Symplectic invariants and symplectic reduction
Details of [V1] p. 706

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The aim of this note is to clarify the proof of the camel problem from [V1] p.706. I wish to thank David Théret for pointing out some shortcomings in the proof.

In this note we extend the notion of G.F.Q.I. by allowing functions $S : N \times (F \times \mathbb{R}^k) \longrightarrow \mathbb{R}$ coinciding with a nondegenerate quadratic form near infinity. Note that F is assumed to be compact. In this note the parentheses indicate the fibre variables.

Then

$$L_S = \{(x, \frac{\partial S}{\partial x}(x,y,\xi)) \mid \frac{\partial S}{\partial p}(x,y,\xi) = 0; \frac{\partial S}{\partial \xi}(x,y,\xi) = 0\}$$

Note that L has a G.F.Q.I. in this generalized sense if and only if it is the reduction of a submanifold of $T^*(N \times F)$, which has a G.F.Q.I. in the former sense, by $C_F = T^*N \times O_F = \{(x,y,X,Y) \mid Y = 0\}$ This is a coisotropic subspace and $(C_F)^\omega = \{(x,y,X,Y) \mid x = 0, X = 0, Y = 0\}$. It follows from this remark that a number of proofs go through, from our familiar setting, corresponding to $F = \{pt\}$, to the general case. In particular existence of a G.F.Q.I. is invariant by Hamiltonian isotopy as we see by extending the isotopy from T^*N to $T^*(N \times F)$. On the other hand, uniqueness is unclear, since we need to extend our notion of stable equivalence (see [Th] for the standard case).

Also we define $c(\alpha, S)$ to be the critical value obtained by minimax from the image by the Thom class of $\alpha \in H^*(N \times F)$. Note that $H^*(N \times F) = H^*(N) \otimes H^*(F)$, so α can be decomposed as $\rho \otimes \sigma$. Then we define $\gamma(S) = c(\mu \otimes \mu, S) - c(1 \otimes 1, S)$.

We may also apply proposition 5.1 from [V1]:

Proposition 0.1. *For $\alpha \in H^*(N \times F)$ we have*

$$c(\alpha \otimes 1, S) \leq \inf_w c(\alpha, S_w) \leq$$

$$\sup_w c(\alpha, S_w) \leq c(\alpha \otimes \mu, S)$$

Definition. Let L_0, L_1 be Lagrange submanifolds. We say that L_0 and L_1 are *gf-homotopic* if and only if there exists a continuous family S_t of functions quadratic at infinity, such that $L_1 = L_{S_1}$, $L_2 = L_{S_2}$

Remarks:

1. The homotopy can be made generic, so that for each t except a finite number of them, S_t is the generating function of some manifold (i.e. satisfies the transversality condition) we have that S_t generates a submanifold L_t. Thus L_0 and L_1 can be connected by a regular homotopy, modulo a finite number of singularities. We still will denote by L_t the set of points defined by S_t.

2. Note that the quadratic form is assumed to be nondegenerate for all t. We could also have assumed that S_t is a fixed quadratic form (independent from t) outside a compact set.

3. Note also that we only assume that L_0 and L_1 have a G.F.Q.I. , we do not require that it should be unique. So L_0 and L_1 need not be isotopic to the zero section.

4. Finally we remark that this property is invariant by symplectic reduction.

Let W be an open set in T^*N, we define an invariant by setting

Definition.

$$\tilde{c}(\alpha, W; 0_N) =$$

$$\sup\{c(\alpha, S) \mid L_S \text{ is gf-homotopic to } 0_N \text{ by a homotopy supported in } W\}$$

Remarks:

1. The support of the isotopy, $\mathrm{supp}(L_t)$, is defined as the closure of $\bigcup_{0 \leq t \leq 1} L_t - 0_N$.

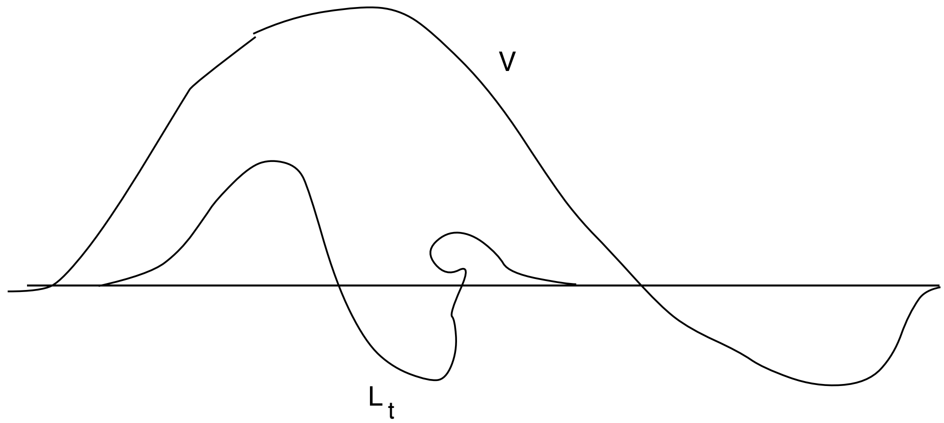

Figure 1:

2. Clearly we have that $\tilde{c}(\alpha, W; 0_N)$ is invariant by symplectic isotopies preserving the zero section.

3. For U an open set in $\mathbb{R}^{2n}$, we have the inequality $c(U) \leq \tilde{c}(\mu, U \times U; \Delta)$. Indeed the graphs of a symplectic map ϕ_t supported in U will be in $U \times U$. We denote by $\tilde{c}(\alpha, U) = \tilde{c}(\alpha, U \times U; \Delta)$. By the previous remark, this is a symplectic invariant.

Theorem 0.2. *Let S_t be a G.F.Q.I. for L_t, R be the G.F.Q.I. of V in T^*N such that $V \cap \operatorname{supp}(L_t) = \emptyset$. Then $c(\alpha, S_1) \leq c(\alpha, R) - c(1, R)$.*

Proof. We look as usual at $c(\alpha, R - S_t)$. This is independent of t, since critical points of $R - S_t$ correspond to points in $V \cap L_t$. Thus the number $c(\alpha, R - S_t)$ must be independent of t, and we have:

$$c(\alpha, R) = c(\alpha, R - S_0) = c(\alpha, R - S_1) \geq c(1, R) + c(\alpha, -S_1)$$

Thus

$$c(\alpha, S_1) \leq c(\alpha, R) - c(1, R).$$

□

We may now state

Proposition 0.3. *Let R be the G.F.Q.I. of N with $N \cap W = \emptyset$. Then*

$$\tilde{c}(\mu, W; O_N) \leq \gamma(R).$$

In particular since for N, graph of ψ with $\psi(U)\cap U=\emptyset$, and L_t supported in $U\times U$, we have $L_t\cap N=\emptyset$, we get:

Proposition 0.4. *If $\psi(U)\cap U=\emptyset$ then*

$$\tilde{c}(\mu,U)\leq\gamma(\psi)$$

Now we consider the following situation. Let U be a domain in $\mathbb{R}^{2n}\times\mathbb{R}^{2m}$. The coordinates are denoted by z,q,p, with $z\in\mathbb{R}^{2n}$ $q\in\mathbb{R}^m$ $p\in(\mathbb{R}^m)^*$. Let $x\in\mathbb{R}^m$ and set

$$U_x=(U\cap(\mathbb{R}^{2n}\times\{x\}\times\mathbb{R}^m)/(\{0\}\times\{x\}\times\mathbb{R}^m)$$

We now make the following compactifications: $\mathbb{R}^{2n}\times\mathbb{R}^{2m}\times\overline{\mathbb{R}^{2n}}\times\overline{\mathbb{R}^{2m}}$ can be identified to $T^*\Delta_{\mathbb{R}^{2n}}\times T^*\Delta_{\mathbb{R}^{2m}}$, where Δ denotes the diagonal. The identification on the factor $\mathbb{R}^{2m}\times\overline{\mathbb{R}^{2m}}$ being given through $(q,p,Q,P)\Longrightarrow(q+Q,\frac{p+P}{2},\frac{p-P}{2},Q-q)$ and we compactify $\Delta_{\mathbb{R}^{2n}}$ to S^{2n} and $\Delta_{\mathbb{R}^{2m}}$ to T^{2m}.

We may then define $c(\rho\otimes\sigma,U)$ for $\rho\in H^*(S^{2n})$ and $\sigma\in H^*(T^{2m})$.

Theorem 0.5.

$$c(\mu\otimes 1,U)\leq\inf_x\gamma(U_x)$$

Proof. Let $S:(S^{2n}\times T^{2m})\times\mathbb{R}^k$ be a G.F.Q.I. for $L\in T^*S^{2n}\times T^*T^{2m}$ with $L=0_{S^{2n}\times T^{2m}}$ outside $U\times U$. More precisely we assume there is a gf-homotopy connecting L with the zero section, having this property. Let us consider S_x, the restriction of S to $S^{2n}\times(\{2x\}\times T^m\times\mathbb{R}^k)$. Note that the "base" was $S^{2n}\times T^{2m}$ and is now S^{2n}. Thus the factor $(\{x\}\times T^m)$ that we expect to be in the base, is now in the fibre.

The submanifold L_{2x} is the reduction of L by $T^*S^{2n}\times\nu^*(\{2x\}\times T^m)=T^*S^{2n}\times(\{2x\}\times T^m)\times(\mathbb{R}^m\times\{0\})$. This is the set $\{(z,Z,q,p,Q,P)\mid q+Q=2x,q-Q=0\}$ or else $\{(z,Z,q,p,Q,P)\mid q=Q=x\}$. Now we claim that if L coincides with the zero section outside $U\times U$, then L_{2x} coincides with the zero section outside $U_x\times U_x$, and there is a gf-homotopy connecting L_{2x} with the zero section. Indeed we have that $(z,Z)\in L_{2x}-(U_x\times U_x)$, if and only if there exist q_0,p_0,Q_0,P_0 such that $(z,Z,q_0,p_0,Q_0,P_0)\in L$ with $q_0=Q_0=x$ and for all p,P we have that $(z,Z,x,p,x,P)\notin U\times U\}$. This last assumption implies that $z=Z$, hence $(z,Z)\in 0_{S^{2n}}$. We just proved that L_{2x} coincides with the zero section away from $U_x\times U_x$. Thus $c(\alpha,S_x)\leq\tilde{c}(\alpha,U_x)$, hence

$$c(\alpha\otimes 1,S)\leq\inf_x\tilde{c}(\alpha,U_x)\leq\inf_x\gamma(\alpha,U_x)$$

As a result

$$c(\mu \otimes 1, U) \leq \inf_x \gamma(U_x).$$

$\square$

References

[Th] D. Théret. *Thèse de Doctorat.* Université de Paris 7, 1996.

[V1] C. Viterbo. Symplectic topology as the geometry of generating functions *Math. Annalen*, 292: 685-710, 1992.