := (T'', F'')$ is a vertex in $\mathscr{V}_2$. By definition, $\mathfrak{v} = (T, F)$ and $(F + e, T - e)$ are adjacent in $\mathscr{H}$. Moreover, $(F + e, T - e)$ and $\mathfrak{v}''$ are adjacent in $\mathscr{H}$. Note that $\mathrm{diff}(T'', T') = r$, and so by the hypothesis of our induction, $\mathfrak{v}''$ and $\mathfrak{v}'$ are connected by a path in $\mathscr{H}$. Thus $\mathfrak{v}$ and $\mathfrak{v}'$ are connected in $\mathscr{H}$, which is a contradiction to the assumption we made. This proves our first claim (I).

As a consequence of (I) we now show that the following claim.

(II) *We have $F \sim_v F'$, i.e., the two partitions $\mathcal{P}(F)$ and $\mathcal{P}(F')$ of V coincide.*

Recall that two spanning 2-forests which induce the same partition of vertices are said equivalent for the the equivalence relation $\sim_v$.

To show this claim, let $\mathcal{P}(F) = \{X, Y\}$ and $\mathcal{P}(F') = \{X', Y'\}$, and suppose for the sake of a contradiction that the two partitions are not equal. The partition $\mathcal{P}(F)$ (resp. $\mathcal{P}(F')$) induces a partition of both X' and Y' (resp. X and Y). One of these four induced partitions has to be non-trivial: by this we mean that, without loss of generality, we can assume for example that $Z := X \cap X'$ and $W := X \cap Y'$ are both non-empty. Since $F[X]$ is connected, there is an edge $e = \{u, v\} \in F$ with $u \in Z$ and $v \in W$. This edge does not belong to F' since it joins a vertex in X' to a vertex in Y', therefore, $e \in T'$. Moreover, since $F' + e$ is is a spanning tree of G. In other words, e is an edge of $E(T') \cap E(F) = E(T') \setminus E(T)$ with $F' + e \in \mathcal{ST}(G)$, which is a contradiction to (I). This proves our claim (II).

Let $\mathcal{P}(F) = \mathcal{P}(F') = \{X, Y\}$. Denote by $\mathcal{P}_X$ the partition of X given by the vertex sets of the connected components of $T[X]$. Also, denote by $\mathcal{P}'_X$ the partition of X induced by the connected components of $T'[X]$. Similarly, define $\mathcal{P}_Y$ and $\mathcal{P}'_Y$. Let $E(\mathcal{P}_X)$ (resp. $E(\mathcal{P}_Y)$) be the set of all edges e of G with end-points in two different members of $\mathcal{P}_X$ (resp. $\mathcal{P}_Y$), respectively. Similarly, define $E(\mathcal{P}'_X)$ and $E(\mathcal{P}'_Y)$.

We now claim.

(III) *All the pairwise intersections $E(T') \cap E(\mathcal{P}_X)$, $E(T') \cap E(\mathcal{P}_Y)$, $E(T) \cap E(\mathcal{P}'_X)$, $E(T) \cap E(\mathcal{P}'_Y)$ are empty.*

Otherwise, without loss of generality, suppose there is an edge $e' \in T'$ with $e' \in E(\mathcal{P}_X)$. Since e' joins two different connected components of $T[X]$, we must have $e' \in F$. The graph $T + e'$ has a cycle, which, once again since e' joins two different connected components of $T[X]$, must include an edge $e \in E(\mathcal{P}(F))$. Since $\mathcal{P}(F) = \mathcal{P}(F')$, we should have $e \in E(T')$.

Let $\mathfrak{v}_1 := (F_1, T_1)$ with $F_1 = T - e$ and $T_1 = F + e$, and $\mathfrak{v}_2 := (T_2, F_2)$ with $T_2 = F_1 + e'$ and $F_2 = T_1 - e'$. By choices we made of e and e', both $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are vertices in $\mathscr{H}$. The three vertices $\mathfrak{v}, \mathfrak{v}_1, \mathfrak{v}_2$ form a path of length two. An easy inspection shows in addition that $\mathrm{diff}(T_2, T') = \mathrm{diff}(T, T') = r + 1$.

Since F_2 contains the edge $e \in E(\mathcal{P}(F))$, we infer that $\mathcal{P}(F_2) \neq \mathcal{P}(F)$. By our assumption, the two vertices $\mathfrak{v}$ and $\mathfrak{v}'$ are not connected in $\mathscr{H}$. This shows that the two vertices $\mathfrak{v}_2$ and $\mathfrak{v}'$ are not connected in $\mathscr{H}$ neither. Applying the above reasoning to $\mathfrak{v}_2$ and $\mathfrak{v}'$, we must therefore have by Claim (II) that $\mathcal{P}(F_2) = \mathcal{P}(F') = \mathcal{P}(F)$, which gives a contradiction. This proves our claim (III).

As an immediate corollary of (III), we get

(IV) *We have the equality of partitions $\mathcal{P}_X = \mathcal{P}'_X$ and $\mathcal{P}_Y = \mathcal{P}'_Y$.*

Indeed, since $E(T') \cap E(\mathcal{P}_X) = \emptyset$, any subset Z' of X with $T'[Z']$ connected should be entirely included in an element of $\mathcal{P}_X$. This in particular, when applied to each $Z' \in \mathcal{P}'_X$, shows that the partition $\mathcal{P}'_X$ is a refinement of $\mathcal{P}_X$. By symmetry, the partition $\mathcal{P}_X$ should be, as well, a refinement of $\mathcal{P}'_X$. Thus, we get the equality of the two partitions $\mathcal{P}_X = \mathcal{P}'_X$. The equality $\mathcal{P}_Y = \mathcal{P}'_Y$ follows similarly.

As an immediate corollary, we get

(V) *The equality $E(\mathcal{P}_X) \sqcup E(\mathcal{P}_Y) = E(\mathcal{P}'_X) \sqcup E(\mathcal{P}'_Y)$ holds.*

We are now ready to finish the proof of the theorem.

By the definition of $\mathscr{H}$, all the vertices $\mathfrak{v}_2 = (T_2, F_2)$ of $\mathscr{H}$ at distance 2 from (T, F) are precisely of the form $T_2 = T - e_1 + e_2$ and $F_2 = F + e_1 - e_2$ for any $e_1 \in E(\mathcal{P}(F))$ and any $e_2 \in E(\mathcal{P}_X) \sqcup E(\mathcal{P}_Y)$. Indeed, if for an edge e_1, we have $F + e_1 \in \mathcal{ST}(G)$, then e_1 should be in $E(\mathcal{P}(F))$. An edge $e_2 \neq e_1$ which belongs to the spanning tree $F + e_1$ must have its end-points either both in X or both in Y. Moreover, if $T - e_1 + e_2 \in \mathcal{ST}(G)$, then the edge e_2 must be in $E(\mathcal{P}_X) \sqcup E(\mathcal{P}_Y)$. To see this, note that otherwise, both the end-points of e_2 would lie in a connected component of $T[X]$ or $T[Y]$, which would imply the existence of a cycle in $T - e_1 + e_2$. This proves one part of the claim. For the other direction, one easily verifies that for a pair of distinct edges $e_1 \in E(\mathcal{P}(F))$ and $e_2 \in E(\mathcal{P}_X) \sqcup E(\mathcal{P}_Y)$, the pair (T_1, F_2) is a vertex of $\mathscr{H}$, which is clearly at distance two from (T, F).

Now by Claim (II), we have $E(P(F)) = E(\mathcal{P}(F'))$, and by Claim (V), we have $E(\mathcal{P}_X) \sqcup E(\mathcal{P}_Y) = E(\mathcal{P}'_X) \sqcup E(\mathcal{P}'_Y)$.

Thus, applying the observation which precedes, for such a vertex $\mathfrak{v}_2 = (T - e_1 + e_2, F + e_1 - e_2)$, the pair $\mathfrak{v}'_2 = (T'_2, F'_2)$ defined by $T'_2 = T' - e_1 + e_2$ and $F'_2 = F' + e_1 - e_2$ is also a vertex of $\mathscr{H}$ which is at distance two from $\mathfrak{v}'$. In addition, we have $\operatorname{diff}(T_2, T'_2) = \operatorname{diff}(T, T') = r + 1$.

Since by our assumption, $\mathfrak{v}$ and $\mathfrak{v}'$ are not connected in $\mathscr{H}$, any pair of vertices $\mathfrak{v}_2$ and $\mathfrak{v}'_2$ obtained as above (i.e., at distance two from and $'$, respectively) are not connected in $\mathscr{H}$.

Since the two spanning trees T and T' are not equal, there is an edge $e_\star \in E(T') \setminus E(T)$. For any choice of e_1, e_2 as above, we have $e_\star \neq e_1, e_2$, and thus we must have $e_\star \in E(T'_2) \setminus E(T_2)$.

Applying the same reasoning to the pair $\mathfrak{v}_2$ and $\mathfrak{v}'_2$, and proceeding inductively on k, we infer that for any vertex $\mathfrak{v}_{2k} = (T_{2k}, F_{2k})$ of $\mathscr{H}$ obtained from (T, F) by an ordered sequence of pivoting involving edges $e_1, e_2, \dots, e_{2k-1}, e_{2k}$, the pair $\mathfrak{v}'_{2k} = (T'_{2k}, F'_{2k})$ obtained from $\mathfrak{v}'$ by pivoting involving the same ordered sequence of edges $e_1, e_2, \dots, e_{2k-1}, e_{2k}$ is a vertex of $\mathscr{H}$, and we have by (I)-(V):

- $\mathcal{P}(F_{2k}) = \mathcal{P}(F'_{2k}) = \{X_{2k}, Y_{2k}\}$ (with X_{2k} and Y_{2k} depending on the sequence of edges $e_1, \dots, e_{2k}$),
- $E(\mathcal{P}_{X_{2k}}) \sqcup E(\mathcal{P}_{Y_{2k}}) = E(\mathcal{P}'_{X_{2k}}) \sqcup E(\mathcal{P}'_{Y_{2k}})$.
- $\operatorname{diff}(\mathfrak{v}_{2k}, \mathfrak{v}'_{2k}) = r + 1$, and $\mathfrak{v}_{2k}$ and $\mathfrak{v}'_{2k}$ are not connected in $\mathscr{H}$.
- $e_* \in E(T'_{2k}) \setminus E(T_{2k})$

Let $\mathscr{H}_0$ be the connected component of $\mathscr{H}$ which contains $\mathfrak{v}$. To get a contradiction, note that all the vertices in $\mathscr{H}_0$ which belong to $\mathscr{V}_2$ appear among the set of vertices $\mathfrak{v}_{2k}$, and we have $e_* \in E(T'_{2k}) \setminus E(T_{2k}) \subset E(F_{2k})$. In other words, for any pair $(A, B) \in \mathscr{V}_2$ which is a vertex of $\mathscr{H}_0$, the two end-points of e_* are both in the same connected component of the spanning 2-forest B. By Propositions 2.10 and 2.11, it follows that the two end-points of e_* are in the same equivalence class for the equivalence relation $\simeq$ we defined for $\mathscr{H}_0$. Since by Proposition 2.12, the partition $\mathcal{P}_\simeq$ coincides with the partition of V into saturated components of G, this leads to a contradiction to the assumption that all the saturated components are singletons. This final contradiction proves the step $r + 1$ of our induction and finishes the proof of the first part of our theorem. □

Proof of Theorem 2.13(2). Part (2) follows directly from part (1): contract all the edges lying in a saturated component in G_0 in order to get the graph $\widetilde{G}_0$. One can verify that in $\widetilde{G}_0$, all the saturated components are singleton, and the edges of $\widetilde{G}_0$ are a disjoint union of the edges of a spanning tree and a spanning 2-forest. Thus by part (1), the graph $\mathscr{H}_{\widetilde{G}_0}$ is connected. There is an isomorphism from $\mathscr{H}_0$ to $\mathscr{H}_{\widetilde{G}_0}$ which sends a pair (A, B) in $\mathscr{V}_0$ to the pair $(\widetilde{A}, \widetilde{B})$ in $\mathscr{H}_{\widetilde{G}_0}$ obtained by contracting all the edges in the trees $T_{j,1}, T_{j,2}$, for $j = 1, \dots, r$. □

3. Proof of Theorem 1.1

For an $r \times t$ matrix X, and subsets $I \subseteq \{1, \dots, r\}$ and $J \subseteq \{1, \dots, t\}$ with $|I| = |J|$, we note by $X_{I,J}$ the square $|I| \times |I|$ submatrix of X with rows and columns in I, J, respectively. If $r \leq t$, and $I = \{1, \dots, r\}$ and $J \subseteq \{1, \dots, t\}$, we simply write X_J instead of $X_{I,J}$.

We use the notation of the introduction: choosing a basis $\gamma_1, \dots, \gamma_h$ for $H_1(G,\mathbb{Z})$, we denote by M the $h \times m$ matrix of the coefficients of γ_i in the standard basis $\{e_i\}_{i=1}^m$ of $\mathbb{R}^m$. Similarly, for the element $\omega \in \mathbb{R}^E$ in the inverse image $\partial^{-1}(\mathbf{p})$ of the vector of external momenta $\mathbf{p} = (\mathbf{p}_v)$, we denote by H_ω the $(h+1)$-dimensional vector subspace of $\mathbb{R}^m$ generated by ω and $H_1(G,\mathbb{R})$. The space H_ω comes with a basis consisting of $\gamma_1, \dots, \gamma_h, \omega$, and we denote by N the $(h+1) \times m$ matrix of the coefficients of this basis in the standard basis $\{e_i\}_{i=1}^m$ of $\mathbb{R}^m$.

By Cauchy-Binet formula, we have

$$\det(NYN^\tau) = \sum_{\substack{I,J \subseteq \{1,\dots,m\} \\ |I|=|J|=h+1}} \det(N_I)\det(Y_{I,J})\det(N_J). \tag{3.1}$$

Since Y is a diagonal matrix, for $I \neq J$, we have $\det(Y_{I,J}) = 0$. Moreover, for $I = J$, we have $\det(Y_{I,I}) = y^I$, where, as usual, we pose $y^I := \prod_{i \in I} y_i$. Therefore, the above sum can be reduced to the following sum

$$\det(NYN^\tau) = \sum_{\substack{I \subset \{1,\dots,m\} \\ |I|=h+1}} \det(N_I)^2 y^I.$$

Similarly, we have

$$\det(MYM^\tau) = \sum_{\substack{I \subset \{1,\dots,m\} \\ |I|=h}} \det(M_I)^2 y^I. \tag{3.2}$$

For a subgraph F in G, by an abuse of the notation, we write F^c (instead of $E \setminus E(F)$) for the set of edges of G not in F.

Lemma 3.1. (1) *For a subset $I \subseteq \{1,\dots,m\}$ of size h, we have $\det(M_I) \neq 0$ if and only if $I = T^c$ for a spanning tree T of G. In this case, we have $\det(M_I)^2 = 1$.*

(2) *For a subset $I \subseteq \{1,\dots,m\}$ of size $h+1$, we have $\det(N_I) = 0$ unless $I = F^c$ for a spanning 2-forest F of G, in which case, we have*

$$\det(N_I)^2 = q(F) = (\sum_{v \in X)} \mathbf{p}_v).(\sum_{v \in Y} \mathbf{p}_v),$$

where $\{X,Y\}$ denotes the partition of V given by F.

Proof of (1). This is folklore. The complement I^c of I has precisely $n-1$ edges, where n is the number of vertices of the graph. If I^c is not the edge set of a spanning tree, then the spanning subgraph of G on vertex set V which contains I^c as the edge set is not connected. Let $\{X,Y\}$ be a partition of V such that all the edges of I^c have either both end-points in X or both end-points in Y. It follows that all the edges $E(X,Y)$ are in I. Without loss of generality, we can assume that these edges are all oriented from X to Y. It follows that for all cycles γ_i, we have $\sum_{e \in E(X,Y)} \gamma_i(e) = 0$, which shows that $\det(M_I) = 0$.

Let now $I = E(T)^c$ for a spanning tree T of G. For any edge $e_i \in I$, the graph $T + e_i$ has a unique cycle γ_i', which in addition contains e_i. The collection of cycles γ_i' for $e_i \in I$ form a basis of $H_1(G,\mathbb{Z})$. Since all the edges of γ_i' different form e_i are in $E(T)$, it follows that the matrix M_I in the basis $\gamma_1', \dots, \gamma_h'$ is the identity matrix. The change of basis matrix from the basis $\{\gamma_i\}_{i=1}^h$ to $\{\gamma_i'\}_{i \in I}$ has determinant 1 or -1, from which the result follows. ∎

Proof of (2). Denote by $e_{i_1}, \dots, e_{i_{h+1}}$ the $(h+1)$ edges of I. Developing $\det(N)$ with respect to the last row (which corresponds to the coefficients of ω), we have

$$\det(N_I) = \sum_{j=1}^{m} (-1)^j \omega(e_i) \det(M_{I \setminus \{e_{i_j}\}}).$$

From the first part, it follows that $\det(N_I) = 0$ if none of $I - e_{i_j}$ is the complement set of edges of a spanning tree, i.e., if I is not of the form F^c for a spanning 2-forest of G. So suppose now that $I = F^c$, denote by $\{X, Y\}$ the partition of V induced by F, and without loss of generality, let $e_{i_1}, \dots, e_{i_r}$ be the set of all the edges in $E(\mathcal{P}(F))$. We can assume that e_i's are all oriented from X to Y. Let $T_j = F \cup \{e_{i_j}\}$ the spanning tree $F \cup \{e_{i_j}\}$ for $j = 1, \dots, r$. It follows that

$$\det(N_I) = \sum_{j=1}^{r} (-1)^j \omega(e_{i_j}) \det(M_{T_j^c}).$$

Since $\partial(\omega) = \mathbf{p}$, and the edges $e_{i_1}, \dots, e_{i_r}$ are oriented from X to Y, it follows that

$$\sum_{j=1}^{r} \omega(e_{i_j}) = \sum_{v \in X} \mathbf{p}_v.$$

So the lemma follows once we prove that $(-1)^j \det(M_{I \setminus \{e_{i_j}\}})$ takes the same value for all $j = 1, \dots, r$. By symmetry, it will be enough to prove $\det(M_{T_1^c}) + \det(M_{T_2^c}) = 0$. By multilinearity of the determinant with respect to the columns, we see that $\det(M_{T_1^c}) + \det(M_{T_2^c}) = \det(P)$ where P is the $h \times h$ matrix with the first column equal to the sum of the first columns of $M_{T_1^c}$ and $M_{T_2^c}$, and the j'th column equal to the j'th column of $M_{T_1^c}$ (which is the same as that of $M_{T_2^c}$), for $j \geq 2$. So it will be enough to show that $\det(P) = 0$. The subgraph $F \cup \{e_{i_1}, e_{i_2}\}$ has a unique cycle γ which contains both e_{i_1}, e_{i_2} from F^c and all the other edges are in F. Writing γ as a linear combination $\gamma = \sum_{j=1}^{h} a_j \gamma_j$ of the cycles γ_j, we show that $(a_1, \dots, a_h)P = 0$. The first coefficient of $(a_1, \dots, a_h)P$ is zero since the cycle γ has e_{i_1} and e_{i_2} with different signs. All the other coordinates of $(a_1, \dots, a_h)P$ are zero since the only edges of γ in F^c are e_{i_1} and e_{i_2}. □

Remark 3.2. The proof of the above lemma shows the following useful property. Suppose that I and J are the complement of the edges of two (vertex-)equivalent 2-forests $F_1 \sim_v F_2$ inducing the partition $V = X \sqcup Y$ of V, respectively. Let $e \in E(\{X, Y\})$ be an edge with one end-point in each of X and Y, so both $T_1 = F_1 \cup \{e\}$ and $T_2 = F_2 \cup \{e\}$ are spanning trees. Then

$$\frac{\det(N_I)}{\det(M_{T_1^c})} = \frac{\det(N_J)}{\det(M_{T_2^c})} = \pm \sum_{e \in E(X,Y)} \omega(e),$$

where e in the above sum runs over all the oriented edges from X to Y. In particular, we have

$$\det(N_I)\det(N_J) = q(F_1)\det(M_{T_1^c})\det(M_{T_2^c}) = q(F_2)\det(M_{T_1^c})\det(M_{T_2^c}). \tag{3.3}$$

From Lemma 3.1 we infer that in the sum (3.1) (resp. (3.2)) above describing $\det(MYM^\tau)$ (resp. $\det(NYN^\tau)$), the only possible non-zero terms correspond to subsets I which are complements of the edges of a spanning tree (resp. spanning 2-forest) of G.

Consider the set-up of Theorem 1.1 as in the introduction, where U is a topological space and $y_1, \dots, y_m : U \to \mathbb{R}_{>0}$ are m continuous functions. Denote by Y the diagonal matrix-valued function on U given by $Y(s) = \mathrm{diag}(y_1(s), \dots, y_m(s))$. Let $\mathbf{p} \in (\mathbb{R})^{V,0}$ be a fixed vector

Define two real-valued functions f_1 and f_2 on U by

$$f_1(s) := \det(MYM^\tau) = \sum_{\substack{T \in \mathcal{ST} \\ I = T^c}} y(s)^I, \text{ and} \tag{3.4}$$

and

$$f_2(s) := \det(NYN^\tau) = \sum_{\substack{F \in \mathcal{SF}_2 \\ I = F^c}} q(F) y(s)^I,$$

at each point $s \in U$. Note that $f_1(s) = \phi(\underline{y}(s))$, for ϕ the first Symanzik polynomial, and $f_2(s) = \psi_G(\omega, \underline{y}(s)))$, for ψ the second Symanzik polynomial of the graph G.

Let now $A : U \to \mathrm{Mat}_{m\times m}(\mathbb{R})$ be a matrix-valued map taking at $s \in U$ the value $A(s)$. Assume that A verifies the two properties

(i) A is a bounded function, i.e., all the entries $A_{i,j}$ of A take values in a bounded interval $[-C, C]$ of $\mathbb{R}$, for some positive constant $C > 0$.
(ii) The two matrices $M(Y + A)M^\tau$ and $N(Y + A)N^\tau$ are invertible at all points $s \in U$.

Define real-valued functions g_1, g_2 on U by $g_1(s) := \det(M(Y + A)M^\tau)$ and $g_2(s) = \det(N(Y + A)N^\tau)$. We have by Cauchy-Binet formula,

$$g_1 = \sum_{\substack{T_1, T_2 \in \mathcal{ST} \\ I = T_1^c, J = T_2^c}} \det(M_I)\det(Y + A)_{I,J}\det(M_J), \text{ and}$$

$$g_2 = \sum_{\substack{F_1, F_2 \in \mathcal{SF}_2 \\ I = F_1^c, J = F_2^c}} \det(N_I)\det(Y + A)_{I,J}\det(N_J).$$

To prove Theorem 1.1, we must show that $g_2/g_1 - f_2/f_1 = O_{\underline{y}}(1)$ on U. Observe first that

Claim 3.3. *There exist constants $c_1, c_2, C > 0$ such that*

$$c_1 f_1(s) \;<\; g_1(s) \;<\; c_2 f_1(s), \tag{3.5}$$

for all points $s \in U$ with $y_1(s), \dots, y_m(s) \geq C$.

Proof. By assumption, all the coordinates of A are bounded functions on U. Developing the determinant $\det(Y + A)_{I,J}$ as a sum (with $\pm$ sign) over permutations of the products of entries of $(Y + A)_{I,J}$, one observes that each term in the sum is the product of a bounded function with a monomial in the y_j's for indices j in a subset of $I \cap J$. For $I \neq J$, these terms become $o(y^I)$. Also for $I = J$, all the terms but the unique one coming from the product of the entries on the diagonal which gives y^I are $o(y^I)$. Since $f_1 = \sum_{T \in \mathcal{ST}} y^{T^c}$, the assertion follows. □

Therefore, in order to prove Theorem 1.1, it will be enough to show that

$$g_2 f_1 - g_1 f_2 = O_{\underline{y}}(f_1^2). \tag{3.6}$$

In considering the terms in $g_2f_1 - g_1f_2$ it will become very convenient to define the bipartite graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$, a variation of the exchange graph introduced in the previous section. The vertex set $\mathfrak{V}$ of $\mathfrak{G}$ is partitioned into two sets $\mathfrak{V}_1$ and $\mathfrak{V}_2$ with

$$\mathfrak{V}_1 := \Big\{(F_1, F_2, T) \,|\, F_1, F_2 \in \mathcal{SF}_2, T \in \mathcal{ST}\Big\},$$

and

$$\mathfrak{V}_2 := \Big\{(T_1, T_2, F) \,|\, T_1, T_2 \in \mathcal{ST}, F \in \mathcal{SF}_2\Big\}$$

There is an edge between $(F_1, F_2, T) \in \mathfrak{V}_1$ and $(T_1, T_2, F) \in \mathfrak{V}_2$ in $\mathfrak{G}$ iff there is an edge $e \in E$ such that $T = F + e$, and $F_1 = T_1 - e$ and $F_2 = T_2 - e$.

Definition 3.4. If $(F_1, F_2, T) \in \mathfrak{V}_1$ and $(T_1, T_2, F) \in \mathfrak{V}_2$ are adjacent in $\mathfrak{G}$, we say $(T_1, T_2, F) \in \mathfrak{V}_1$ is obtained from (F_1, F_2, T) by *pivoting involving the edge* e (with $E(T) \setminus E(F) = \{e\}$).

Define two weight functions $\xi, \zeta : \mathfrak{V} \to C^0(U, \mathbb{R})$ on the vertices of $\mathfrak{G}$ as follows. For $(F_1, F_2, T) \in \mathfrak{V}_1$, let

$$\xi(F_1, F_2, T) := \det(Y + A)_{F_1^c, F_2^c}\; y^{T^c},$$
$$\zeta(F_1, F_2, T) := \det(N_{F_1^c}) \det(N_{F_2^c})\, \xi(F_1, F_2, T),$$

and for $(T_1, T_2, F) \in \mathfrak{V}_2$, define

$$\xi(T_1, T_2, F) := \det(Y + A)_{T_1^c, T_2^c}\; y^{F^c},$$
$$\zeta(T_1, T_2, F) := \det(M_{T_1^c}) \det(M_{T_2^c}) q(F)\, \xi(T_1, T_2, F).$$

Note that these weights are precisely the terms which appear in the products g_2f_1 and g_1f_2; we have

$$g_2f_1 = \sum_{(F_1,F_2,T)\in\mathfrak{V}_1} \zeta(F_1, F_2, T), \tag{3.7}$$

and

$$g_1f_2 = \sum_{(T_1,T_2,F)\in\mathfrak{V}_2} \zeta(T_1, T_2, F). \tag{3.8}$$

We have the following

Claim 3.5.
- *For any $(F_1, F_2, T) \in \mathfrak{V}_1$, we have*
$$\xi(F_1, F_2, T) = O_{\underline{y}}(y^{F_1^c \cap F_2^c} y^{T^c}).$$
- *For any $(T_1, T_2, F) \in \mathfrak{V}_2$, we have*
$$\xi(T_1, T_2, F) = O_{\underline{y}}(y^{T_1^c \cap T_2^c} y^{F^c}).$$
- *For two adjacent vertices $(F_1, F_2, T) \in \mathfrak{V}_1$ and $(T_1, T_2, F) \in \mathfrak{V}_2$, we have*
$$\xi(F_1, F_2, T) = \xi(T_1, T_2, F) + O_{\underline{y}}(f_1^2).$$

Proof. The first two assertions are straightforward. To prove the last one, let e be the unique edge in $T \setminus F$. We have

$$\det(Y + A)_{F_1^c, F_2^c} = y_e \det(Y + A)_{T_1^c, T_2^c} + O_{\underline{y}}(y^{T_1^c}),$$

Multiplying both sides by y^{T^c} gives

$$\xi(F_1, F_2, T) = \xi(T_1, T_2, F) + O_{\underline{y}}(y^{T^c} y^{T_1^c}) = O_{\underline{y}}(f_1^2).$$

□

We now define ordinary and special vertices of $\mathfrak{V}$. Roughly speaking, special vertices are those vertices whose contributions to g_2f_1 and g_1f_2 are small, so roughly speaking, they can be ignored in proving the theorem. The contribution is then made only by ordinary vertices, and this will be understood by the results we proved for the exchange graph in the previous section.

Definition 3.6.
- A triple $(F_1, F_2, T) \in \mathfrak{V}_1$ is called *special* if $F_1 \not\sim_v F_2$, i.e., if the partition of vertices induced by F_1 is different from the one induced by F_2. Otherwise, it is called *ordinary*.
- A triple $(T_1, T_2, F) \in \mathfrak{V}_2$ is called *special* if there exists either $e \in E(T_1) \setminus E(T_2)$ or $e \in E(T_2) \setminus E(T_1)$ such that $F + e$ is a spanning tree. Otherwise, it is called *ordinary*.

The following observations are crucial for the proof of our theorem. They show that connected components of $\mathfrak{G}$ which contain special vertices have only "light weight" vertices.

Claim 3.7. (1) *For any special vertex $\mathfrak{w}$ in $\mathfrak{V}$, we have*

$$\xi(\mathfrak{w}) = O_{\underline{y}}(f_1^2).$$

(2) *For any vertex $\mathfrak{v} \in \mathfrak{V}$ connected by a path in $\mathfrak{G}$ to a special vertex $\mathfrak{u}$, we have*

$$\xi(\mathfrak{v}) = O_{\underline{y}}(f_1^2).$$

Proof. (1) If $\mathfrak{w} = (F_1, F_2, T) \in \mathfrak{V}_1$, then since $F_1 \not\sim_v F_2$, there exists an edge $e \in F_2$ such that $T_1 = F_1 + e$ is a tree. In this case, since $e \notin F_2^c$, we have $F_1^c \cap F_2^c \subseteq T_1^c$, and so we have by Claim 3.5,

$$\xi(F_1, F_2, T) = O_{\underline{y}}(y^{F_1^c \cap F_2^c} y^{T^c}) = O_{\underline{y}}(y^{T_1^c} y^{T^c}) = O_{\underline{y}}(f_1^2).$$

Similarly, let $\mathfrak{w} = (T_1, T_2, F) \in \mathfrak{V}_2$ be special, and assume without loss of generality that there is an edge $e \in E(T_1) \setminus E(T_2)$ such that $T = F + e$ is a spanning tree. Since $e \notin T_2$, we have $\{e\} \cup (T_1^c \cap T_2^c) \subseteq T_2^c$, which shows that

$$y_e y^{T_1^c \cap T_2^c} = O_{\underline{y}}(y^{T_2^c}).$$

Observing that $y^{F^c} = y_e y^{T^c}$, and applying Claim 3.5, we get

$$\xi(T_1, T_2, F) = O_{\underline{y}}(y^{T_1^c \cap T_2^c} y^{F^c}) = O_{\underline{y}}(y^{T_1^c \cap T_2^c} y_e y^{T^c}) = O_{\underline{y}}(y^{T_2^c} y^{T^c}) = O_{\underline{y}}(f_1^2).$$

(2) This follows from (1) and the third assertion in Claim 3.5. □

Definition 3.8. Let $\mathbf{p} \in \mathbb{R}^{V,0}$ be the vector of external momenta. For any ordinary vertex $\mathfrak{u} = (T_1, T_2, F) \in \mathfrak{V}_2$, define $q(\mathfrak{u}) := q(F)$. For any ordinary vertex $\mathfrak{v} = (F_1, F_2, T) \in \mathfrak{V}_1$ (so with $F_1 \sim_v F_2$), define $q(\mathfrak{v}) := q(F_1) = q(F_2)$.

As corollary of the above claims, we get

Corollary 3.9.
- *Let $\mathcal{G}$ be a connected component of $\mathfrak{G}$. If $\mathcal{G}$ contains a special vertex, then for any vertex $\mathfrak{v} \in \mathfrak{V}(\mathcal{G})$, we have*

$$\zeta(\mathfrak{v}) = O_{\underline{y}}(f_1^2).$$

- *Let $\mathcal{G}$ be a connected component of $\mathfrak{G}$ entirely composed of ordinary vertices. There exists a real-valued function ρ defined on U such that for any vertex $\mathfrak{w}$ of $\mathcal{G}$, we have*

$$\zeta(\mathfrak{w}) = q(\mathfrak{w})\rho + O_{\underline{y}}(f_1^2).$$

Proof. The first assertion already follows from Claim 3.7.

We prove the second part. So let $\mathcal{G}$ be a component entirely composed of ordinary vertices. Note that $\mathcal{G}$ contains both vertices in $\mathfrak{V}_1$ and $\mathfrak{V}_2$. Let $\mathfrak{u}_0 = (T_{0,1}, T_{0,2}, F_0)$ be a vertex of $\mathcal{G}$, with $T_{0,1}$ and $T_{0,2}$ spanning trees and F_0 a spanning 2-forest of G. Define

$$\rho := \det(M_{T^c_{0,1}}) \det(M_{T^c_{0,2}})\xi(\mathfrak{u}_0).$$

Let now $\mathfrak{v} = (F_1, F_2, T) \in \mathfrak{V}_1$ and $\mathfrak{u} = (T_1, T_2, F) \in \mathfrak{V}_2$ be two vertices of G. Suppose that $\mathfrak{u}$ and $\mathfrak{v}$ are adjacent, and let e be the edge in E with $T = F + e$, $T_1 = F_1 + e$ and $T_2 = F_2 + e$. By assumption, we have $F_1 \sim_v F_2$. By Equation (3.3), we have

$$\det(N_{F^c_1}) \det(N_{F^c_2}) = \det(M_{T^c_1}) \det(M_{T^c_2})q(F_1).$$

Since this is true for all vector of momenta, connectivity of $\mathcal{G}$ implies that for all vertices $\mathfrak{u} = (T_1, T_2, F)$ of $\mathcal{G}$, we should have

$$\det(M_{T^c_1}) \det(M_{T^c_2}) = \det(M_{T^c_{0,1}}) \det(M_{T^c_{0,2}}).$$

On the other hand, we already noted that for any pair of adjacent vertices, we have

$$\xi(\mathfrak{v}) = \xi(\mathfrak{u}) + O_{\underline{y}}(f_1^2). \tag{3.9}$$

By connectivity of $\mathcal{G}$, this shows that for any vertex $\mathfrak{u}$ as above, we have

$$\xi(\mathfrak{u}) = \xi(\mathfrak{u}_0) + O_{\underline{y}}(f_1^2).$$

Multiplying both sides of this equation by $\det(M_{T^c_1}) \det(M_{T^c_2})$, gives

$$\det(M_{T^c_1}) \det(M_{T^c_2})\xi(\mathfrak{u}) = \rho + O_{\underline{y}}(f_1^2), \tag{3.10}$$

which finally implies that

$$\zeta(\mathfrak{u}) = q(\mathfrak{u})\rho + O_{\underline{y}}(f_1^2),$$

which proves the claim for all vertices of $\mathcal{G}$ in $\mathfrak{V}_2$.

To prove the result for $\mathfrak{v} \in \mathfrak{V}_1$, note that multiplying both sides of Equation 3.9 by $\det(N_{F^c_1}) \det(N_{F^c_2})$, and using Equation (3.3), $\det(N_{F^c_1}) \det(N_{F^c_2}) = \det(M_{T^c_1}) \det(M_{T^c_2})q(F_1)$, we infer that

$$\zeta(\mathfrak{v}) = \det(M_{T^c_1}) \det(M_{T^c_2})q(F_1)\xi(\mathfrak{u}) + O_{\underline{y}}(f_1^2).$$

By Equation 3.10, this becomes

$$\zeta(\mathfrak{v}) = q(\mathfrak{v})\rho + O_{\underline{y}}(f_1^2),$$

and the claim follows for all vertices $\mathfrak{v}$ of $\mathcal{G}$ which lie in $\mathfrak{V}_1$. □

The following proposition finally allows us to prove Theorem 1.1.

Proposition 3.10. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected component of $\mathfrak{G}$ with vertex set $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$, with $\mathcal{V}_i = \mathcal{V} \cap \mathfrak{V}_i$. Suppose that $\mathcal{G}$ is entirely composed of ordinary vertices. Then we have*

$$\sum_{\mathfrak{u} \in \mathcal{V}_1} q(\mathfrak{u}) = \sum_{\mathfrak{w} \in \mathcal{V}_2} q(\mathfrak{w}).$$

We will give the proof of this proposition in the next section. Let us first explain how to deduce Theorem 1.1 assuming this result.

Proof of Theorem 1.1. We have to show that $g_2f_1-g_1f_2 = O_{\underline{y}}(f_1^2)$. Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1), \dots, \mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N)$ be all the connected components of $\mathfrak{G}$. For each $i = 1, \dots, N$, denote by $\mathcal{V}_{i,1}$ $\mathcal{V}_{i,2}$ the intersection of $\mathcal{V}_i$ with $\mathfrak{V}_1$ and $\mathfrak{V}_2$ respectively. Using Equations (3.7) and (3.8), we can write

$$
\begin{aligned}
g_2f_1 - g_1f_2 &= \sum_{\mathfrak{v}\in\mathfrak{V}_1}\zeta(\mathfrak{v}) - \sum_{\mathfrak{u}\in\mathfrak{V}_2}\zeta(\mathfrak{u})\\
&= \sum_{i=1}^{N}\Big(\sum_{\mathfrak{v}\in\mathcal{V}_{i,1}}\zeta(\mathfrak{v}) - \sum_{\mathfrak{u}\in\mathcal{V}_{i,2}}\zeta(\mathfrak{u})\Big).
\end{aligned}
$$

For each $1 \le i \le N$, we have the following two possibilities. Either, $\mathcal{G}_i$ contains a special vertex, in which case we have $\zeta(\mathfrak{w}) = O_{\underline{y}}(f_1^2)$ for all $\mathfrak{w} \in \mathcal{V}(\mathcal{G}_i)$. In particular,

$$
\sum_{\mathfrak{v}\in\mathcal{V}_{i,1}}\zeta(\mathfrak{v}) - \sum_{\mathfrak{u}\in\mathcal{V}_{i,2}}\zeta(\mathfrak{u}) = O_{\underline{y}}(f_1^2).
$$

Or, $\mathcal{G}_i$ contains only ordinary vertices. In this case, by Corollary 3.9, there exists a real-valued function ρ_i such that $\zeta(\mathfrak{v}) = q(\mathfrak{v})\rho_i + O_{\underline{y}}(f_1^2)$ for all vertices $\mathfrak{v}$ of $\mathcal{G}_i$. We must then have

$$
\begin{aligned}
\sum_{\mathfrak{v}\in\mathcal{V}_{i,1}}\zeta(\mathfrak{v}) - \sum_{\mathfrak{u}\in\mathcal{V}_{i,2}}\zeta(\mathfrak{u}) &= \rho_i\Big(\sum_{\mathfrak{v}\in\mathcal{V}_{i,1}}q(\mathfrak{v}) - \sum_{\mathfrak{u}\in\mathcal{V}_{i,2}}q(\mathfrak{u})\Big) + O_{\underline{y}}(f_1^2)\\
&= O_{\underline{y}}(f_1^2) \qquad \text{(by Proposition 3.10).}
\end{aligned}
$$

Thus, $g_2f_1 - g_1f_2 = O_{\underline{y}}(f_1^2)$ and the theorem follows. □

3.1. **Proof of Proposition 3.10.** Recall that for a partition $\mathcal{P}$ of V into sets $X_1, \dots, X_k$, we denote by $E(\mathcal{P})$ the set of all edges in G with end-points lying in two different sets among X_is. For a spanning 2-forest F, the partition of V into the vertex sets of the two connected components of F is as before denoted by $\mathcal{P}(F)$.

Let $\mathcal{G}$ be a connected component of $\mathfrak{G}$ which is entirely composed of ordinary vertices. Let $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ be the vertex set of $\mathcal{G}$ with $\mathcal{V}_i \subset \mathfrak{V}_i$, for $i = 1, 2$. We will give a complete description of the structure of $\mathcal{G}$ using the structure theorem we proved for the exchange graph, which in particular allows to prove Proposition 3.10.

Define equivalence relations $\equiv_1, \equiv_2, \equiv_3$ on the set of vertices V of G as follows. For two vertices $u, v \in V$,

- we say $u \equiv_1 v$ if for any $(T_1, T_2, F) \in \mathcal{V}_2$, both vertices u and v lie in the same connected component of $T_1 \setminus E(\mathcal{P}(F))$.

Similarly,

- we say $u \equiv_2 v$ if for any $(T_1, T_2, F) \in \mathcal{V}_2$, both vertices u and v lie in the same connected component of $T_2 \setminus E(\mathcal{P}(F))$.

And finally,

- we say $u \equiv_3 v$ if for any $(F_1, F_2, T) \in \mathcal{V}_1$, both vertices u and v lie in the same connected component of $T \setminus E(\mathcal{P}(F_1))$.

Note that since $\mathcal{G}$ does not contain any special vertex, we have $F_1 \sim_v F_2$ for all $(F_1, F_2, T) \in \mathcal{V}_1$. In particular, $T \setminus E(\mathcal{P}(F_1)) = T \setminus E(\mathcal{P}(F_2))$.

The following statements are analogous to the statements of Lemma 2.9 and Proposition 2.10 for the exchange graph.

Lemma 3.11. *Let F be a spanning 2-forest in G. Let T be a spanning tree of G. Suppose two vertices $u, v \in V$ are in two different connected components of $T \setminus E(\mathcal{P}(F))$. There exists and edge $e \in E(\mathcal{P}(F)) \cap E(T_1)$ such that u and v are not connected in $T - e$.*

Proof. Denote by S_u and S_v the two connected components of $T \setminus E(\mathcal{P}(F))$ which contain u and v, respectively. There is a path joining S_u to S_v in T. Since $S_u \neq S_v$, it contains an edge $e \in E(\mathcal{P}(F))$. For such an edge e, u and v are not connected in $T - e$. □

The previous lemma allows to prove the following claim.

Claim 3.12. *The three equivalence relations $\equiv_1$, $\equiv_2$, $\equiv_3$ are the same.*

Proof. To prove that $\equiv_1$ and $\equiv_2$ are the same, suppose for the sake of a contradiction that $u \equiv_1 v$ but $u \not\equiv_2 v$ for two vertices u and v in V. This implies the existence of $(T_1, T_2, F) \in \mathcal{V}_2$ such that

- the two vertices u and v are both in X with $\mathcal{P}(F) = \{X, X^c\}$.
- u and v are in the same connected component of $T_1[X]$, and they are in two different connected components of $T_2[X]$.

Applying the previous lemma, there exists an edge $e \in E(T_2) \cap E(\mathcal{P}(F))$ such that u and v lie in two different connected components of $F_2 = T_2 - e$. Since (T_1, T_2, F) is not special, and $e \in E(\mathcal{P}(F))$, we have $e \in T_1$. In particular, u, v are in the same connected component of $F_1 = T_1 - e$. We have proved that $\mathcal{P}(F_1) \neq \mathcal{P}(F_2)$, i.e., the triple (F_1, F_2, T) obtained from (T_1, T_2, F) by pivoting involving e is special. This contradicts the assumption on $\mathcal{G}$ (that it does not contain special vertices), and proves our claim.

We now prove that $\equiv_1$ and $\equiv_3$ are similar. Suppose for the sake of a contradiction that this is not the case. Let $u, v \in V$ be two vertices with $u \equiv_3 v$ but $u \not\equiv_1 v$ (the other case $u \not\equiv_3 v$ but $u \equiv_1 v$ has a similar treatment that we omit). This implies the existence of $(T_1, T_2, F) \in \mathcal{V}_2$ such that u, v belong to two different connected components of $T_1 \setminus E(\mathcal{P}(F))$. Applying the previous lemma, we infer the existence of an edge $e \in E(T_1) \cap E(\mathcal{P}(F))$ such that u and v are not connected in $T_1 - e$. Pivoting involving e gives a triple (F_1, F_2, T) such that u and v lie in two different connected components of F_1. In particular, it follows that $u \not\equiv_3 v$, which is a contradiction. This proves the claim. □

We denote by $\equiv$ the equivalence relation on vertices induced by $\equiv_i$. As in Proposition 2.11, we have the following remark.

Remark 3.13. Note that if u and v are two vertices with $u \not\equiv v$, there exists $(F_1, F_2, T) \in \mathcal{V}_1$ such that u and v lie in different connected components of F_i. Similarly, there exists $(T_1, T_2, F) \in \mathcal{V}_2$ such that u, v lie in different connected components of F.

Denote by $\mathcal{P}_\equiv = \{X_1, \dots, X_k\}$ the partition of V induced by the equivalence classes X_i of $\equiv$. Note that pivoting only involves edges in $E(\mathcal{P}_\equiv)$, i.e, those edges which are not contained in any of the sets $X_1, \dots, X_k$. By connectivity of $\mathcal{G}$, it follows that for each i, there are three trees $\tau_{i,1}, \tau_{i,2}, \tau_{i,3}$ on the vertex set X_i such that for any $(F_1, F_2, T) \in \mathcal{V}_1$ and any $(T_1, T_2, F) \in \mathcal{V}_2$, we have

$$T_1[X_i] = F_1[X_i] = \tau_{i,1}, \ T_2[X_i] = F_2[X_i] = \tau_{i,2}, \ T[X_i] = F[X_i] = \tau_{i,3}.$$

In other words, the subtrees $\tau_{i,1}, \tau_{i,2}, \tau_{i,3}$ are the "constant" part of the elements in $\mathcal{G}$.

(To see that $T_1[X_i]$ is a tree, consider two vertices u, v of X_i, and let P be the unique path in T_1 which connectes u to v. By Remark 3.13, all the vertices of P are in the same equivalence class X_i, i.e., $T_1[X_i]$ is connected, and so it is a tree. The other cases follow by a similar argument.)

We now prove

Claim 3.14. *For any* $(T_1, T_2, F) \in \mathcal{V}_2$*, we have*

$$T_1 \setminus \Big(\bigcup_{i=1}^{k} E(\tau_{i,1})\Big) = T_2 \setminus \Big(\bigcup_{i=1}^{k} E(\tau_{i,2})\Big).$$

In other words, the edges of T_1 *and* T_2 *which lie outside all* X_i*s are the same.*

Proof. Let $e = \{u, v\}$ be an edge of T_1 with u and v lying in two different equivalence classes X_i and X_j. By Remark 3.13, there exists $(T_1', T_2', F') \in \mathcal{V}_1$ such that u and v belong to two different sets of the partition $\mathcal{P}(F)$. By connectivity of $\mathcal{G}$, the edges in $E(T_1) \cup E(F)$ are the same as those in $E(T_1') \cup E(F')$. Since $e \notin E(F')$ and $e \in E(T_1)$, we must have $e \in E(T_1')$. Since (T_1', T_2', F') is ordinary, we infer $e \in E(T_2')$. By connectivity of $\mathcal{G}$, and the way the edges are defined (which requires pivoting involving the same edge for the two trees in any vertex of $\mathcal{V}_2$), we must have $e \in E(T_2)$, and the claim follows. □

Let $\mathfrak{v} = (T_1, T_2, F) \in \mathcal{V}_2$. Let

$$E_{1,2}(\mathfrak{v}) := E(T_1) \cap E(\mathcal{P}_{\equiv}) = E(T_2) \cap E(\mathcal{P}_{\equiv}), \qquad \text{and}$$
$$E_3(\mathfrak{v}) := E(F) \cap E(\mathcal{P}_{\equiv}).$$

(Note that the equality of the two sets in the definition of $E_{1,2}()$ follows from Claim 3.14.)

Obviously, we have

$$E(T_1) = E_{1,2}(\mathfrak{v}) \sqcup \bigsqcup_{i=1}^{k} E(\tau_{i,1}) \ , \ E(T_2) = E_{1,2}(\mathfrak{v}) \sqcup \bigsqcup_{i=1}^{k} E(\tau_{i,2}) \ , \text{ and}$$
$$E(F) = E_3(\mathfrak{v}) \cup \bigcup_{i=1}^{k} E(\tau_{i,1}).$$

Define the multiset

$$E_{\mathcal{G}} := E_{1,2}(\mathfrak{v}) \sqcup E_3(\mathfrak{v}).$$

By the definition of the edges in the graph $\mathfrak{G}$, and connectivity of (the connected component) $\mathcal{G}$, $E_{\mathcal{G}}$ is independent of the choice of $\mathfrak{v} \in \mathcal{V}_2$. In addition, if for $\mathfrak{u} \in \mathcal{V}_1$, we define $E_{1,2}(\mathfrak{u}) = E(F_1) \cap E(\mathcal{P}_{\equiv})$, and $E_3(\mathfrak{u}) = E(T) \cap E(\mathcal{P}_{\equiv})$, we should have $E_{\mathcal{G}} = E_{1,2}(\mathfrak{u}) \sqcup E_3(\mathfrak{u})$.

Define an (auxiliary) multigraph $G_0 = (V_0, E_0)$ obtained by contracting each equivalence class X_i to a vertex x_i and having the multiset of edges $E_0 = E_{\mathcal{G}}$. More precisely, G_0 has the vertex set $V_0 = \{x_1, \dots, x_k\}$, and an edge $\{x_i, x_j\}$ for any edge $e = \{u, v\}$ in the multiset $E_{\mathcal{G}}$ which joins a vertex $u \in X_i$ to a vertex $v \in X_j$. By an abuse of the notation, we identify E_0 with $E_{\mathcal{G}}$.

Each $\mathfrak{v} = (T_1, T_2, F) \in \mathcal{V}_2$ gives a pair $(T_{\mathfrak{v}}, F_{\mathfrak{v}})$ that we denote by $\pi(\mathfrak{v})$ consisting of a spanning tree $T_{\mathfrak{v}}$ of G_0 with edges $E_{1,2}(\mathfrak{v})$ and a spanning 2-forest $F_{\mathfrak{v}}$ of G_0 with edge set $E_3(\mathfrak{v})$. As a multiset, we have $E_0 = E(T_{\mathfrak{v}}) \sqcup E(F_{\mathfrak{v}})$. Similarly, each $\mathfrak{u} = (F_1, F_2, T) \in \mathcal{V}_1$ gives a pair $\pi(\mathfrak{u}) = (F_{\mathfrak{v}}, T_{\mathfrak{v}})$ consisting of a spanning 2-forest $F_{\mathfrak{v}}$ and a spanning tree $T_{\mathfrak{v}}$ of G_0 with edge sets $E_{1,2}(\mathfrak{u})$ and $E_3(\mathfrak{u})$, respectively.

We will describe $\mathcal{G}$ in terms of the multigraph G_0. Let $\mathscr{H}_0 = (\mathscr{V}_0, \mathscr{E}_0)$ be the exchange graph associated to the multigraph G_0 as in Section 2. Recall that the vertex set $\mathscr{V}_0$ of $\mathscr{H}_0$ is the disjoint union of two sets $\mathscr{V}_{0,1}$ and $\mathscr{V}_{0,2}$, where

$$\mathscr{V}_{0,1} := \Big\{ (F,T) \,\Big|\, F \in \mathcal{SF}_2(G_0), T \in \mathcal{ST}(G_0),\ E(F) \sqcup E(T) = E_0 \Big\},$$

and

$$\mathscr{V}_{0,2} := \Big\{ (T,F) \,\Big|\, T \in \mathcal{ST}(G_0), F \in \mathcal{SF}_2(G_0),\ E(F) \sqcup E(T) = E_0 \Big\}.$$

There is an edge in $\mathscr{E}_0$ connecting $(F,T) \in \mathscr{V}_{0,1}$ to $(T',F') \in \mathscr{V}_{0,2}$ if (T',F') *is obtained from* (F,T) *by pivoting involving an edge* $e \in E_0$, i.e., if $F = T' - e$ and $F' = T - e$.

With this notation, we get an application $\pi : \mathcal{V} \to \mathscr{V}_0$. By what we have proved so far, it is clear that π is injective. By the definition of edges in $\mathfrak{G}$ and $\mathscr{H}_0$, π induces a homomorphism of graphs $\pi : \mathcal{G} \to \mathscr{H}_0$. In addition, any pivoting in $\mathscr{H}_0$ involving an edge $e \in E_0 = E_{\mathcal{G}}$ can be lifted to pivoting involving the same edge e in $\mathcal{G}$. This proves that π induces an isomorphism onto (its image) a connected component of $\mathscr{H}_0$.

Proposition 3.15. *The exchange graph* $\mathscr{H}_0$ *is connected. As a consequence, the projection map* π *is an isomorphism.*

Proof. By the discussion preceding the proposition, we only need to show that $\mathscr{H}_0$ is connected. Since the multigraph G_0 is a disjoint union of a spanning tree and a spanning forest, we will get this latter statement from the first part of Theorem 2.13 by observing that the only saturated non-empty subsets of vertices of G_0 are singletons.

To see this, let S be a saturated component of G_0. Note that $G_0[S]$ is connected, and no pivoting in G_0 involves the edge set of S in G_0. By the injectivity of the projection map $\pi : \mathcal{G} \to \mathscr{H}_0$, and the observation we made that pivoting involving an edge e in $\mathcal{G}$ corresponds to pivoting involving the same edge e in $\mathscr{H}_0$, it follows that no edge of S is involved in pivoting in $\mathcal{G}$.

For the sake of a contradiction, suppose that S has size at least two, and let x_i and x_j be two different vertices of S which are connected by an edge e. The edge e connected two vertices u_i and u_j in G, such that $u_i \in X_i$ and $u_j \in X_j$. For a triple $\mathfrak{v} = (T_1, T_2, F) \in \mathcal{V}_2$, the edge e belongs either to both T_1 and T_2, or it belongs to F. In either case, since no pivoting involves e, by connectivity of $\mathcal{G}$ and Remark 3.13, it follows that $u_i \equiv u_j$ for the equivalence defined by $\mathcal{G}$. This implies that any vertex in X_i is equivalent to any vertex in X_j. This is impossible since X_i and X_j are two different equivalence classes in $\mathcal{P}$. This final contradiction implies that $|S| = 1$ and the proposition follows. □

We can now prove Proposition 3.10.

Proof of Proposition 3.10. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected component of $\mathfrak{G}$ which consists entirely of ordinary vertices. Let G_0 be the multigraph we associated to $\mathcal{G}$, and $\pi : \mathcal{G} \to \mathscr{H}_0$ be the isomorphism constructed above.

For $(F,T) \in \mathscr{V}_{0,1}$, we have $(T,F) \in \mathscr{V}_{0,2}$, and by definition, we have

$$q(\pi^{-1}(F,T)) = q(\pi^{-1}(T,F)).$$

Since π is an isomorphism, it follows that

$$\sum_{\mathfrak{u} \in \mathcal{V}_2} q(\mathfrak{u}) = \sum_{(T,F) \in \mathscr{V}_{0,2}} q(\pi^{-1}(T,F)) \;=\; \sum_{(F,T) \in \mathscr{V}_{0,1}} q(\pi^{-1}(F,T)) = \sum_{\mathfrak{u} \in \mathcal{V}_1} q(\mathfrak{u}),$$

and the proposition follows. □

The proof of Theorem 1.1 is now complete.

4. Proof of Theorem 1.2

In this section we explain how to derive Theorem 1.2 from Theorem 1.1. The presentation here is heavily based on the results and notations of [1], to which we refer for the missing details.

First we recall the set-up. Let Δ be a small open disc around the origin in $\mathbb{C}$, and denote by $\Delta^* = \Delta \setminus \{0\}$ the punctured disk. Let $S = \Delta^{3g-3}$. Let C_0 be a stable curve of arithmetic genus g, and let $G = (V, E)$ be the dual graph of C_0. Denote by $h \leq g$ the genus of G, so we have $h = |E| - |V| + 1$. The versal analytic deformation of C_0 over S is denoted by $\pi : \mathcal{C} \to S$. The fibres of π are smooth outside a normal crossing divisor $D = \bigcup_{e \in E} D_e \subset S$, which has irreducible components indexed by the set of edges of G (which are in bijection with the singular points of C_0). Let U be the complement of the divisor D in S, that we identify with $U = (\Delta^*)^E \times \Delta^{3g-3-|E|}$. Let

$$\widetilde{U} := \mathbb{H}^E \times \Delta^{3g-3-|E|} \longrightarrow U. \tag{4.1}$$

be the universal cover of U. The projection map $\widetilde{U} \to U$ is given by $z_e \mapsto \exp(2\pi i z_e)$ in the first factors corresponding to the edges of G, and is the identity on the remaining factors.

Suppose that we have two collections

$$\sigma_1 = \{\sigma_{l,1}\}_{l=1,\dots,n}, \quad \sigma_2 = \{\sigma_{l,2}\}_{l=1,\dots,n}$$

of sections $\sigma_{l,i} \colon S \to \mathcal{C}$ of π, for $1 \leq l \leq n$ and $i = 1, 2$. By regularity of $\mathcal{C}$, these sections cannot pass through double points of C_0, and for each l, $\sigma_{l,i}(S) \cap C_0$ lies in a unique irreducible component X_{v_l} of C_0, which corresponds to a vertex v_l of the dual graph G. We assume that the sections $\sigma_{l,1}$ and $\sigma_{l,2}$ are distinct on C_0, which implies, after shrinking S if necessary, that σ_1 and σ_2 are disjoint as well.

Let $\underline{\mathbf{p}}_1 = \{\mathbf{p}_{l,1}\}_{l=1}^n \in (\mathbb{R}^D)^{n,0}$ and $\underline{\mathbf{p}}_2 = \{\mathbf{p}_{l,2}\}_{l=1}^n \in (\mathbb{R}^D)^{n,0}$ be two collections of external momenta satisfying the conservation law (1.1). Using the labelings of sections and the external momenta, we associate each marked point $\sigma_{l,i}$ with $\mathbf{p}_{l,i} \in \mathbb{R}^D$, and denote by $\underline{\mathbf{p}}_1^G = (\mathbf{p}_{v,1}^G)$ and $\mathbf{p}_2^G = (\mathbf{p}_{v,2}^G)$ the restriction of $\underline{\mathbf{p}}_1$ and $\underline{\mathbf{p}}_2$ to the graph G: for each vertex v of G, the vector $\mathbf{p}_{v,i}^G$ is the sum of all the momenta $\mathbf{p}_{l,i}$ with $v_l = v$. In this way, at any point $s \in S$, we get two $\mathbb{R}^D$-valued degree zero divisors on the curve C_s that we denote by $\mathfrak{A}_s$ and $\mathfrak{B}_s$: they are defined by

$$\mathfrak{A}_s := \sum_{l=1}^n \mathbf{p}_{l,1}\sigma_{l,1}(s), \qquad \mathfrak{B}_s := \sum_{l=1}^n \mathbf{p}_{l,2}\sigma_{l,2}(s).$$

This gives us the real valued function on U which sends the point s of U to $\langle \mathfrak{A}_s, \mathfrak{B}_s \rangle$, where $\langle . , . \rangle$ denotes the archimedean height pairing between $\mathbb{R}^D$-valued degree zero divisors, see the introduction and [1] for the definition of the height pairing and the extension to $\mathbb{R}^D$-valued divisors defined by means of the given Minkowski bilinear form.

We are interested in understanding the behaviour of the function $s \mapsto \langle \mathfrak{A}_s, \mathfrak{B}_s \rangle$ close to the origin $0 \in S \setminus U$. This can be carried out using the nilpotent orbit theorem in Hodge theory, c.f. [1]. We can reduce to the case where the external momenta are all integers, and in this case, the divisors $\mathfrak{A}_s$ and $\mathfrak{B}_s$ having integer coefficients at any point s, the Archimedean height pairing between $\mathfrak{A}_s$ and $\mathfrak{B}_s$ can be described in terms of a biextension mixed Hodge structure, c.f. [5, 1]. Denoting by $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ the biextension mixed Hodge structure associated to the pair $\mathfrak{A}_s$ and $\mathfrak{B}_s$, the family $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ fit together into an admissible variation of mixed Hodge structures. An explicit description of the period map for the variation of the biextension mixed Hodge structures $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ was obtained in [1]. We briefly recall this now.

Fix base points $s_0 \in U$ and $\tilde{s}_0 \in \widetilde{U}$ lying above s_0, and choose a symplectic basis

$$a_1, \dots, a_g, b_1, \dots, b_g \in H_1(C_{s_0}, \mathbb{Z}) = A_0 \oplus B_0.$$

Shrinking S if necessary, the inclusion $C_{s_0} \hookrightarrow \mathcal{C}$ gives a surjective specialisation map

$$\mathrm{sp}\colon H_1(C_{s_0}, \mathbb{Z}) \to H_1(\mathcal{C}, \mathbb{Z}) \simeq H_1(C_0, \mathbb{Z}).$$

Denote by $A \subset H_1(C_{s_0}, \mathbb{Z})$ the subspace spanned by the vanishing cycles a_e, one for each $e \in E$. We have the exact sequence

$$0 \to A \to H_1(C_{s_0}, \mathbb{Z}) \xrightarrow{\mathrm{sp}} H_1(C_0, \mathbb{Z}) \to 0,$$

and we define $A' = A + \mathrm{sp}^{-1}(\bigoplus_{v \in V} H_1(X_v, \mathbb{Z})) \subseteq H_1(C_{s_0}, \mathbb{Z})$. We have

$$H_1(C_{s_0}, \mathbb{Z})/A' \simeq H_1(G, \mathbb{Z}). \tag{4.2}$$

Changing the symplectic basis if necessary, we suppose that the space of vanishing cycles A is generated by $a_1, \dots, a_h \in A$, and that $b_1, \dots, b_h$ generate $H_1(C_{s_0}, \mathbb{Z})/A' \simeq H_1(G, \mathbb{Z})$ as in (4.2).

For $i = 1, 2$, let $\Sigma_{i,s} = \{\sigma_{1,i}(s), \dots, \sigma_{n,i}(s)\}$, and set $\Sigma_s = \Sigma_{1,s} \cup \Sigma_{2,s}$ and $\Sigma_i = \bigcup_s \Sigma_{i,s}$. By choosing loops that do not meet the points in Σ_{s_0}, we lift the classes a_j and b_j, $j = 1, \dots, g$ to elements of $H_1(C_{s_0} \setminus \Sigma_{s_0}, \mathbb{Z})$. By an abuse of the notation, we denote by a_j and b_j these new classes as well. This symplectic basis can be spread out to a basis

$$a_{1,\tilde{s}}, \dots, a_{g,\tilde{s}}, b_{1,\tilde{s}}, \dots, b_{g,\tilde{s}}$$

of $H_1(C_{\tilde{s}} \setminus \Sigma_{\tilde{s}}, \mathbb{Z})$, for any $s \in U$ and any $\tilde{s} \in \widetilde{U}$ over s. The elements $a_{i,\tilde{s}}$ only depend on s and not on $\tilde{s}$; we will also denote them by $a_{i,s}$. If there is no risk of confusion, we drop $\tilde{s}$, and use simply a_i and b_i.

In addition, we have a collection of 1-forms $\{\omega_i\}_{i=1,\dots,g}$ on $\pi^{-1}(U) \subset \mathcal{C}$ such that the forms $\{\omega_{i,s} \coloneqq \omega_i|_{C_s}\}_{i=1,\dots,g}$, for each $s \in U$, are a basis of the holomorphic differentials on C_s and

$$\int_{a_{i,s}} \omega_{j,s} = \delta_{i,j}. \tag{4.3}$$

The period matrix for the curve C_s is given by $\left(\int_{b_{i,s}} \omega_{j,s} \right)$.

Choose now an integer valued 1-chain $\gamma_{\mathfrak{B}_{s_0}}$ on $C_{s_0} \setminus \Sigma_{1,s_0}$ with $\mathfrak{B}_{s_0}$ as boundary. Adding a linear combination of the b_j if necessary, we further assume that

$$\langle a_i, \gamma_{\mathfrak{B}_{s_0}} \rangle = 0. \tag{4.4}$$

We spread the class

$$[\gamma_{\mathfrak{B}_{s_0}}] \in H_1(C_{s_0} \setminus \Sigma_{1,s_0}, \Sigma_{2,s_0}, \mathbb{Z})$$

of $\gamma_{\mathfrak{B}_{s_0}}$ to classes $\gamma_{\mathfrak{B}_{\tilde{s}}}$.

Similarly, we obtain a 1-form $\omega_{\mathfrak{A}}$ on $\pi^{-1}(U) \setminus \Sigma_1$ such that each restriction $\omega_{\mathfrak{A},s} := \omega_{\mathfrak{A}}|_{C_s}$ is a holomorphic form of the third kind with residue $\mathfrak{A}_s$. Adding to $\omega_{\mathfrak{A}}$ a linear combination of the ω_i if needed, we can suppose that $\omega_{\mathfrak{A}}$ is normalised so that

$$\int_{a_i,s} \omega_{\mathfrak{A},s} = 0, \quad i = 1, \dots, g. \tag{4.5}$$

Denote by $\mathrm{Row}_g(\mathbb{C}) \simeq \mathbb{C}^g$ and $\mathrm{Col}_g(\mathbb{C}) \simeq \mathbb{C}^g$ the g-dimensional vector space of row and column matrices, and let

$$\widetilde{X} := \mathbb{H}_g \times \mathrm{Row}_g(\mathbb{C}) \times \mathrm{Col}_g(\mathbb{C}) \times \mathbb{C}.$$

We have the following description of the period map from [1].

Proposition 4.1 ([1])**.** *The period map of the variation of mixed Hodge structures $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ is given by*

$$\begin{aligned} \widetilde{\Phi} \colon \widetilde{U} &\longrightarrow \widetilde{X} \\ \tilde{s} &\longmapsto \Big(\big(\int_{b_{i,\tilde{s}}} \omega_{j,s} \big)_{i,j}, \big(\int_{\gamma_{\mathfrak{B},\tilde{s}}} \omega_{j,s} \big)_j, \big(\int_{b_{i,\tilde{s}}} \omega_{\mathfrak{A},s} \big)_i, \int_{\gamma_{\mathfrak{B},\tilde{s}}} \omega_{\mathfrak{A},s} \Big). \end{aligned}$$

We now explain the action of the logarithm of monodromy map N_e, for $e \in E$, c.f. [1].

As before, each vanishing cycle $a_e \in H_1(C_{s_0}, \mathbb{Z})$ for $e \in E$ can be lifted in a canonical way to a cycle a_e in $H_1(C_{s_0} \setminus \Sigma_{s_0}, \mathbb{Z})$.

In this homology group, we can write

$$a_e = \sum_i c_{e,i} a_i + \sum_l d_{e,l,1} \gamma_{l,1} + \sum_l d_{e,l,2} \gamma_{l,2}, \tag{4.6}$$

with $\gamma_{l,i}$ denoting a small enough negatively oriented loop around the point $\sigma_{l,i}(s_0)$. Note that the coefficients $c_{e,i}$ are zero for $i > h$ (by the choice of the symplectic basis $\{a_i, b_i\}$).

By Picard-Lefschetz formula, we deduce from (4.4) and (4.6) that

$$N_e(b_i) = -\langle b_i, a_e \rangle a_e = c_{e,i} a_e, \tag{4.7}$$

$$N_e(\gamma_{\mathfrak{B}_{s_0}}) = -\langle \gamma_{\mathfrak{B}_{s_0}}, a_e \rangle a_e = -a_e \sum_l \mathbf{p}_{l,2} d_{e,l,2}. \tag{4.8}$$

Using (4.6), (4.5) and (4.3), we can compute the integral of the forms ω_j and $\omega_{\mathfrak{A}}$ with respect to the vanishing cycles, giving

$$\int_{a_e} \omega_j = c_{e,j}, \quad \int_{a_e} \omega_{\mathfrak{A}_{s_0}} = \sum_l \mathbf{p}_{l,1} d_{e,l,1}. \tag{4.9}$$

From (4.7), (4.8) and (4.9), we get

$$N_e(\int_{b_i}\omega_{j,s_0}) = -\langle b_i, a_e\rangle\int_{a_e}\omega_{j,s} = c_{e,i}c_{e,j}\,;$$
$$N_e(\int_{b_i}\omega_{\mathfrak{A}_{s_0}}) = -\langle b_i, a_e\rangle\int_{a_e}\omega_{\mathfrak{A}_{s_0}} = c_{e,i}\sum_l \mathbf{p}_{l,1}d_{e,l,1}\,;$$
$$N_e(\int_{\gamma_{\mathfrak{B}_{s_0}}}\omega_{j,s_0}) = -\langle \gamma_{\mathfrak{B}_{s_0}}, a_e\rangle\int_{a_e}\omega_{j,s} = -c_{e,j}\sum_l \mathbf{p}_{l,2}d_{e,l,2}\,;$$
$$N_e(\int_{\gamma_{\mathfrak{B}_{s_0}}}\omega_{\mathfrak{A}_{s_0}}) = -\langle \gamma_{\mathfrak{B}_{s_0}}, a_e\rangle\int_{a_e}\omega_{\mathfrak{A}_{s_0}} = -\Big(\sum_l \mathbf{p}_{l,1}d_{e,l,1}\Big)\Big(\sum_k \mathbf{p}_{k,2}d_{e,k,2}\Big).$$

For each $e\in E$, the logarithm of the monodromy N_e is given by

$$N_e = \begin{pmatrix} 0 & 0 & \underline{\mathbf{p}}_2\widetilde{W}_e & \underline{\mathbf{p}}_2\Gamma_e{}^t\underline{\mathbf{p}}_1\\ 0 & 0 & \widetilde{M}_e & \widetilde{Z}_e{}^t\underline{\mathbf{p}}_1\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix},$$

where the matrices $\widetilde{M}_e$, $\widetilde{W}_e$, $\widetilde{Z}_e$, and Γ_e are given by

$$(\widetilde{M}_e)_{i,j} = c_{e,i}c_{e,j}, \qquad (\widetilde{W}_e)_{l,j} = -c_{e,j}d_{e,l,2}, \qquad (\widetilde{Z}_e)_{i,l} = c_{e,i}d_{e,l,1}, \qquad (\Gamma_{k,l}) = -d_{e,k,2}d_{e,l,1}.$$

One verifies that the matrix $\widetilde{M}_e$ is the $h\times h$ matrix M_e filled with zeros to a $g\times g$ matrix, where M_e is the matrix of the symmetric bilinear form $\langle . \rangle_e$ in the basis $b_1,\dots,b_h$ of $H_1(G,\mathbb{Z})$. Similarly, one sees that the matrix $\widetilde{W}_e$ (resp. $\widetilde{Z}_e$) is obtained from a matrix W_e (resp. Z_e) that has only h columns (resp. rows) by extension with zeros. The entries of these matrices are given as follows. The choice of the path $\gamma_{\mathfrak{B}}$ provides a preimage ω_2 for the vector $\mathbf{p}_2^G$ in $\mathbb{Z}^E$, obtained by counting the number of times with sign that $\gamma_{\mathfrak{B}}$ crosses the vanishing cycle a_e. Similarly, $\omega_{\mathfrak{A}}$ gives a preimage ω_1 for $\mathbf{p}_1^G$ in $\mathbb{C}^E$ whose e-th component, for $e\in E(G)$, is given by $\int_{a_e}\omega_{\mathfrak{A}}$.

With these preliminaries, we can now state the expression of the height pairing in terms of the period map. Let us separate the variables which correspond to the edges of the graph G as s_E. Any point s of U then can be written as $s = s_E\times s_{E^c}$, where s_{E^c} denotes all the other $3g-3-|E|$ coordinates. Denoting the coordinates in the universal cover $\widetilde{U}$ by z_e, the projection $\widetilde{U}\to U$ is given in these coordinates by

$$s_e = \begin{cases}\exp(2\pi i z_e), & \text{for } e\in E,\\ z_e, & \text{for } e\notin E.\end{cases}$$

The following expression for the height pairing is obtained in [1].

Proposition 4.2 ([1])**.** *There exists $h_0>0$ and a holomorphic map $\Psi_0 : U\to\widetilde{X}$,*

$$\Psi_0(s) = (\Omega_0(s), W_0(s), Z_0(s), \rho_0(s)),$$

such that introducing

$$y_e = \mathrm{Im}(z_e) = \frac{-1}{2\pi}\log|s_e|,$$

the height pairing is given by

$$(4.10)\quad \langle \mathfrak{A}_s, \mathfrak{B}_s\rangle = -2\pi\,\mathrm{Im}(\rho_0) - \sum_{e\in E} 2\pi y'_e \underline{\mathbf{p}}_2 \Gamma_e\, {}^t\underline{\mathbf{p}}_1 + 2\pi\Big(\mathrm{Im}(W_0) + \sum_{e\in E} y'_e \underline{\mathbf{p}}_2 \widetilde{W}_e\Big)\cdot\Big(\mathrm{Im}(\Omega_0) + \sum_{e\in E} y'_e \widetilde{M}_e\Big)^{-1} \cdot\Big(\mathrm{Im}(Z_0) + \sum_{e\in E} y'_e \widetilde{Z}_e\, {}^t\underline{\mathbf{p}}_1\Big),$$

where $y'_e = y_e - h_0$.

We have

Theorem 4.3. *There exists a bounded function* $h\colon U \to \mathbb{R}$ *such that after shrinking the radius of* Δ *if necessary, we can write the height pairing as*

(4.11)

$$\langle \mathfrak{A}_s, \mathfrak{B}_s\rangle \;=\; -\sum_{e\in E} 2\pi y_e \underline{\mathbf{p}}_2 \Gamma_e\, {}^t\underline{\mathbf{p}}_1 + 2\pi\Big(\sum_{e\in E} y_e \underline{\mathbf{p}}_2 W_e\Big)\Big(\sum_{e\in E} y_e M_e\Big)^{-1}\Big(\sum_{e\in E} y_e Z_e\, {}^t\underline{\mathbf{p}}_1\Big) + h(s).$$

This theorem was proved in [1] using normlike functions in the terminology of [2, Section 3.1]. We now give a proof based on Theorem 1.1.

We treat first the case $g = h$ and explain later how to reduce to this case. Note that the case $g = h$ corresponds to all irreducible components of C_0 being of genus zero.

Proof of Theorem 4.3 in the case $g = h$. We use the notations of Proposition 4.2. Since ρ_0 is a holomorphic function on $S = \Delta^{3g-3}$, after shrinking the radius of Δ if necessary, we can assume that $\mathrm{Im}(\rho_0)$ is bounded. In addition, since h_0 is constant, the difference between $y'_e \underline{\mathbf{p}}_2 \Gamma_e\, {}^t\underline{\mathbf{p}}_1$ and $y_e \underline{\mathbf{p}}_2 \Gamma_e\, {}^t\underline{\mathbf{p}}_1$ is constant for each e. So we only need to prove that the third term in the right hand side of equation (4.10) is, up to a bounded function, equal to the second term in the right hand side of (4.11).

First, we can reduce to the case where $\mathbf{p}_i$ are real valued, c.f. [1]. Using the bilinearity of the right hand side term in (4.11), we can reduce to the case $\mathbf{p}_1 = \mathbf{p}_2$.

Let $H = H_1(G, \mathbb{R})$, and let $\omega \in \mathbb{R}^E$ be given by $\mathbf{p}$, and denote $H_\omega \supset H$ the subspace generated by H and ω as in Section 1.2. Let $\alpha = \sum_e y_e\langle\,,\rangle_e$ be the bilinear form on $\mathbb{R}^E$.

For a matrix of the form,

$$T = \begin{pmatrix} L & W \\ {}^{\mathrm{t}}W & S\end{pmatrix}$$

where L is an invertible $h\times h$ matrix, W is a (column) vector of dimension h, and S is a scalar, recall that the Schur complement of L is given by

$$T/L := -\,{}^{\mathrm{t}}W L^{-1} W + S,$$

and it verifies the equation

$$\frac{\det T}{\det L} = -\,{}^{\mathrm{t}}W L^{-1} W + S.$$

Using these observations, the expression on the right hand side of (4.11) is the ratio $2\pi \det(\alpha|_{H_\omega})/\det(\alpha|_H)$, for the basis of H (resp. H_ω) given by $B = \{b_1, \dots, b_h\}$ (resp. $B_\omega = \{b_1, \dots, b_h, \omega\}$). Similarly, the expression on the right hand side of Proposition 4.2

at any point s of U is the ratio $2\pi\det(\alpha|_{H_\omega}+\beta(s)|_{H_\omega})/\det(\alpha|_H+\beta(s)|_H)$ for a bilinear form $\beta(s)$ on H_ω (given by W_0, Z_0, Ω_0, h_0, and Γ_e, W_e, Z_e, M_e), calculated using the basis B and B_ω of H and H_ω).

By boundedness of W_0, Z_0, Ω_0, and h_0, $\beta(s)$ lies in a compact subset of the space of bilinear forms on H_ω. Fixing a complmenet H' to H_ω, i.e., $H_\omega+H'=\mathbb{R}^m$, and extending $\beta(s)$ trivially (by zero) to $\mathbb{R}^m$, we can assume that $\beta(s)$ is the restriction to H_ω of a bilinear form $\widetilde{\beta}(s)$ on $\mathbb{R}^m$, and that $\widetilde{\beta}(s)$ lie in a compact subset of the space of bilinear forms on $\mathbb{R}^m$ for $s\in U$.

Let M (resp. N) be the $h\times m$ (resp. $(h+1)\times m$) matrix of the coefficients of the basis B (resp. B_ω) in the standard basis of $\mathbb{R}^m$. Let $Y=\mathrm{diag}(y_1,\dots,y_m)$ be the diagonal $m\times m$ matrix of α in the standard basis of $\mathbb{R}^m$.

Let $A: U\to \mathrm{Mat}_{m\times m}(\mathbb{R})$ be the matrix-valued map taking at $s\in U$ the value $A(s)$ equal to the matrix of the bilinear form $\widetilde{\beta}(s)$ in the standard basis of $\mathbb{R}^m$.

Theorem 4.3 in the case $g=h$ now follows from Theorem 1.2, which is the statement that the difference $\det(N(Y+A)N^\tau)/\det(M(Y+A)M^\tau)-\det(NYN^\tau)/\det(MYM^\tau)$ is $O_{\underline{y}}(1)$. □

We now show how to reduce the treat the general case by reducing to a case similar to the case $g=h$ treated above.

Proof of Theorem 4.3, general case. Suppose $g>h$. Let $\mathscr{W}:=\mathrm{Im}(W_0)-\sum_{e\in E}y_e\underline{\mathbf{p}}_2\widetilde{W}_e$, and write $\mathscr{W}=(\mathscr{W}_1,\mathscr{W}_2)$ with $\mathscr{W}_1$ the vector of the h first coordinates. Similarly, write $\mathscr{Z}=\mathrm{Im}(Z_0)+\sum_{e\in E}y_e\widetilde{Z}_e\,{}^t\underline{\mathbf{p}}_1$, and write $\mathscr{Z}={}^t(\mathscr{Z}_1,\mathscr{Z}_2)$ with $\mathscr{Z}_1$ the first vector of the h first coordinates.

Let $\mathscr{M}=\mathrm{Im}(\Omega_0)+\sum_{e\in E}y_e\widetilde{M}_e$, and write

$$\mathscr{M}=\begin{pmatrix}\mathscr{M}_{11} & \mathscr{M}_{12}\\ \mathscr{M}_{21} & \mathscr{M}_{22}.\end{pmatrix} \tag{4.12}$$

Theorem 4.3 now follows from Proposition 4.4, similar to the proof of the case $g=h$ given above. □

Proposition 4.4. *We have*

$$\mathscr{W}\mathscr{M}^{-1}\mathscr{Z}-\Big(\sum_{e\in E}y_e\underline{\mathbf{p}}_2W_e\Big)\Big(\sum_{e\in E}y_eM_e\Big)^{-1}\Big(\sum_{e\in E}y_eZ_e\,{}^t\underline{\mathbf{p}}_1\Big)=O_{\underline{y}}(1).$$

Proof. Let $\mathscr{N}=\mathscr{M}^{-1}$, and write

$$\mathscr{N}=\begin{pmatrix}\mathscr{N}_{11} & \mathscr{N}_{12}\\ \mathscr{N}_{21} & \mathscr{N}_{22}\end{pmatrix} \tag{4.13}$$

with $\mathscr{N}_{11}$ and $\mathscr{N}_{22}$ square matrices of size $h\times h$ and $(g-h)\times(g-h)$, respectively. Writing

$$\mathscr{W}\mathscr{N}\mathscr{Z}=\mathscr{W}_1\mathscr{N}_{11}\mathscr{Z}_1+\mathscr{W}_1\mathscr{N}_{12}\mathscr{Z}_2+\mathscr{W}_2\mathscr{N}_{21}\mathscr{Z}_1+\mathscr{W}_2\mathscr{N}_{22}\mathscr{Z}_2,$$

in order to prove Claim 4.4, we prove

$$\mathscr{W}_1\mathscr{N}_{11}\mathscr{Z}_1-\Big(\sum_{e\in E}y_e\underline{\mathbf{p}}_2W_e\Big)\Big(\sum_{e\in E}y_eM_e\Big)^{-1}\Big(\sum_{e\in E}y_eZ_e\,{}^t\underline{\mathbf{p}}_1\Big)=O_{\underline{y}}(1),$$

and

$$\mathscr{W}_1\mathscr{N}_{12}\mathscr{Z}_2=O_{\underline{y}}(1),\quad \mathscr{W}_2\mathscr{N}_{21}\mathscr{Z}_1=O_{\underline{y}}(1),\quad \mathscr{W}_2\mathscr{N}_{22}\mathscr{Z}_2=O_{\underline{y}}(1).$$

For $y_1,\dots,y_m$ large enough, since $\mathscr{M}_{12},\mathscr{M}_{21},\mathscr{M}_{22}$ are bounded, we have the following expressions:

$$\mathscr{N}_{11}=(\mathscr{M}_{11}-\mathscr{M}_{12}\mathscr{M}_{22}^{-1}\mathscr{M}_{21})^{-1}\qquad,\qquad \mathscr{N}_{22}=(\mathscr{M}_{22}-\mathscr{M}_{21}\mathscr{M}_{11}^{-1}\mathscr{M}_{12})^{-1}$$
$$\mathscr{N}_{12}=-\mathscr{M}_{11}^{-1}\mathscr{M}_{12}(\mathscr{M}_{22}-\mathscr{M}_{21}\mathscr{M}_{11}^{-1}\mathscr{M}_{12})^{-1},\qquad \text{and}$$
$$\mathscr{N}_{21}=-\mathscr{M}_{22}^{-1}\mathscr{M}_{21}(\mathscr{M}_{11}-\mathscr{M}_{12}\mathscr{M}_{22}^{-1}\mathscr{M}_{21})^{-1}.$$

Note that $\mathscr{M}_{22}(s)=\Omega_{0,22}(s)$ for $s\in U$, and by our assumption on U, the matrices $\mathscr{M}_{22}^{-1}(s)$ lies in a compact set for $s\in U$. Thus, $\mathscr{N}_{11}=\mathscr{A}(s)+\sum_e y_eM_e$ for an $h\times h$ matrix-valued map $\mathscr{A}$ on U taking values in a compact set provided that $y_1,\dots,y_m$ are large. It follows from the result in the case $g=h$ that

$$\mathscr{W}_1\,\mathscr{N}_{11}\mathscr{Z}_1-\Big(\sum_{e\in E}y_e\underline{\mathbf{p}}_2W_e\Big)\Big(\sum_{e\in E}y_eM_e\Big)^{-1}\Big(\sum_{e\in E}y_eZ_e\,{}^t\underline{\mathbf{p}}_1\Big)=O_{\underline{y}}(1).$$

The boundedness of the other three quantities can be proved similarly. For example, to treat the term $\mathscr{W}_1\,\mathscr{N}_{12}\mathscr{Z}_2$, we observe first that $\mathscr{C}=\mathscr{M}_{12}(\mathscr{M}_{22}-\mathscr{M}_{21}\mathscr{M}_{11}^{-1}\mathscr{M}_{12})^{-1}$ lies in a bounded compact set provided that $y_1,\dots,y_m$ are large enough. We have

$$\mathscr{W}_1\,\mathscr{N}_{12}\mathscr{Z}_2=-\mathscr{W}_1\mathscr{M}_{11}^{-1}\mathscr{C}=-\mathscr{W}_1\mathscr{M}_{11}^{-1}\Big(\mathscr{C}_1-\mathscr{C}_2\Big),$$

with $\mathscr{C}_2=\sum_{e\in E}y_eZ_e\,{}^t\underline{\mathbf{p}}_1$ and $\mathscr{C}_1=\mathscr{C}+\mathscr{C}_2$.

Applying the result in the case $g=h$, we have for both the quantities for $k=1,2$

$$\mathscr{W}_1\,\mathscr{N}_{11}\mathscr{C}_k-\Big(\sum_{e\in E}y_e\underline{\mathbf{p}}_2W_e\Big)\Big(\sum_{e\in E}y_eM_e\Big)^{-1}\Big(\sum_{e\in E}y_eZ_e\,{}^t\underline{\mathbf{p}}_1\Big)=O_{\underline{y}}(1).$$

Taking now their difference shows what we wanted to prove. □

To conclude the proof of Theorem 1.2, we remark that by [1], the expression on the right hand side of Theorem 4.3 is precisely the right hand side term in Theorem 1.2.

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